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INTRODUCTION OF DUAL-HORIZON PERIDYNAMICS AND NONLOCAL OPERATOR METHOD

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Abstract. This paper provides a concise introduction to the nonlocal theories of Dual-Horizon Peridynamics (DH-PD) and the Nonlocal Operator Method (NOM), including their primary concepts and fundamental governing equations. In addition, a dual-horizon bond-based peridynamics equipped with variational damage is derived. A numerical example is presented to explore their advantages in crack simulation.

1 INTRODUCTION

Peridynamics (PD) is a nonlocal solid mechanics theory introduced by Silling in year 2000, focusing on solving fracture problems [1, 2]. Its core feature is nonlocality, where the state of a material point depends on the information of other material points within a certain neighborhood. This nonlocal nature allows explicit modeling of direct interactions between material points, enabling intuitive simulation of cracks by cutting these interactions.

Since its inception, PD has assumed constant nonlocality for material points to ensure symmetric interactions. However, PD can also be regarded as a meshless solid mechanics computational method. It is well known that to improve simulation accuracy while reducing computational cost, meshless methods and finite element methods (FEM) can use non-uniform discretization, with dense meshes in critical regions and sparse meshes elsewhere. Similarly, we hope PD can adopt non-uniform discretization for material points. However, traditional PD formulations cannot achieve this efficiently.

To overcome this limitation, the Dual-Horizon Peridynamics (DH-PD) formulation emerged. A key feature of DH-PD is the non-uniform discretization of material points in the solid domain, where the size of the nonlocal neighborhood of a material point is defined individually [3, 4]. This improves computational efficiency and provides a theoretical basis for multiscale analysis in PD. Despite being initially derived from Newton's Second Law as a nonlocal analog, DH-PD can also be derived variationally, where the dual-horizon concept naturally arises [5]. DH-PD has been applied widely and verified by numerous scholars from various perspectives.

As a generalization of PD, DH-PD is com-

patible with all PD formulations (bond-based PD, state-based PD, and non-ordinary state-based PD). However, DH-PD is limited to solid domains, and the concept of nonlocality has not been fully explored. A core concept in DH-PD is the nonlocal gradient. From the perspective of partial differential operators, PD mainly focuses on nonlocal gradients. Many other partial differential operators, such as curl, divergence, and higher-order gradients, exist mathematically. The Nonlocal Operator Method (NOM) has been proposed as an extension of DH-PD, systematically defining various orders of nonlocal derivatives and deriving nonlocal integral equations using classical variational principles [6–8].

The remainder of the paper is organized as follows. Section 2 briefly reviews the theories and concepts of DH-PD. In Section 3, an outline of nonlocal operator method is described. In Section 4, a numerical example for the simulation of dynamic fractures is provided. Some outlooks are draw in Section 5.

2 DUAL-HORIZON PERIDYNAMICS

The idea of DH-PD is the non-uniform nonlocality, where the horizon domain of each material point depends on itself and need not match the sizes of other points in domain [3, 4]. This is akin to the non-uniform discretization in numerical methods like FEM and meshless methods. The fundamental principles underlying dual-horizon peridynamics pertain to the notions of horizon domain and dual-horizon domain.

Horizon domain of $\boldsymbol{x}$ is a finite-size neighborhood of material point $\boldsymbol{x}$.

$$H_{\boldsymbol{x}} = \{\boldsymbol{x}' \big| ||\boldsymbol{x}' - \boldsymbol{x}|| \leq \delta\} \tag{1}$$

where δ is the radius of the neighborhood.

Dual-horizon domain is defined as a union of points whose horizons include $\boldsymbol{x}$, denoted by

$$H'_{\boldsymbol{x}} = \{\boldsymbol{x}' \big| \boldsymbol{x} \in H_{\boldsymbol{x}'}\} \tag{2}$$

The definition of dual-horizon implies $\boldsymbol{x}' \in H'_{\boldsymbol{x}} \leftrightarrow \boldsymbol{x} \in H_{\boldsymbol{x}'}$.

