10^{th} International Conference on Fracture Mechanics of Concrete and Concrete Structures
FraMCoS-X
G. Pijaudier-Cabot, P. Grassl and C. La Borderie (Eds)
https://doi.org/10.21012/FC10.237725

FISHNET STATISTICS FOR QUASIBRITTLE MATERIALS WITH NACRE-LIKE ALTERNATING SERIES AND PARALLEL LINKS: DESIGN FOR FAILURE PROBABILITY $< 10^{-6}$

ZDENĚK P. BAŽANT* **(presenter) and WEN LUO**†

*Northwestern University
Evanston, IL USA
e-mail: z-bazant@northwestern.edu

§Northwestern University
Evanston, IL USA
e-mail:wenluo2016@u.northwestern.edu

Key words: Structure Reliability, Nacreous Materials, Biomimetic Materials, Probability Distribution of Strength, Failure Risk, Statistical Size Effect, Monte Carlo Simulations, Fracture Mechanics

Abstract. The failure probability of engineering structures such as bridges, airframes and MEMS ought to be 10^{-6}. This is a challenge. For perfectly brittle and ductile materials obeying the Weibull or Gaussian failure probability distribution functions (pdf) with the same coefficient of variation, the distances from the mean strength to 10-6 differ by about 2:1. For quasibrittle or architectured materials such as concrete, composites, tough ceramics, rocks, ice, foams, bone or nacre, this distance can be anywhere in-between. Hence, a new theory is needed. The lecture begins with a review of the recent formulation of Gauss-Weibull statistics derived from analytical nano-macro scale transitions and equality of probability and frequency of interatomic bond ruptures governed by activation energy. Extensions to the lifetime pdf based on subcritical crack growth is pointed out. Then, motivated by the nanoscale imbricated lamellar architecture of nacre, a new probability model with alternating series and parallel links, resembling a diagonally-pulled fishnet, has been developed. After the weakest-link and fiber-bundle models, it is the third model tractable analytically. It allows for a continuous transition between Gaussian and Weibull distributions, and is strongly size-dependent. The original fishnet model for strength of fishnet with brittle links is extended to quasibrittle links and is handled by order statistics. The size effect on the mean fishnet strength is a new kind of Type 1 size effect. It is found to consist of a series of intermediate asymptotes of decreasing slope and can be used for calibrating the fishnet distribution. Finally it is observed that random particulate materials such a concrete may follow the fishnet statistics in the low probability range. Comparisons with experimental histograms and size-effect tests support the theory.

This plenary conference lecture reviews recent studies at Northwestern University dealing with materials whose microstructure has a nacreous lamellar architecture. It focusses on the fishnet statistical model which, after the weakest-link and fiber-bundle models, became two years ago the third failure probability model tractable analytically. The lecture is based on a recent apercu of the theory that has just been published as an inaugural paper in the Proceedings of Royal Society A [1], and so only a succinct account of the theory is presented here.

1 INTRODUCTION

The amazingly robust mechanical properties of nacre-like imbricated (staggered) lamellar structures have been extensively studied over the past two decades [2–9, e.g.]. Good understanding of the deterministic toughening mechanism and of the critical role of the hierarchical fine-scale structure in enhancing material toughness has been achieved, and fostered the advent of novel bioinspired materials. These studies, however, were mostly deterministic and provided only the mean behavior. For nacreous material architectures, no probability distribution of strength up to the far left fail existed until the fishnet idea came up two years ago.

Capturing the far-out tail of failure probability 10^{-6} is the ultimate goal, whose main ideas first developed in detail in [10], compactly presented in [11], and recently broadly reviewed in [1]. Brittle material constituents were considered in the initial work. Subsequently, the failure probability of nacreous materials with progressively softening constituents has been analyzed [12]. Inferring the tail strength distribution via size effect tests has recently been studied and explained extensively in [13].

Designing structures using nacre-mimetic materials typically requires knowing their probability strength distribution up to the tail with probability close to $P_f = 10^{-6}$ per lifetime, which is generally the level of safety required for engineering structures such as bridges, aircraft, MEMS, etc [14,15]. It guarantees the failure risks of engineering structures to be four orders magnitude lower than other risks that people inevitably take (e.g., car driving), and leads to be of about the same level of risk as being killed, e.g., by a lightning or a falling tree. Such low tail probabilities can hardly be determined by histogram testing of millions of specimens or structures. Therefore, one needs a realistic and accurate mathematical model for the strength distribution, to be verified only indirectly, by other predictions depending on the tail, among which the size effect is most important.

