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A NOVEL CONTINUUM DAMAGE MODEL FOR QUASI-BRITTLE MATERIALS: APPLICATION TO CONCRETE

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Abstract. The macroscopic behaviour of concrete is quasi-brittle, and its fracture behaviour is greatly influenced by the fracture process zone (FPZ). The experimental studies on plain concrete show that the load-displacement response of the concrete is characterized by an initial elastic phase followed by a nonlinear behaviour up to peak load and subsequent non-linear softening response. Continuum Damage Mechanics (CDM) is a widely used approach for modeling the fracture behavior of quasi-brittle materials. In CDM, softening is represented by stiffness degradation, modeled using a monotonically decreasing damage parameter. This damage variable is characterized through area reduction in the cross-section, degradation of elastic stiffness, microcrack density, etc. However, the evolution equations in most of the models are not consistent with their physical meaning. In this work, we relate damage to a probability measure that modifies the load-carrying area or volume (in a diffused sense). The damage variable evolves in a manner analogous to transition probability density in non-conserved processes, like those observed in killed diffusion processes. The evolution equation consists of a killing rate term that controls the rate of damage/degradation. The structure of the killing rate is such that it consists of a term that controls initial elastic behaviour, and a parameter that controls the rate of fracture. To ensure the monotonic reduction in this variable, the killing rate is ensured to always be a positive quantity. Softening in the load-displacement response is captured similarly to that of CDM through a gradual reduction in stress in the linear momentum balance equation. We validate our model by reproducing the crack pattern and load-displacement responses observed in corresponding experimental studies of plain concrete.

1 Introduction

Fracture is a prevalent failure mode in engineering materials like glass, concrete, rocks, ceramics, etc. and failure is typically preceded by the formation of intricate crack trajectories. Predicting complex crack patterns requires high-fidelity numerical tools, which may involve high computational costs. Fracture and damage theories that form the basis for these numerical tools must have sufficiently sound physical and mathematical foundations to reduce this cost and increase the ease of numerical implementation. Similar to [1], a mathematically rigorous treatment can augment the physical understanding of the bond-breaking or the degradation process for the same.

This article proposes a novel regularised integral continuum damage model for quasi-brittle fracture that draws inspiration from a measure-theoretic perspective on the Continuum Damage mechanics (CDM). Degradation or damage is associated with a change of measures, with the degradation function appearing as a Radon-Nikodym derivative and damage physically representing breaking of the macroscopic bonds between material points that transfer force. The damage evolution has a closed-form solution and does not need to be numerically solved through a computationally intensive process.

Existing damage models can be classified into two main categories: continuum and discrete approaches. A key distinction between them is how they handle displacement jumps resulting from discontinuities. Discrete approaches consider the displacement jump due to discontinuity in the formulation, while continuum approaches smear it out.

Initial discrete approaches employed Finite Element Methods (FEM) with re-meshing to simulate crack growth. However, this method had a significant limitation: the crack path was constrained to follow element edges, leading to high mesh dependency in the predicted crack paths. Extended finite element (XFEM) [2] doesn't require re-meshing; However, prior knowledge about the crack path for enrichment is needed. Discrete models often necessitate complex crack-tracking algorithms, making them less suitable for intricate problems like 3D crack propagation. Other discrete methods include cohesive crack model [3] depending on traction separation law, interelement separation model [4], cracking particle method [5] etc.

In contrast, continuum-based theories like the Microplane and crack band models are particularly well-suited for materials like concrete [6, 7]. Microplane models define constitutive equations using vectors representing stress and strain on randomly oriented planes, differing from classical continuum approaches, where second and fourth order tensors are used. However, It has been pointed out that the Microplane model needs a restriction on Poisson's ratio to be 0.25. Furthermore, it faces challenges in achieving thermodynamic consistency, as the Microplane response governs both elastic and inelastic behaviour.