According to reference [2], the governing equations of conventional peridynamics are

$$\int_{H_{\boldsymbol{x}}} (\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'} - \boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}})\, dV_{\boldsymbol{x}'} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}_{\boldsymbol{x}} \tag{3}$$

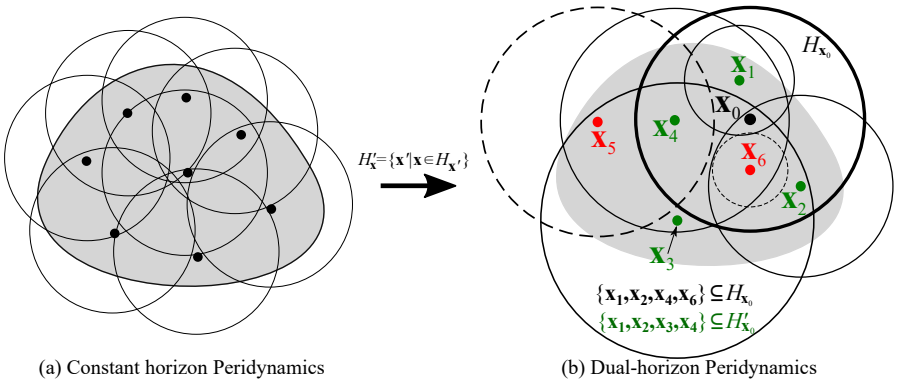
where $\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}$ is the bond-force density between material point $\boldsymbol{x}$ and $\boldsymbol{x}'$. The body force density is represented by the symbol $\boldsymbol{b}$, while the density is denoted by ρ. The displacement field is denoted by the symbol $\boldsymbol{u}$.

With the notation of dual-horizon domain [4], the governing equations of dual-horizon peridynamics is

$$\int_{H_{\boldsymbol{x}}} \boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}\, dV_{\boldsymbol{x}'} - \int_{H'_{\boldsymbol{x}}} \boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}\, dV_{\boldsymbol{x}'} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}_{\boldsymbol{x}} \tag{4}$$

The dual-horizon peridynamics is a straightforward adaptation of the traditional peridynamics. It facilitates the use of peridynamics for non-uniform horizons in all material points. The present configuration bears resemblance to the non-uniform discretization technique employed in finite element methods, thereby leading to a substantial enhancement in computational efficacy.

The dual-horizon peridynamics encompasses the conventional constant-horizon peridynamics as a specific instance. The table below outlines a comparison between peridynamics and dual-horizon peridynamics. The three peridynamics models under consideration are bond-based peridynamics (BB-PD), state-based peridynamics (OSB-PD), and nonordinary state-based peridynamics (NOSB-PD), each with their respective abbreviations.



(a) Constant horizon Peridynamics (b) Dual-horizon Peridynamics

Figure 1: From constant horizon peridynamics to dual-horizon peridynamics.

Table 1: Comparison of constant horizon peridynamics and dual-horizon peridynamics.

Model	Peridynamics	Dual-horizon peridynamics
Equation	$\int_{H_{\boldsymbol{x}}}(\boldsymbol{f}_{\boldsymbol{x}'}-\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}})\,dV_{\boldsymbol{x}'}+\boldsymbol{b}=\rho\ddot{\boldsymbol{u}}_{\boldsymbol{x}}$	$\int_{H_{\boldsymbol{x}}}\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}\,dV_{\boldsymbol{x}'}-\int_{H'_{\boldsymbol{x}}}\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}\,dV_{\boldsymbol{x}'}+\boldsymbol{b}=\rho\ddot{\boldsymbol{u}}_{\boldsymbol{x}}$
BB-PD	$\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}-\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}=cs_{\boldsymbol{x}\boldsymbol{x}'}\boldsymbol{n},\forall\boldsymbol{x}'\in H_{\boldsymbol{x}}$	$\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}=C(\delta_{\boldsymbol{x}})\,s_{\boldsymbol{x}\boldsymbol{x}'}\boldsymbol{n},\forall\boldsymbol{x}'\in H_{\boldsymbol{x}}$ $\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}=C(\delta_{\boldsymbol{x}'})\,s_{\boldsymbol{x}\boldsymbol{x}'}(-\boldsymbol{n}),\forall\boldsymbol{x}'\in H'_{\boldsymbol{x}}$
OSB-PD	$\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}=\underline{t}\langle\boldsymbol{\xi}\rangle\boldsymbol{n}$ $\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}=\underline{t}'\langle-\boldsymbol{\xi}\rangle(-\boldsymbol{n}),\forall\boldsymbol{x}'\in H_{\boldsymbol{x}}$	$\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}=\underline{t}\langle\boldsymbol{\xi}\rangle\boldsymbol{n},\forall\boldsymbol{x}'\in H_{\boldsymbol{x}}$ $\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}=\underline{t}'\langle-\boldsymbol{\xi}\rangle(-\boldsymbol{n}),\forall\boldsymbol{x}'\in H'_{\boldsymbol{x}}$
NOSB-PD	$\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}=\underline{\boldsymbol{T}}[\boldsymbol{x}'-\boldsymbol{x}]$, $\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}=\underline{\boldsymbol{T}}'[\boldsymbol{x}-\boldsymbol{x}'],\forall\boldsymbol{x}'\in H_{\boldsymbol{x}}$	$\boldsymbol{f}_{\boldsymbol{x}\boldsymbol{x}'}=\underline{\boldsymbol{T}}[\boldsymbol{x}'-\boldsymbol{x}],\forall\boldsymbol{x}'\in H_{\boldsymbol{x}}$ $\boldsymbol{f}_{\boldsymbol{x}'\boldsymbol{x}}=\underline{\boldsymbol{T}}'[\boldsymbol{x}-\boldsymbol{x}'],\forall\boldsymbol{x}'\in H'_{\boldsymbol{x}}$