Here, a diagonally pulled fishnet is introduced for the statistical modeling, providing a sufficiently realistic simplification of the connectivity of nacre's microstructure. Same as the weakest-link model, the failure probability of fishnet, P_f, is obtained by calculating its counterpart—the survival probability, $1 - P_f$. But, in contrast to the weakest-link model, the fishnet survival probability receives additional (additive) contributions from failure occurring when one, two, three, ... fishnet links have failed (depending on structure shape) before the maximum load. These additional survival probabilities greatly enhance the structure strength for $P_f < 10^{-6}$, compared to the the weakest-link model. They also provide a gradual transition from the weakest link model towards the fiber bundle model.

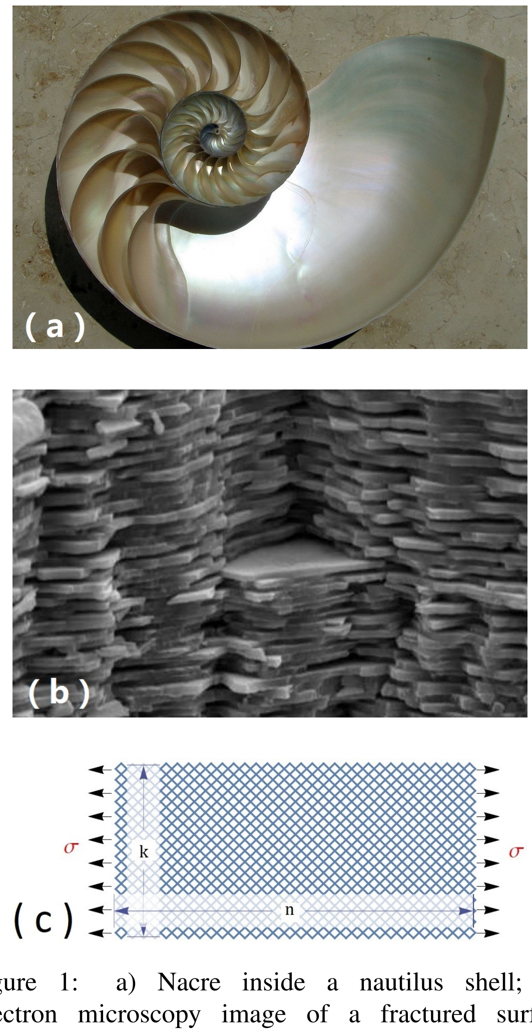


Figure 1: a) Nacre inside a nautilus shell; b) Electron microscopy image of a fractured surface of nacre (both (a) and (b) are from Wikipedia; https://en.wikipedia.org/wiki/Nacre); c) Simplified fishnet structure

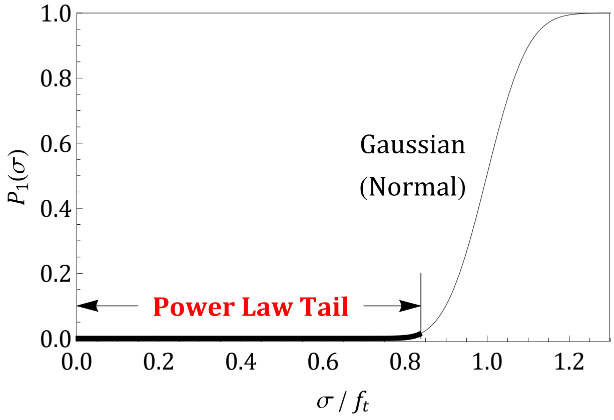
The analytical predictions of failure probability are here verified by million Monte-Carlo simulations for each of many loading cases. Monte Carlo simulations of nacreous structures have previously been conducted with the random fuse model (RFM) [5, 16], in which the brittle bonds in the structure are simplified as a lattice of resisters with random burnout thresholds. The RFM simulates the gradual failure of a resister network under increasing voltage. This is partly similar to the failure process of quasibrittle elastic material under controlled uniaxial load.

To calculate the maximum loads of the system of fishnet links, a simple finite element (FE) program for a pin-jointed truss is formulated (in MatLab). For each of many shapes and sizes of the fishnet, the maximum loads are calculated for about 1 million input samples of randomly generated strengths of the links, based on the assumption that the link strength follows the grafted Gauss-Weibull distribution (see [15]), previously derived from Kramers' transition rate theory for activation energy controlled breaks of interatomic bonds, and by asymptotic analysis of probability tails in scale transitions from nano to macro.. Running each set of about 1 million FE solutions takes a few days. With such a large number of random samples, the resulting strength histograms become visually indistinguishable from the theoretical cumulative probability density function (cdf) of failure probability P_f, derived analytically in [10].