In CDM, stiffness evaluation is addressed by introducing a damage parameter to quantify material deterioration. Damage can be defined at various scales in CDM. At the microscale, it might involve the rupture of atomic bonds. At the mesoscale, damage can manifest as the nucleation, growth, and coalescence of microscopic voids. At the macroscale, damage is often associated with the propagation of cracks. These phenomena collectively contribute to material degradation [8]. The physical interpretation of the damage parameter varies across studies. It has been linked to factors such as effective load-carrying area [9], variation in elastic modulus [10] micro crack density [11], etc. This diversity highlights the challenge of defining a univocal damage variable. Moreover, many existing studies lack consistency in their damage evolution equations.

Local continuum damage models may be physically inconsistent due to their assumption of scale-independent behaviour. Micromechanical arguments [12] support the need for nonlocal CDM, citing two key reasons: (a) The release of stored energy from a single microcrack depends on the macroscopic continuum's average strain, and (b) Microcracks interact with each other. Most engineering materials have an associated characteristic length scale of heterogeneity. Consequently, non-local models are essential for both physical accuracy and numerical stability.

Non-local models can be categorized into two types: (a) Integral-type models, which define the non-local field through a weighted average over a small neighbourhood that depends on physical quantities like the length scale of heterogeneity, and (b) Gradient-enhanced models, which primarily originate from integral approaches. These models require small length scale parameters, allowing the higher gradient of the non-local variable to be disregarded.

The phase field method(PFM) is a gradient-enhanced model. While PFM is popular for modeling phase transformations, has been adapted for fracture mechanics [13]. However, this adaptation lacks a clear physical interpretation for the phase field variable [1]. This necessitates redefining the free energy functional and developing specific numerical techniques, potentially leading to challenges in accurately capturing fracture behavior

A recent study [1] proposed a nonlocal damage model within the graph-based finite element analysis (GraFEA) framework [14]. In this approach, each element is assumed to have a finite number of potential crack planes, and the evolution of the damage variable (which represents the survival probability of the crack plane) is governed by a master equation [15]. However, the master equation doesn't follow any known probability evolution equations. This work introduces a novel non-local CDM framework that leverages the concept of a Radon-Nikodym derivative to model damage. By treating damage as a probability measure, we provide a clearer physical interpretation and avoid the limitations associated with traditional phase-field approaches. This approach offers a more efficient and accurate way to simulate damage evolution in quasi-brittle materials

The remainder of this article is structured as follows. In §2, we discuss the underlying rationale for our approach and the connection between degradation and changes of measures. Also, the governing PDE and dissipation inequality is presented. In §3, we reproduce key experimental observations in concrete fracture. Finally, we conclude our contribution in §4.

2 Theoretical framework

Let Ω_0 denote the cracked domain as illustrated in Fig. 1. The boundary of the Ω_0, Γ is partitioned into Γ_t on which external tractions $\tilde{\boldsymbol{t}}$ are specified and Γ_u on which displacements $\tilde{\boldsymbol{u}}$ are prescribed.

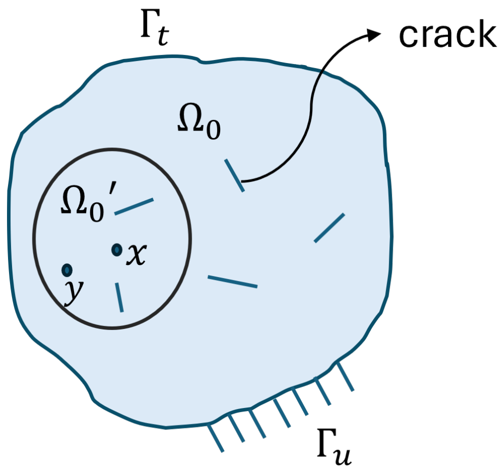

Figure 1: Deformable body with domain Ω_0 and boundary Γ

In Continuum Damage Mechanics(CDM), the energy within a given domain Ω_0, for an undamaged body can be typically expressed as

$$\mathcal{E} = \int_{\Omega_0} \psi_0 d\Omega_0 \tag{1}$$

where ψ_0 is the free energy density of the intact body. we assume ψ_{dam} as the free energy density of the damaged body and that $d\hat{\Omega}_0$ be the modified incremental measure in the presence of damage.