2.1 Bond-based peridynamics embedded with damage

A distinctive feature of nonlocal models is the flexibility in describing cracks. Based on certain fracture criteria, interactions within the nonlocal neighborhood can be easily severed. In traditional local models, equations are defined at a point, and since points lack physical dimensions, interactions with surrounding points cannot be quantitatively severed. Nonlocal models, however, introduce an integration domain that explicitly represents interactions between material points. Interactions with physical entities can be easily modified or severed. For example, in PD, the most common fracture criterion is the maximum tensile strain criterion. When the strain between material points exceeds a specified value, the bond is permanently cut. However, bond breaking has side effects, such as computational instability due to conditional judgments. These can be mitigated mathematically by calculating the damage for strain energy density of a bond.

The strain energy carried on a bond $\boldsymbol{x}_{ij}$ is

$$\phi_{ij}:=\phi(\boldsymbol{u}_{ij},\boldsymbol{x}_{ij})=\tfrac{1}{2}C_i s_{ij}^2,\ s_{ij}=\frac{\|\boldsymbol{x}_{ij}+\boldsymbol{u}_{ij}\|}{\|\boldsymbol{x}_{ij}\|}-1 \tag{5}$$

where C_i is the coefficient of bond stretch defined in horizon domain $H_{\boldsymbol{x}_i}$, $\boldsymbol{u}_{ij}:=\boldsymbol{u}_j-\boldsymbol{u}_i$ is the relative displacement. Bond damage occurs in tensile deformation and no damage happens for compressive deformation. Let $\langle s_{ij}\rangle_{\pm}$ be defined as

$$\langle s_{ij}\rangle_{\pm}=\frac{s_{ij}\pm|s_{ij}|}{2} \tag{6}$$

The "positive"/"negative" decomposition of bond strain energy is defined as

$$\phi_{ij}^{\pm}=\frac{1}{2}C_i\langle s_{ij}\rangle_{\pm}^2 \tag{7}$$

Based on the variational damage model [9], the damage of bond $\boldsymbol{x}_{ij}$ can be defined as

$$d_{ij}:=d(\boldsymbol{x}_{ij})=\frac{\phi_{ij}^{+}}{\phi_{ij}^{+}+G_c/\ell}. \tag{8}$$

where G_c/ℓ is the critical energy density carried by a bond.