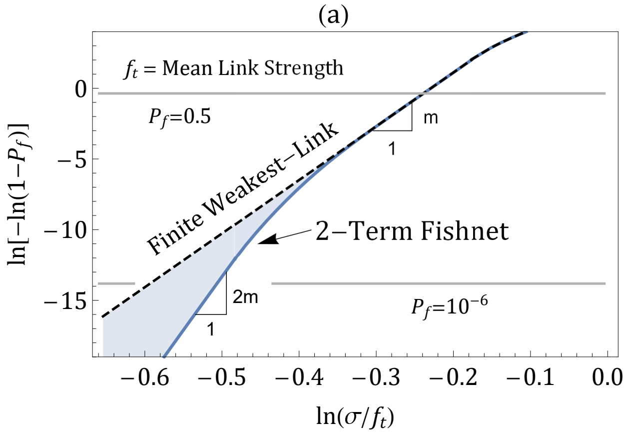
For the purpose of statistical analysis, the longitudinal load transmission must be realistically simplified. Almost no load gets transmitted between the ends of adjacent lamellae in one row, and virtually all the load gets transmitted by shear resistance of ultra-thin biopolymer layers within the overlap of parallel lamellae. The links of the lamellae in adjacent rows many be imagined as the lines connecting the lamellae centroids.

2 FAILURE PROBABILITY OF FISHNET MODEL

We consider the case of load control, for which the failure load is the maximum load, σ_{max}. We analyze rectangular fishnets with k rows and n columns, containing $N = k \times n$ links (Fig. 1.c), loaded uniformly by uniaxial stress σ imposed at the ends of rows. Let $P_f(\sigma)$ be the failure probability of fishnet loaded by σ, and $X(\sigma)$ the total number of links failed at the end of experiment under constant load σ. This means that $X(\sigma)$ is measured when no more damages occur.



(a)



(b)

Figure 2: a) Cumulative distribution function (cdf) of failure for a single link with mean $f_t = 10.016$ MPa and CoV = 7.8%); b) Comparison of P_f (in Weibull scale) between the finite weakest-link model and the fishnet model with first 2 terms in the expansion of Eq. 1

The failed links may be contiguous or scattered discontinuously. The events $\{X(\sigma) = r\}, r = 1, 2, 3, ...$ are mutually exclusive (or disjoint). So, to obtain the survival probability of the whole fishnet, the corresponding survival probabilities, $P_{S_r}(\sigma)$, must be summed;

$$1-P_f(\sigma) = P_{S_0}(\sigma) + P_{S_1}(\sigma) + P_{S_2}(\sigma) + \cdots \quad (1)$$

$$+ \mathrm{Prob}(\mathrm{X}(\sigma) \geq \mathrm{k} \text{ and structure still safe}) \quad (2)$$

where $P_f(\sigma) = \mathrm{Prob}(\sigma_{\max} \leq \sigma)$; σ_{max} = nominal strength of structure; and $P_{S_r}(\sigma) = \mathrm{Prob}(\mathrm{X}(\sigma) = r),\ r = 0, 1, 2, ...$

To get a better upper bound on structure strnegth, we now include the second term in Eq.(1), i.e., $1-P_f(\sigma) = P_{S_0}(\sigma)+P_{S_1}(\sigma)$ where σ = average longitudinal stress in the cross section, the same in every section. For the sake of simplicity, we further assume that: 1) the stress redistribution affects only a finite number, ν_1, of links in a finite neighborhood of the first failed link in which $\lambda_i > 1.1$, and 2) factor λ_i is treated as constant, $\lambda_i = \eta_a^{(1)}$ (> 1) within this neighborhood, taken either as the weighted average of all redistribution factors (to get the best estimate), or as the maximum of these factors (to preserve an upper bound on P_f). With this simplification,

$$P_{S_1}(\sigma) = NP_1(\sigma)[1-P_1(\sigma)]^{N-\nu_1-1}[1-P_1(\eta_a^{(1)}\sigma)]^{\nu_1} \quad (3)$$

Here N means that failure can start in any one of the N links, which gives N mutually exclusive cases. The two bracketed terms mean that the failure of one of the N links must occur jointly with the survival of: (i) each of the remaining $(N - \nu_1 - 1)$ links with stress σ, and of (ii) each of the remaining ν_1 links with redistributed stress $\eta_a^{(1)}$. Analysis shows that the second term of fishnet statistics P_{S_1} increases the terminal slope of strength probability distribution in Weibull scale by the factor of exactly 2.