In CDM, ψ_{dam} can be written using the degradation function (ϕ) as,

$$\psi_{\text{dam}} = \phi\psi_0 \tag{2}$$

Using the modified incremental measure, the Energy can now be written as

$$\mathcal{E} = \int_{\hat{\Omega}_0} \psi_{\text{dam}} d\hat{\Omega}_0 = \int_{\Omega_0} \phi\psi_0 d\Omega_0 \tag{3}$$

Here, the degradation function can also regarded as the Radon-Nikodym derivative. We assume that between the material points, the mechanical force is transferred through the macroscopic bonds and that the material degradation progresses through the snapping of these macroscopic bonds. We relate damage to the probability measure (P), which measures the number density of the bonds that are not broken (undamaged). We assume that the evolution of these macroscopic bonds is diffusive, and hence, the evolution equation of the probability density function (ϕ) follows the Fokker-Planck equation. Adding a suitable killing term, we obtain the probability density function over a time intervel dt as,

$$\begin{aligned} \phi\left(|\boldsymbol{y}-\boldsymbol{x}|\right) = \frac{1}{(2\pi\beta^2 dt)^{\frac{n}{2}}} \times \\ \exp\left\{ -\int_t^{t+dt} G ds - \frac{|\boldsymbol{y}-\boldsymbol{x}|^2}{2\beta^2 dt} \right\} \end{aligned} \tag{4}$$

where $\boldsymbol{x}$ is the material point, $\boldsymbol{y}$ is the neighbouring point, n is the dimension of $\boldsymbol{y}$, β^2 is the diffusion coefficient, and G is the killing rate.

To ensure the monotonic reduction in the damage variable and to satisfy the thermodynamic restrictions, the killing rate is ensured to always be a positive quantity. The form of the killing rate is given in eqn. (5) that it consists of a term that controls initial elastic behaviour, and a parameter that controls the rate of fracture, and a term which vanishes during unloading.

$$G = G_0 \langle \frac{\sigma_{\text{eq}}}{\sigma_{\text{c}}} - 1 \rangle \langle \dot{\sigma}_{\text{eq}} \rangle \tag{5}$$

where, $\langle A \rangle$ denotes the Macaulay brackets of A which is defined as $\frac{A+|A|}{2}$ and σ_{eq} is the equivalent stress which represents the current loading condition. The equivalent stress is calculated following the method outlined in [16]. The expression for the equivalent stress is given by Eqn. 6

$$\sigma_{\text{eq}} = \sqrt{2\psi^+ E} \tag{6}$$

where ψ^+ is the part of ψ contributing to damage and E is the Young's modulus. ψ^+ can be defined through various menthods available in literature [17, 18]. In the proposed model, we have considered the spectral decomposition [19].

The dependencies of ψ are as defined in Eqn.14. The free energy function is given by Eqn. 17

In this study, we consider infinitesimal deformations. Let us denote the stress and strain tensor as $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, respectively. At a material point $\boldsymbol{x}$ the small strain tensor $\boldsymbol{\epsilon}$ as a function of displacements $\boldsymbol{u}$ is given as,

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right) \tag{7}$$

To accurately model crack initiation and propagation, it is essential to consider the non-local effects that influence the fracture process [1]. Hence, we consider the stress and strain as the non-local quantities using the transition density. Let Ω_0' be the sub-domain used to define the non-local or the mean stress and strains

$$\boldsymbol{\sigma}^*(\boldsymbol{x}) = \int_{\Omega_0'} \boldsymbol{\sigma}(\boldsymbol{y}) \phi\left(|\boldsymbol{y}-\boldsymbol{x}|\right) d\Omega_0' \tag{8}$$

$$\boldsymbol{\epsilon}^*(\boldsymbol{x}) = \int_{\Omega_0'} \boldsymbol{\epsilon}(\boldsymbol{y}) \phi\left(|\boldsymbol{y}-\boldsymbol{x}|\right) d\Omega_0' \tag{9}$$

The internal power can then be expressed as:

$$P_{int} = \int_{\Omega_0} \boldsymbol{\sigma}^* : \dot{\boldsymbol{\epsilon}} d\Omega_0 \tag{10}$$

