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TENSORIAL CONTINUUM DAMAGE MODEL FOR CONCRETE

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Abstract. A tensorial continuum damage model for concrete which can correctly predict the failure stress states and failure modes in general multiaxial stress states is presented. The model is thermodynamically consistent and is based on proper expressions for the specific Gibbs free energy and the complementary form of the dissipation potential. Damaging of the material is described by a symmetric positive definite second order damage tensor. Invariant theory is used in construction of the potential functions which guarantees that the proper symmetry behaviour is satisfied and no artificial symmetrization operations need not to be done. The failure surface is formulated in a way that it mimics the behaviour of the well-known Ottosen's four parameter failure surface. The predictions of the proposed model are compared to the Concrete Damaged Plasticity (CDP) model available in the commercial finite element software Abaqus in uniaxial and equibiaxial cases. The CDP model is calibrated against the uniaxial test results. However, for the CDP model the strain in the loading direction in the biaxial case starts to deviate from the experimental results already before the peak stress, while the present model yields accurate prediction. The constitutive model is implemented as an UMAT subroutine to be used in Abaqus.

1 INTRODUCTION

Concrete is one of the most commonly used construction materials, and has been subjected to a significant amount of research including failure behaviour in consideration of different loading conditions and micro-structural defects such as voids, inhomogeneities and micro-cracks. Typical characteristics of the failure behaviour of concrete include gradual loss of the elasticity, volume dilatancy and strain softening, which are consequences of the propagation and coalescence of micro-cracks leading ultimately to the material failure [10, 41]. The mechanical failure of concrete is induced mainly by cracking in tension due to propagation of the most critical micro-crack, and by crushing in compression due to the interaction of distributed micro-cracks [2]. This fact shows up in uniaxial compression as axial splitting along the direction of compression. Schreyer [38] observed that the classical stress criteria do not have the flexibility to reflect the failure modes for various stress states and thus cannot predict axial-splitting. Only few continuum models can predict the compressive axial-splitting, e.g. [2, 17, 24, 33, 38, 42, 46], although the physics of axial-splitting have been studied quite extensively, see e.g. [18] and the references therein.

A huge number of different constitutive models based on plasticity theory have been presented for concrete, see e.g. [11, 15, 44]. Damage descriptions by means of scalar, vectorial or tensorial damage variables have been utilized. Scalar damage variables are easy to implement, and hence widely applied, e.g. [14, 20, 27, 34]. Mazars and Pijaudier-Cabot [31] presented relationships between isotropic damage and fracture mechanics theories. The damage of rock-like materials, however, is definitely anisotropic due to the orientation of micro-cracks depending on the stress state. This feature can be described only by vectorial or tensorial damage variables. Vectorial damage variables have been used in [4, 5, 9, 24, 32] and second- or higher-order damage tensors in [1, 10, 12, 23, 28, 33, 36, 39, 45, 47, 49].

Several authors have combined plasticity and damage to model the failure of concrete [6, 14, 19, 21, 22, 43, 47]. On the other hand, since the inelastic behaviour of concrete is rather due to damage and micro-cracking than plastic deformations, as presented by [30, 31], models based merely on damage and micro-cracking have been formulated regularly.

The well-known Barcelona model, using a scalar damage variable for the degradation of both the volumetric and distortional elasticity, separately, was presented in [29]. Lee and Fenves [26, 27] revised the model by using two independent scalar damage variables to represent properly the cyclic behaviour of concrete. Their model has been implemented in the commercial finite element software Abaqus by [8]. A similar approach is carried out by [16], and a selection of 3D concrete models is reviewed in [35, 40]. As an alternative approach the microplane approach has got much attention, see e.g. [3, 40].

Ottosen proposed in 1977 [37] a model which captures the relevant features of concrete failure under various multiaxial stress states. In a recent study [48], the Ottosen model is formulated as an elasto-plastic model with a non-associated flow rule. Combination of the Ottosen model with damage mechanics with two damage parameters is given by Contrafatto and Cuomo [7].

In the present paper, a model for concrete, which was proposed in [42], is briefly summarized. In developing this model the following items were emphasized. The model should be able to capture the basic brittle failure modes: a) axial splitting along the direction of unconfined uniaxial compression, b) damaging perpendicular to the direction of uniaxial unconfined tension. It should also predict similar failure stresses to the Ottosen model and be thermodynamically consistent.