Particularly important are the implications for structural safety. In Fig.1.c, the horizontal line for $P_f = 10^{-6}$ marks the maximum failure probability that is tolerable for engineering design. In this typical case, for constant N, the strength for $P_f = 10^{-6}$ is seen to increase by 10.5% when passing from the weakest-link failures to fishnet failures, while, at fixed strength, the P_f is seen to decrease about 25-times. The P_f decrease depends on the fishnet configurations and on P_1. but is generally more than 10-times greater. This is an enormous safety advantage of the imbricated lamellar microstructure, which comes in addition to the advantages for mean strength previously identified by deterministic studies.

Further improvement can be obtained by including the third term of the sum in Eq.(1). This term may be split into two parts, $P_{S_2} = P_{S_{21}} + P_{S_{22}}$, which are mutually exclusive, and thus additive. They represent the survival probabilities when the next failed link is, or is not, adjacent to the previously failed link. For detailed derivation, see [10].

3 MONTE CARLO FAILURE SIMULATIONS

A rectangular fishnet truss, with k rows and n columns of identical links, has been simulated by a finite element program (in MatLab). For computational stability, the fishnet is loaded under displacement control, by incrementing equal longitudinal displacements u_0 at the right boundary. At the left boundary, the horizontal displacement is zero. The boundary nodes slide freely in the transverse direction.

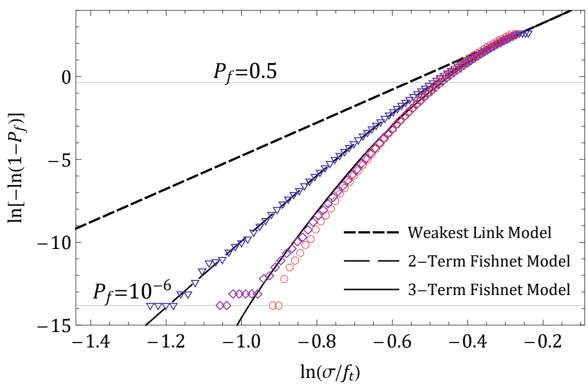


Figure 3: Normalized histogram of 10^6 Monte Carlo realizations (σ_{max}) compared with the probability density functions of the weakest-link, 2-term fishnet and 3-term fishnet models converted into cumulative probability distribution and plotted on the Weibull paper. f_t = 9.87 MPa is the mean strength of one link and CoV = 9.87%.

According to the arguments in [15, 17–19], based on nano-mechanics and scale transitions, the cumulative distribution function (cdf) of strength of each link, $P_1(\sigma)$, is assumed to be a Gaussian (or normal) distribution with a Weibull tail of exponent m grafted on the left at failure probability P_g (for $\sigma \to 0$, the cdf $\propto \sigma^m$). The strength of each of $N = k \times n$ links is generated randomly according to $P_1(\sigma)$. The autocorrelation length of the link strength field is assumed to be equal to the link size and, therefore, is not considered.

To verify the analytical two- or three-term statistics, respectively, the cases in which more than one, or two, links failed prior to the maximum load have been deleted from the set of about 1 million simulations of a fishnet having 16×32 links, CoV = 0.987 of P_1, and grafting point at P_g = 0.09. This is equivalent to omitting in Eq.(1) all the terms except the first two or three, respectively.

The remaining histograms ($\sigma_{max}^{(1)}$ and $\sigma_{max}^{(2)}$) are compared with the analytical cdf in Fig.3b (Fig.3a shows, for all simulations of σ_{max}, only the histogram). Despite simplifications, such as using a uniform redistribution ratio η and not distinguishing link failures at the boundary from those in the interior, the agreement is excellent. This validates the analytical solution.

Fig.3 shows, for comparison, also the histograms of all the Monte Carlo simulations, which correspond to the complete sum in Eq.(1). Note that, in this case, the three-term model, and even the two-term model, give a satisfactory estimate of fishnet cdf.

Shape Effect

Consider now the effect of the fishnet shape, or aspect ratio k/n. Fig. 4 shows the histograms obtained by random simulations (again about a million each) for fishnets with N = 256 links when their dimensions $k \times n$ are varied from 128×2, which represents the weakest-link chain (or series coupling), to 2×128, and is equivalent to the fiber bundle (or parallel coupling, with mechanics-based load sharing, i.e., equal extensions of all fibers). Obviously, the shape effect is very strong. However, fishnets with $k \gg n$ and rigid-body boundary displacements are usually not relevant to practical situations.