The above expression is similar to nonlocal elasticity (see, e.g. [20]). In our work, ϕ is related to damage and its evaluation incorporates the degradation mechanism during damage evolution. Assuming that there is no external body force acting, the external power can be expressed using the following expression

$$P_{ext} = \int_{\partial\Omega_0} \tilde{\boldsymbol{t}} \cdot \boldsymbol{v} d\Gamma \tag{11}$$

using the divergence theorem and invoking the virtual power balance principle($P_{ext} = P_{int}$), the linear momentum balance equation can be written as,

$$\nabla \cdot \boldsymbol{\sigma}^* = 0 \quad \text{on } \Omega_0 \tag{12a}$$

$$\boldsymbol{\sigma}^* \boldsymbol{n} = \tilde{\boldsymbol{t}} \quad \text{on } \Gamma \tag{12b}$$

Where $\boldsymbol{n}$ is unit outward normal to the boundary Γ.

The energy imbalance can be written as:

$$P_{int} - \int_{\Omega_0} \dot{\psi} d\Omega_0 \geq 0 \tag{13}$$

Assuming the form of free energy density as

$$\psi = \psi(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^*) \tag{14}$$

Taking the material-time derivative of the free energy density, and using the self-adjoint property of the non-local operator (refer to Eq. 5 in [20], we obtain the constitutive relation and the reduced dissipation inequality as :

$$\boldsymbol{\sigma}^* = \frac{\partial \psi}{\partial \boldsymbol{\epsilon}} + \int_{\Omega_0'} \frac{\partial \psi}{\partial \boldsymbol{\epsilon}^*}(\boldsymbol{y})\phi\left(|\boldsymbol{y}-\boldsymbol{x}|\right) d\Omega_0' \quad (15)$$

$$\frac{\partial \psi}{\partial \boldsymbol{\epsilon}^*} : \int_{\Omega_0'} \boldsymbol{\epsilon}(\boldsymbol{y})\dot{\phi}(|\boldsymbol{y}-\boldsymbol{x}|)|d\Omega_0' \leq 0 \quad (16)$$

The form of the free energy considered as

$$\psi(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^*) = \frac{\lambda}{2}(\mathrm{tr}\boldsymbol{\epsilon})(\mathrm{tr}\boldsymbol{\epsilon}^*) + \mu\boldsymbol{\epsilon} : \boldsymbol{\epsilon}^* \quad (17)$$

where, λ and μ are Lame's parameters. Hence, we obtain the constitutive relation on $\boldsymbol{\sigma}^*$ as

$$\boldsymbol{\sigma}^* = \lambda(\mathrm{tr}\boldsymbol{\epsilon}^*) + 2\mu\boldsymbol{\epsilon}^* \quad (18)$$

3 Numerical results

In this section, we consider our constitutive theory to validate the wedge splitting test for plain concrete. A wedge is inserted between opposing faces of the specimen. Applying a compressive load to the wedge forces it deeper, inducing tensile stresses that initiate and propagate cracks. The specimen's geometry along with loading and boundary conditions are shown in Fig. 2. The thickness of specimen is 400 mm [21, 22]. This is a plane stress problem. Two horizontal splitting displacements u_r are imposed on the upper lateral faces. The reaction force F_r and the crack mouth opening displacement (CMOD) are measured during loading. The following material properties are chosen: $G_0 = 0.0385mm^2/N$, Young's modulus $E = 28300MPa$, tensile strength $\sigma_c = 2.12MPa$, and Poisson's ratio $\nu = 0.18$. The mesh size in the vicinity of the crack is approximately 1.5 mm and it gradually increases to 100 mm. (see Fig. 3)