2 Constitutive model

This conference paper gives a brief overview of the recent continuum damage based constitutive model proposed in [42]. The model can capture the basic brittle failure modes and predict multiaxial failure stresses identical to those predicted by the Ottosen failure surface. Formulation of the model is based on continuum thermodynamics and pertinent choices for the two potential functions: the specific Gibbs free energy (Gibbs function) and the dual form of the dissipation potential, which are developed by using the invariant theory of scalar tensor-valued functions.

The reversible behaviour is modelled by the free energy function which depends on two symmetric second order tensors, the stress $\boldsymbol{\sigma}$ and the damage tensor $\boldsymbol{D}$. In this particular case the integrity basis contains ten invariants Adopting only the terms which are linear in $\boldsymbol{D}$ and quadratic in $\boldsymbol{\sigma}$, the mechanical part of the free energy function is given as

$$\rho_0\psi^{\mathrm{c}}(\boldsymbol{\sigma},\boldsymbol{D},\kappa)=\frac{1}{4G}\left[\operatorname{tr}(\boldsymbol{s}^2)+\operatorname{tr}(\boldsymbol{s}^2\boldsymbol{D})\right]+\frac{1}{18K_{\mathrm{b}}}(1+\chi\operatorname{tr}\boldsymbol{D})(\operatorname{tr}\boldsymbol{\sigma})^2+\int_0^{\kappa}\int_0^{\kappa'}g(\kappa'')\mathrm{d}\kappa''\mathrm{d}\kappa', \quad (1)$$

where G and K_{b} stand for the shear and bulk modulus, χ is a dimensionless positive material parameter, κ is a scalar internal variable describing the hardening/softening behaviour and ρ_0 is the density. The deviatoric stress tensor is denoted as $\boldsymbol{s}=\boldsymbol{\sigma}-\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I}$. The integrand $g(\kappa)$ is a four-parameter rational function [42]

$$g(\kappa)=\frac{H_0}{\kappa_0}\frac{h_2\left(\kappa/\kappa_0\right)^2-2h_1\left(\kappa/\kappa_0\right)-1}{\left[h_2\left(\kappa/\kappa_0\right)^2+1\right]^2}. \quad (2)$$

For description of the irreversible damaging behaviour, the dissipation potential is defined by the thermodynamic forces $\boldsymbol{Y}$ and K describing the damage and hardening, respectively. The dissipation potential is now defined as

$$\varphi(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon})=\mathrm{I}(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon}), \quad (3)$$

where I denotes the indicator function that restricts the stresses inside the elastic domain [13], and it is defined as

$$\mathrm{I}(\boldsymbol{Y},\boldsymbol{K};\boldsymbol{\sigma},\boldsymbol{\varepsilon})=\begin{cases}0, & \text{if } (\boldsymbol{Y},K)\in\Sigma\\ +\infty, & \text{if } (\boldsymbol{Y},K)\notin\Sigma,\end{cases} \quad (4)$$

where Σ is a convex set defining the admissible elastic domain for $(\boldsymbol{Y},K)$ and bounded by the damage criterion $f(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon})\le 0$.

The failure surface is in line with the Ottosen model [37] and since the failure modes of deformation are essential, the damage surface is now formulated as

$$f(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon})=\frac{AJ_2}{\sigma_{\mathrm{c}}}+\Lambda(\theta)\sqrt{J_2}+BI_1-(\sigma_{\mathrm{c0}}+K)+\operatorname{tr}\left[\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{Y}-\boldsymbol{Y}(\boldsymbol{\sigma})\right)\right], \quad (5)$$

where $J_2=\frac{1}{2}\operatorname{tr}(\boldsymbol{s}^2), I_1=\operatorname{tr}\boldsymbol{\sigma}$ and σ_{c}, σ_{c0} are the uniaxial compressive strength and damage initiation stress. The shape factor $\Lambda(\theta)$ depends on the Lode angle $\theta=(1/3)\arccos(3\sqrt{3}\,J_3/(2J_2^{3/2}))$ and determines the size and shape of the failure surface on the deviatoric plane [37, 42] and $J_3=\det(\boldsymbol{s})$. Shape of the failure locus changes from almost triangular shape at low hydrostatic compressive stress states to more rounded shape at high hydrostatic compressive stress states, see Fig. 1. The symmetric positive definite second-order tensor $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ is defined as

$$\boldsymbol{A}=\frac{1}{1+\beta_1\langle\operatorname{tr}\,\boldsymbol{\sigma}\rangle/\sigma_{\mathrm{t}}}\left(\frac{\boldsymbol{\varepsilon}_+}{\|\boldsymbol{\varepsilon}_+\|}+\beta_2\boldsymbol{I}\right) \quad (6)$$