4 SIZE EFFECT

Fishnet statistics shows a different size effect on the median nominal strength of geometrically similar rectangular fishnets. To derive the size effect relation, the geometric scaling of a rectangular fishnet has been split into transverse and longitudinal scalings: transverse scaling increases the width (orthogonal to loading direction) while fixing its length; and longitudinal scaling increases the length (parallel to loading direction) while keeping its width constant.

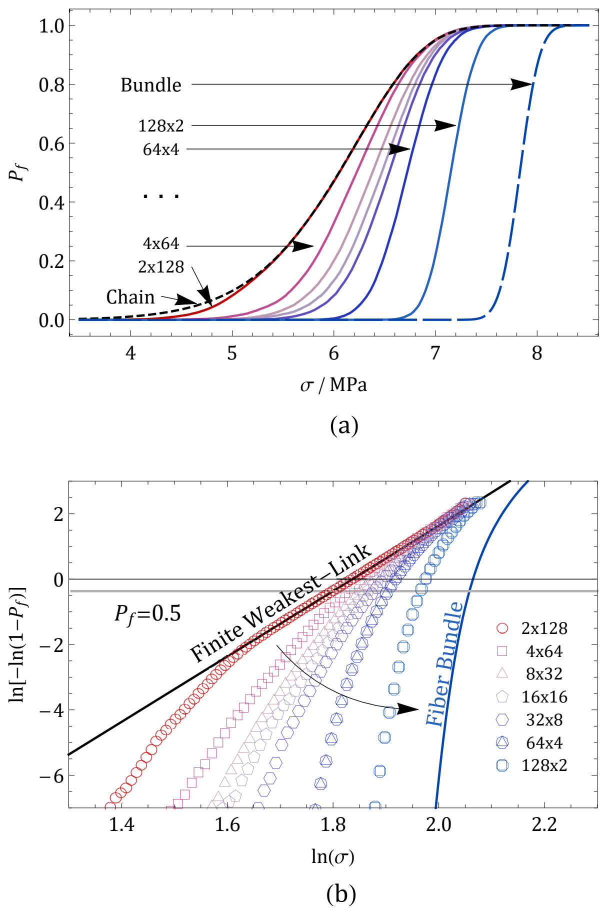


Figure 4: Change of failure probability of a fishnet pulled horizontally caused by varying the aspect ratio k/n gradually from $1 : N$ to $N : 1$ at constant number of links (Weibull scale): a) Monte Carlo simulations showing the transition of P_f as the aspect ratio of fishnet is changed ($N = 256$); b) The same data re-plotted on Weibull paper. $f_t = 9.87$ MPa is the mean strength of one link and CoV=9.87%.

Proportional combination of the effect of both transverse and longitudinal scaling then yields the two-dimensional geometric scaling and its size effect:

$$\ln \sigma_{0.5} = X = \frac{\ln\ln 2 - y_0 - \ln D}{m_0(1 + c\ln D)} + x_0 \quad (4)$$

Detailed derivation can be found in [13]. Eq. 4 is the median size effect relation that describes the relation between the logarithms of the median strength $X = \ln\sigma$ and the dimensionless size, D. There are four parameters in this relation; they are: x_0, y_0, c and m_0, where (x_0, y_0) is the coordinate of the point of rotation, Q, under transverse scaling; c is the rate of slope increase for histograms under transverse scaling; m_0 is the apparent Weibull modulus for the reference size fishnet. Note that m_0 is generally not the same as the Weibull modulus of the link strength distribution, and depends on the shape of the chosen reference size fishnet.

With Eq. (4), one could infer the strength distribution from the median size effect obtained from experiments or Monte Carlo simulations. Compared to the direct estimation of the histogram, which in general would require tens of thousands of test repetitions for $P_f > 10^{-3}$ (and, in theory, 10^7 repetitions for $P_f > 10^{-6}$), the size effect method requires only a few (typically not more than 6) tests for each of three or four structure sizes, to obtain a tight upper bound on the failure probability distribution.

Fig. 5 shows the median size effect curves using the median strengths of the samples obtained from the histograms of the Monte Carlo simulations. The optimum fit of the data by Eq. 4 is also shown for comparison. The size effect relation is monotonically decreasing and has a convex curvature, which is a general feature of the type-1 size effect [15, 20, 21].