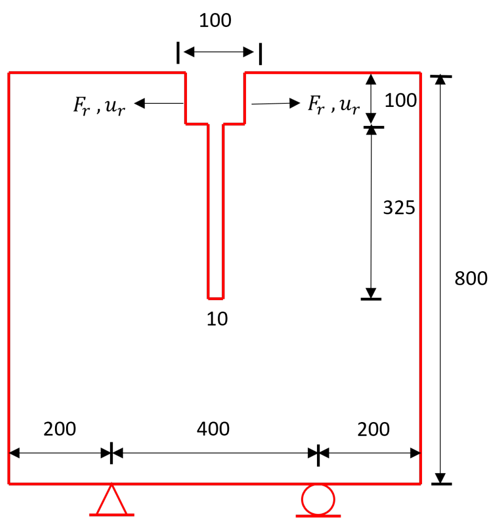

Figure 2: Geometry and boundary conditions for wedge splitting test (all dimensions are in mm)

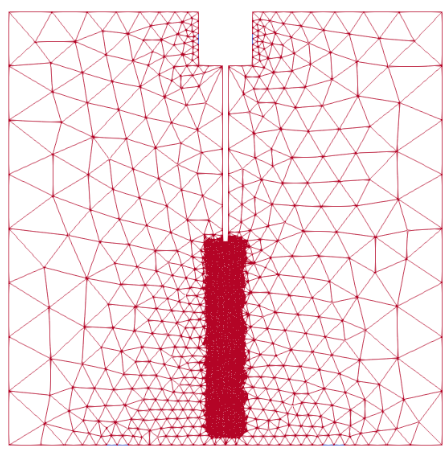

Figure 3: FE mesh with the average element size of 1.5 mm around the region where the crack is expected to propagate

The simulated crack path is shown in the Fig. 4 and the pattern closely matches with experimental observations.

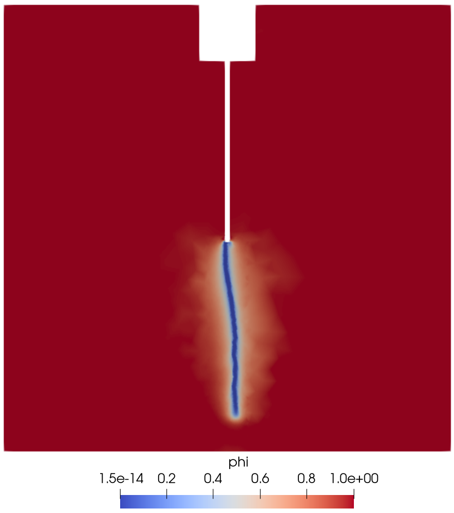

Figure 4: contour plot of damage

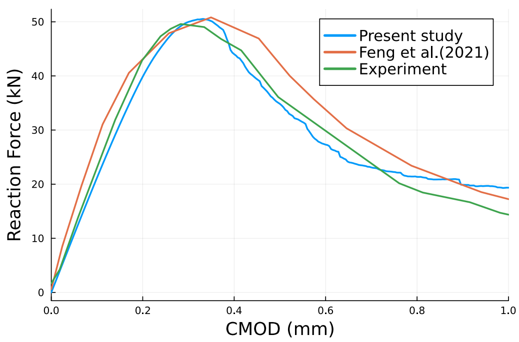

Figure 5: Comparisons of Force–CMOD curves with experiments and PFM

The predicted and the experimental force-displacement response are shown along with the response of Cohesive Phase Field model ([22]) in Fig. 5. The predicted response compare well with the experimental results.

4 Conclusions

We propose a new approach for modeling damage in the framework of CDM. We introduce a probability measure by recognizing the similarity between the degradation function in CDM and the Radon-Nikodym derivative in measure theory. The evolution of this probability measure is modelled as a killed diffusion process with a positive killing rate. Our model interprets damage as breaking bonds between material points (i.e., the ability to transfer the forces between two material points). The problem was framed so that the availble closed-form solution in the killed diffusion process for the evolution of PDF can be used. This formulation allows for a closed-form solution of the damage evolution, eliminating the need for computationally expensive micro-force balance equations. Along with the substantive computational efficiency, this model also shows physical transparency. The ad-hoc history-dependent routes that are used to account for the irreversibility of the damage can be eliminated by considering an appropriate killing rate in our method. The efficacy of the proposed formalism is manifested by performing numerical experiments on quasi-brittle damage using wedge splitting test.