and $\boldsymbol{\varepsilon}_+$ is the positive part of the elastic strain tensor, i.e. $\boldsymbol{\varepsilon}_+=\sum_{i=1}^{3}\langle\varepsilon_i\rangle\boldsymbol{\phi}_i\otimes\boldsymbol{\phi}_i$, where ε_i, $i=1,\dots,3$ are the eigenvalues of the elastic strain tensor, $\boldsymbol{\phi}_i$ stands for the corresponding eigenvectors, β_1,β_2 are positive non-dimensional parameters and σ_{t} is the uniaxial tensile strength. The norm of a second order tensor tensor is defined in a standard way, i.e. $\|\boldsymbol{\varepsilon}_+\|=\sqrt{\operatorname{tr}(\boldsymbol{\varepsilon}_+^2)}$, and $\langle\bullet\rangle$ denotes the Macaulay brackets.

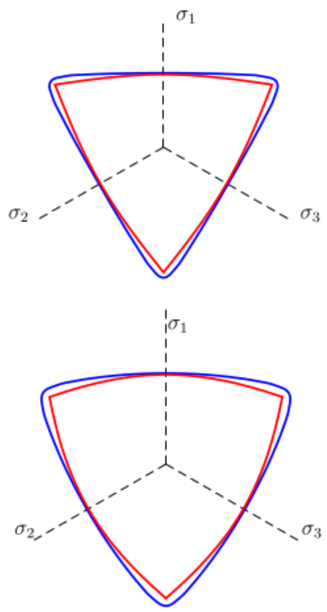

Figure 1: Shape of the failure locus on the deviatoric plane. Above on the π-plane ($\sigma_{\mathrm{m}}=0$) and at $\sigma_{\mathrm{m}}=-\sigma_{\mathrm{c}}$ below. Blue curve corresponds to the Ottosen model and red one to the Barcelona model.

The thermodynamic force $\boldsymbol{Y}$ dual to the damage rate is obtained from the definition

$$\boldsymbol{Y}(\boldsymbol{\sigma}) = \rho_0 \frac{\partial \psi^c}{\partial \boldsymbol{D}} = \frac{1}{4G}\boldsymbol{s}^2 + \frac{\chi}{18K_\mathrm{b}}(\mathrm{tr}\ \boldsymbol{\sigma})^2\boldsymbol{I}. \tag{7}$$

and the hardening variable K is

$$K = -\rho_0 \frac{\partial \psi^c}{\partial \kappa} = -\int_0^\kappa g(\kappa')\mathrm{d}\kappa' = H_0 \frac{h_1 (\kappa/\kappa_0)^2 + (\kappa/\kappa_0)}{h_2 (\kappa/\kappa_0)^2 + 1}, \tag{8}$$

Positive definiteness of tensor $\boldsymbol{A}$ in (5) together with the requirement that $\chi \geq 0$ in (7) guarantees that the dissipation power is non-negative.

Using the approach presented in [42] results in the constitutive equations

$$\boldsymbol{\varepsilon} = \rho_0 \frac{\partial \psi^c}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{D}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{Y}} = \dot{\lambda}\boldsymbol{A}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \qquad \dot{\kappa} = -\dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda}. \tag{9}$$

The multiplier $\dot{\lambda}$ is determined from the consistency condition $\dot{f}(\boldsymbol{Y}, K; \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = 0$. It can be easily seen that the failure mode is correct in unconfined uniaxial tension and compression, see [42, Sec. 3.2].

3 IDENTIFICATION OF MATERIAL PARAMETERS

The Ottosen parameter set A, B, k_1 and k_2 in the failure surface are adjusted to represent the damage surface corresponding with the four failure states [37, 42]. Since damage starts to develop at fairly low stress levels, the damage initiation stress in compression σ_{c0} needs to be selected. To obtain the parameters χ, κ_0, β_1 and β_2, the uniaxial compression and tension as well as the biaxial test results are needed. It should be emphasized that the twelve model parameters $(A, B, k_1, k_2, K_\infty, H_0, h_1, h_2, \kappa_0, \chi, \beta_1, \beta_2)$ can be obtained by using closed form expressions from the data shown in [42, Table 1] (strength values in uniaxial compression and tension, biaxial compression, one extra point from compressive meridian (I_1, J_2)-values, damage initiation stress in uniaxial compression, strain values at the peak stress in uniaxial compression, strain at uniaxial tension failure and one stress and corresponding strain value on the post-peak range. Complete procedure to obtain the parameter values is described in [42, Sect. 3.3].