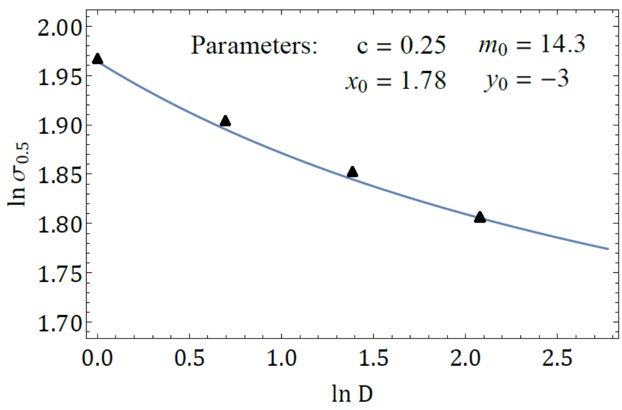


Figure 5: Optimum fit of the size effect relation ($\ln\sigma_{0.5}$ vs $\ln D$) using the sample median strength obtained from Monte Carlo simulations. The data points are the sample median strengths for the 4 typical sizes and the curve is the optimum fit.

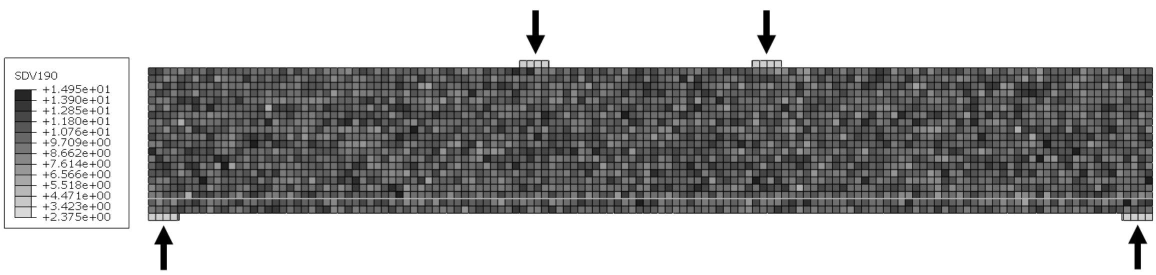


Figure 6: Shear failure of randomized RC beams under 4-point loading. The darker the color, the stronger the element (in terms of strength and stiffness).

5 RAMIFICATIONS TO CONCRETE, RC BEAM SHEAR, MASONRY AND OTHER QUASIBRITTLE MATERIALS

The microscale connectivity of particulate quasibrittle materials such as concrete provide provides complicated pathways for force transmission which include alternating lateral transfer of longitudinal loads. This observation suggests that these materials may exhibit partial fishnet connectivity in their load-transfer path, and thus the weakest link action may be partially enriched by fishnet action to some extent.

It surely does not occur in concrete and tough (or toughened) ceramics at failure probability level $P > 0.005$ because Weibull's (1939) histograms testing of about 10,000 specimens of concrete demonstrated that, in Weibull scale, a long lower tail of the histogram is perfectly straight. But that is so only for $P_f > 0.005$.

Nobody knows how the histogram for concrete or tough ceramics would look for the $3\frac{1}{2}$ orders of magnitude below. It could be that a fishnet type deviation comes into play. Probably the only way to answer this question would be by size effect testing of specimens of positive geometry. If so, there could be a safety gain at 10^{-6}, compared to an Weibullian extrapolation by the weakest link model.

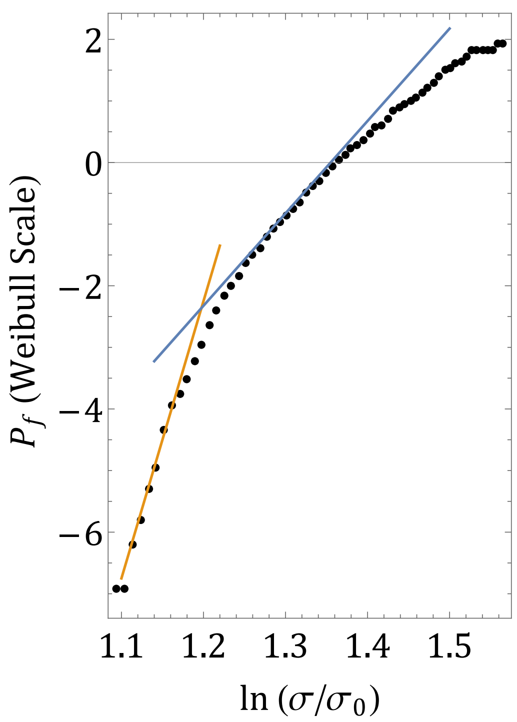


Figure 7: Histogram of nominal strengths of RC beams under shear failure (4-point loading). Sample size = 1000.