REFERENCES

[1] P Thamburaja, K Sarah, A Srinivasa, and JN Reddy. Fracture modelling of plain concrete using non-local fracture mechanics and a graph-based computational framework. *Proceedings of the Royal Society A*, 477(2252):20210398, 2021.

[2] Ted Belytschko and Tom Black. Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5):601–620, 1999.

[3] Arne Hillerborg, Mats Modéer, and P-E Petersson. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and concrete research*, 6(6):773–781, 1976.

[4] X-P Xu and A Needleman. Numerical simulations of dynamic crack growth along an interface. *International journal of fracture*, 74:289–324, 1996.

[5] Timon Rabczuk and Ted Belytschko. Cracking particles: a simplified meshfree method for arbitrary evolving cracks. *International journal for numerical methods in engineering*, 61(13):2316–2343, 2004.

[6] Zdeněk P Bažant and Byung H Oh. Crack band theory for fracture of concrete. *Matériaux et construction*, 16:155–177, 1983.

[7] Ferhun C Caner and Zdeněk P Bažant. Microplane model m7 for plain concrete. ii: Calibration and verification. *Journal of Engineering Mechanics*, 139(12):1724–1735, 2013.

[8] Sumio Murakami. *Continuum damage mechanics: a continuum mechanics approach to the analysis of damage and fracture*, volume 185. Springer Science & Business Media, 2012.

[9] Lasar Kachanov. *Introduction to continuum damage mechanics*, volume 10. Springer Science & Business Media, 2013.

[10] Jean Lemaître and Jean-Louis Chaboche. Phenomenological aspect of damage failure. *J Méc Appl*, 2(3), 1978.

[11] Jacinto Ulloa, Jef Wambacq, Roberto Alessi, Esteban Samaniego, Geert Degrande, and Stijn Francois. A micromechanics-based variational phase-field model for fracture in geomaterials with brittle-tensile and compressive-ductile behavior. *Journal of the Mechanics and Physics of Solids*, 159:104684, 2022.

[12] Zdeněk P Bažant. Why continuum damage is nonlocal: Micromechanics arguments. *Journal of Engineering Mechanics*, 117(5):1070–1087, 1991.

[13] Christian Miehe, Martina Hofacker, and Fabian Welschinger. A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering*, 199(45-48):2765–2778, 2010.

[14] Parisa Khodabakhshi, JN Reddy, and Arun Srinivasa. Grafea: a graph-based finite element approach for the study of damage and fracture in brittle materials. *Meccanica*, 51:3129–3147, 2016.

[15] AR Srinivasa, HY Shin, P Thamburaja, and JN Reddy. Multiple cracking model in a 3d grafea framework. *Continuum Mechanics and Thermodynamics*, 33:1409–1428, 2021.

[16] Jean Lemaitre. A continuous damage mechanics model for ductile fracture. 1985.

[17] Christian Steinke, Johannes Storm, and Michael Kaliske. Energetically motivated crack orientation vector for phase-field fracture with a directional split. *International Journal of Fracture*, 237(1):15–46, 2022.

[18] Shuaifang Zhang, Wen Jiang, and Michael R Tonks. Assessment of four strain energy decomposition methods for phase field fracture models using quasi-static and dynamic benchmark cases. *Materials Theory*, 6(1):6, 2022.

[19] Christian Miehe, Fabian Welschinger, and Martina Hofacker. Thermodynamically consistent phase-field models of fracture: Variational principles and multi-field fe implementations. *International journal for numerical methods in engineering*, 83(10):1273–1311, 2010.

[20] Castrenze Polizzotto. Nonlocal elasticity and related variational principles. *International Journal of Solids and Structures*, 38(42-43):7359–7380, 2001.

[21] Bernhard Günter Trunk. *Einfluss der Bauteilgrösse auf die Bruchenergie von Beton*. PhD thesis, ETH Zurich, 1999.

[22] Ye Feng, Jiadi Fan, and Jie Li. Endowing explicit cohesive laws to the phase-field fracture theory. *Journal of the Mechanics and Physics of Solids*, 152:104464, 2021.