4 RESULTS

Predictions of the model are compared to the well documented experimental data for concrete by Kupfer et al. [25] and the values used in calibration of the model are shown in [42, Tables 1 and 2] as well as the current model parameters. The biaxial tests in [25] were performed with specimens having dimensions 20 cm $\times$ 20 cm $\times$ 5 cm and the model results are compared to the tests having unconfined uniaxial compressive strength of 30.9 MPa, maximum aggregate size 15 mm and the water-cement ratio 0.9.

Results for the equibiaxial compression test $(\sigma_{11} = \sigma_{22})$ are shown in Fig. 2.

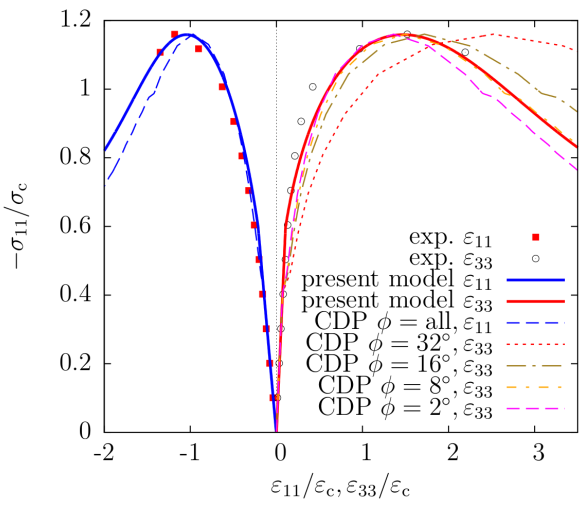

Figure 2: Stress-strain behaviour in equibiaxial compression $(\sigma_{11} = \sigma_{22})$ [42, Fig. 8] with experimental results from [25]. The Abaqus CDP model responses are shown for four values of the dilatation angle. It should be noticed that different dilatation angle gives the best fit to experimental data in comparison to unconfined uniaxial compression (see [42, Fig. 4]) for the CDP model.

The best similarity to the experimental results for the Abaqus CDP model is obtained when the dilatation angle is $8°$, in which case the $\sigma_{11}, \epsilon_{33}$-curve is almost identical to the response of the present model. However, for the CPD model the strain in the loading direction starts to deviate from the experimental results already before the peak stress.

As a result of the present model, the dominant damage develops perpendicularly to the plane of loading as it can be seen from Fig. 3.

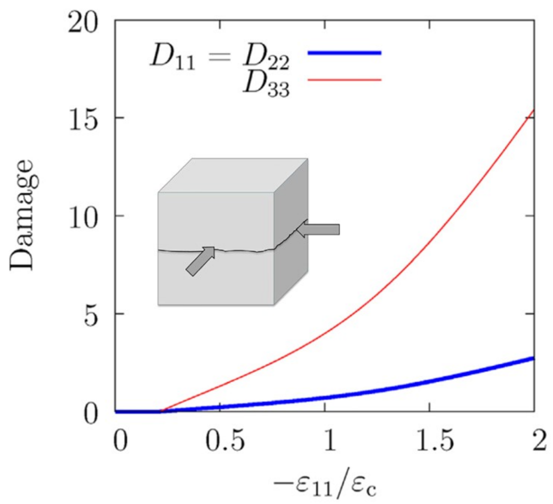

Figure 3: Damage-strain behaviour, damage is the largest in the 33-direction, i.e. the fracture mode corresponds splitting along the compressive plane illustrated [42, Fig.9].

5 CONCLUSIONS

A thermodynamically consistent formulation to model anisotropic damage for concrete and other quasi-brittle materials is presented. The model is based on proper expressions for the specific Gibbs free energy and the complementary form of the dissipation potential. Damaging of the material is described by a symmetric positive definite second order damage tensor. In this approach the values of the components in the damage tensor do not have an upper bound, thus facilitating a continuous numerical implementation. Especially, the failure surface is formulated in such a way that it will mimic the behaviour of the well-known Ottosen's four parameter failure surface. Although the formulation is basically non-associated, it follows closely the one for the standard dissipative solid.

One significant feature of the proposed model is its ability to model the failure modes correctly. In uniaxial compression, the mode of failure is axial splitting, i.e. the damaged zones are aligned parallel to the direction of the compressive stress. In tension, the failure takes place in a plane perpendicular to the applied stress. The obtained results are in accordance with the well-known experimental results found in literature.

To obtain a well behaving numerical model avoiding pathological mesh-dependency, some regularization strategy should be included in the model. Fassin et al. [12] regularized a CDM-model where damage is described by a second order tensor by using only a scalar variable.

Future research will also be focused on describing the plastic and cyclic behaviour of the material.

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