In particular, for various quasibrittle materials, Eq.(3) could be applied by determining the equivalent values of N, ν and $\eta_a^{(1)}$, where $N =$ equivalent number of links, $\nu =$ equivalent size of stress redistribution zone and $\eta_a^{(1)} =$ equivalent stress redistribution ratio.

Fig.6 shows one random realization of the strength and modulus of reinforced concrete (RC) beam under four-point shear loading. Microplane model M7 is here used for the constitutive law of concrete, in which Young's modulus and scaling parameter k_1 are randomized based on a Gauss-Weibull grafted distribution. The histogram of the simulations (normalized to cdf in Weibull scale) is shown in Fig.7. It is clear that the general pattern of strength distribution is very similar to that shown in Fig.3 for the fishnet. The slope of the curve at the lower tail has increased slightly more than 3 times compared to the upper tail. Therefore, a 3-term fishnet model could be enough to capture the failure probability (i.e. strength distribution) of the beam.

Finally note that the microstructures of nacre and of the brick-and-mortat masonry are quite similar. Thus the fishnet statistics could also be applied to masonry, which bears the same brick-and-mortar structure as nacre.

6 CONCLUSIONS

1. The failure statistics of nacre-like material with imbricated (or staggered) lamellar microstructure under longitudinal tension can be approximately modeled by square or rectangular fishnets pulled diagonally.
2. The probability distribution of fishnet strength, including the far-out left tail, can be calculated from the survival probability representing a sum of those corresponding to the failure of zero, one, two, three, etc., links prior to the overall failure. The series converges rapidly—the faster the greater the coefficient of variation (CoV) of scatter of each link.
3. The terms of this series represent various combinations of joint probabilities of survival and additive probabilities of failure for disjoint events. Near the zone of failed links, the link survival probabilities must be modified according to the mechanical stress redistribution due to previously failed links.
4. There is no fixed-size representative vol-

ume element of material (RVE), in contrast to the weakest-link model for Type 1 quasibrittle failures of particulate materials. The size of the zone of failed links at maximum load grows with the CoV of link strength.

5. The size effect law is similar, though not the same, as in quasibrittle Type 1 finite weakest-link model. The longitudinal scaling leads to a vertical upward shift of the distribution in the Weibull scale, while the transverse scaling leads to a counter-clockwise rotation of the distribution about a fixed point located below the median. Combination of scalings in two separate directions gives the size effect relation. The nominal strength of fishnet at the same width-to-length ratio decreases significantly with the fishnet size. At the Failure probability 10^{-6}, a reverse size effect on structure strength is possible.
6. The fishnet shape, i.e., the width-to-length aspect ratio, has a major effect on the probability distribution of strength, which contrasts with to finite weakest-link model for Type 1. The greater this ratio, the higher is the safety margin, i.e., the greater is the strength at the failure probability level $P_f = 10^{-6}$. As the aspect ratio is increased from 0 to ∞, the fishnet gradually transits for the weakest-link chain to the fiber bundle as the limit cases.
7. The fishnet model is verified by about a million Monte Carlo simulations of failure for each of many loading cases. The simulations were run for each of many different aspect ratios, link strength CoVs and fishnet sizes.
8. There now exist three basic, analytically tractable, statistical models for the strength of materials and structures:

   - the fiber bundle model (parallel coupling),
   - the weakest-link chain model (series coupling), and
   - the fishnet model (mixed, or imbricated coupling)

   The third one includes the first two as the limit cases.
9. A similar steepening of the distribution slope at the lower end of Weibull scale plot can also be achieved by the chain-of-bundles model, but only if a convenient intuitive non-mechanical load-sharing rule is empirically postulated for each bundle, and if the specimen length is subdivided by chosen cross sections into statistically independent segments of suitable length, corresponding to each bundle. However, the imbricated (staggered) lamellar connectivity cannot be captured.
10. At present, the experimental results, for any material, do note reveal the strength distribution for failure probability less than 0.005, which is $3\frac{1}{2}$ orders of magnitude above what is needed, i.e., 10^{-6}. It is likely that the fishnet statistics might be at least partially applicable to other quasibrittle materials such as concrete or brittle ceramics. As for masonry, its structures is similar to nacre and so it is likely to follow the fishnet statistics. Monte Carlo simulations of shear failure of RC beams also suggest partial fishnet action in the lower probability tail.

Acknowledgment: Financial support under ARO Grant W91INF-19-1-0039 to Northwestern University is gratefully acknowledged.

REFERENCES

[1] Zdeněk P Bažant. Design of quasibrittle materials and structures to optimize strength and scaling at probability tail: an apercu. *Proceedings of the Royal Society A*, 475(2224):20180617, 2019.

[2] Huajian Gao, Baohua Ji, Ingomar L Jäger, Eduard Arzt, and Peter Fratzl. Ma-

terials become insensitive to flaws at nanoscale: lessons from nature. *Proceedings of the national Academy of Sciences*, 100(10):5597–5600, 2003.

[3] RZ Wang, Z Suo, AG Evans, N Yao, and IA Aksay. Deformation mechanisms in nacre. *Journal of Materials Research*, 16(9):2485–2493, 2001.

[4] Yue Shao, Hong-Ping Zhao, Xi-Qiao Feng, and Huajian Gao. Discontinuous crack-bridging model for fracture toughness analysis of nacre. *Journal of the Mechanics and Physics of Solids*, 60(8):1400–1419, 2012.

[5] Zsolt Bertalan, Ashivni Shekhawat, James P Sethna, and Stefano Zapperi. Fracture strength: Stress concentration, extreme value statistics, and the fate of the weibull distribution. *Physical Review Applied*, 2(3):034008, 2014.

[6] Roberto Ballarini and Arthur H Heuer. Secrets in the shell the body armor of the queen conch is much tougher than comparable synthetic materials. what secrets does it hold? *American Scientist*, 95(5):422–429, 2007.

[7] F Barthelat and HD Espinosa. An experimental investigation of deformation and fracture of nacre–mother of pearl. *Experimental mechanics*, 47(3):311–324, 2007.

[8] Abhishek Dutta and Srinivasan Arjun Tekalur. Crack tortuousity in the nacreous layer–topological dependence and biomimetic design guideline. *International Journal of Solids and Structures*, 51(2):325–335, 2014.

[9] S Kamat, X Su, R Ballarini, and AH Heuer. Structural basis for the fracture toughness of the shell of the conch strombus gigas. *Nature*, 405(6790):1036–1040, 2000.

[10] Wen Luo and Zdeněk P Bažant. Fishnet model for failure probability tail of nacre-like imbricated lamellar materials. *Proceedings of the National Academy of Sciences*, 114(49):12900–12905, 2017.

[11] Wen Luo and Zdeněk P Bažant. Fishnet statistics for probabilistic strength and scaling of nacreous imbricated lamellar materials. *Journal of the Mechanics and Physics of Solids*, 109:264–287, 2017.

[12] Wen Luo and Zdeněk P Bažant. Fishnet model with order statistics for tail probability of failure of nacreous biomimetic materials with softening interlaminar links. *Journal of the Mechanics and Physics of Solids*, 121:281–295, 2018.

[13] Wen Luo and Zdeněk P Bažant. Fishnet statistical size effect on strength of materials with nacreous microstructure. *Journal of Applied Mechanics*, in press, 2019.

[14] Ronald Aylmer Fisher and Leonard Henry Caleb Tippett. Limiting forms of the frequency distribution of the largest or smallest member of a sample. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 24, pages 180–190. Cambridge University Press, 1928.

[15] Zdenek P Bazant and Jia-Liang Le. *Probabilistic Mechanics of Quasibrittle Structures: Strength, Lifetime, and Size Effect*. Cambridge University Press, 2017.

[16] Mikko J Alava, Phani KVV Nukala, and Stefano Zapperi. Statistical models of fracture. *Advances in Physics*, 55(3-4):349–476, 2006.

[17] Zdeněk P Bažant and Sze-Dai Pang. Mechanics-based statistics of failure risk of quasibrittle structures and size effect on safety factors. *Proceedings of the National Academy of Sciences*, 103(25):9434–9439, 2006.

[18] Zdeněk P Bažant and Sze-Dai Pang. Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. *Journal of the Mechanics and Physics of Solids*, 55(1):91–131, 2007.

[19] Zdeněk P Bažant, Jia-Liang Le, and Martin Z Bazant. Scaling of strength and lifetime probability distributions of quasibrittle structures based on atomistic fracture mechanics. *Proceedings of the National Academy of Sciences*, 106(28):11484–11489, 2009.

[20] Zdenek P Bazant. *Scaling of structural strength*. Butterworth-Heinemann, 2005.

[21] Mohammad Rasoolinejad and Zdeněk P Bažant. Size effect of squat shear walls extrapolated by microplane model M7. *ACI Structural Journal*, page in press, 2019.