In order to satisfy the non-reversibility of damage evolution, we introduce the historical maximal "positive" strain energy density at point $\boldsymbol{x}$ in the time interval $[0,T]$, i.e.

$$H_{ij} := \max_{t\in[0,T]} \phi_{ij}^{+}(s_{ij}(\boldsymbol{x},t)). \tag{9}$$

Then the damage state for bond $\boldsymbol{x}_{ij}$ can be rewritten as

$$d_{ij} := d(\boldsymbol{x}_{ij}) = \frac{H_{ij}}{H_{ij}+G_c/\ell}. \tag{10}$$

Therefore, the bond force embedded with bond damage in the governing equation can be expressed as

$$\boldsymbol{f}_{ij}^{d} = C_i \left(\frac{\langle s_{ij}\rangle_{+}}{(1+H_{ij}\ell/G_c)^2} + \langle s_{ij}\rangle_{-} \right) \frac{\boldsymbol{x}_{ij}+\boldsymbol{u}_{ij}}{\|\boldsymbol{x}_{ij}+\boldsymbol{u}_{ij}\|} \tag{11}$$

The governing equation of bond-based peridynamics embedded with bond damage becomes

$$\rho\ddot{\boldsymbol{u}}_i = \int_{H_{\boldsymbol{x}_i}} \boldsymbol{f}_{ij}^{d} dV_j - \int_{H'_{\boldsymbol{x}_i}} \boldsymbol{f}_{ji}^{d} dV_j + \boldsymbol{b} \tag{12}$$

$$d_{ij} = \frac{H_{ij}}{H_{ij}+G_c/\ell}, d_i = \frac{\int_{H_{\boldsymbol{x}_i}} d_{ij}\, dV_j}{\int_{H_{\boldsymbol{x}_i}} 1\ dV_j} \tag{13}$$

In above equations, the damage depends solely on the historical energy density of bonds and the bond-cutting process is avoided.

2.2 Force accumulation in dual-horizon PD

Despite the introduction of the dual-horizon, the numerical implementation did not incur any additional expenses. The dual-horizon bond forces can be automatically accumulated during the computation of forces from the horizons of other material points. This concept is demonstrated through a straightforward illustration.

Take the particles in Fig.1(b) for an example, $H_{\boldsymbol{x}_0} = \{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_4,\boldsymbol{x}_6\}, H'_{\boldsymbol{x}_0} = \{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4\}$. The bond force density on a bond between two material points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is denoted by $\boldsymbol{f}_{ij}$. The symbol ΔV_i represents the volume of a specific material point denoted by $\boldsymbol{x}_i$. According to the idea of dual-horizon, the process of forces summation is listed as follows:

Internal forces from $H_{\boldsymbol{x}_0}$:

- Add $\boldsymbol{f}_{01}\Delta V_0\Delta V_1$ to $\boldsymbol{x}_0$, add reaction force $-\boldsymbol{f}_{01}\Delta V_0\Delta V_1$ to $\boldsymbol{x}_1$
- Add $\boldsymbol{f}_{02}\Delta V_0\Delta V_2$ to $\boldsymbol{x}_0$, add reaction force $-\boldsymbol{f}_{02}\Delta V_0\Delta V_2$ to $\boldsymbol{x}_2$
- Add $\boldsymbol{f}_{04}\Delta V_0\Delta V_4$ to $\boldsymbol{x}_0$, add reaction force $-\boldsymbol{f}_{04}\Delta V_0\Delta V_4$ to $\boldsymbol{x}_4$
- Add $\boldsymbol{f}_{06}\Delta V_0\Delta V_6$ to $\boldsymbol{x}_0$, add reaction force $-\boldsymbol{f}_{06}\Delta V_0\Delta V_6$ to $\boldsymbol{x}_6$

Internal forces from $H'_{\boldsymbol{x}_0}$ are calculated when calculating bond forces in $H_{\boldsymbol{x}_1}, H_{\boldsymbol{x}_2}, H_{\boldsymbol{x}_3}, H_{\boldsymbol{x}_4}$:

- In $H_{\boldsymbol{x}_1}$, add $\boldsymbol{f}_{10}\Delta V_1\Delta V_0$ to $\boldsymbol{x}_1$, add reaction force $-\boldsymbol{f}_{10}\Delta V_1\Delta V_0$ to $\boldsymbol{x}_0$
- In $H_{\boldsymbol{x}_2}$, add $\boldsymbol{f}_{20}\Delta V_2\Delta V_0$ to $\boldsymbol{x}_2$, add reaction force $-\boldsymbol{f}_{20}\Delta V_2\Delta V_0$ to $\boldsymbol{x}_0$
- In $H_{\boldsymbol{x}_3}$, add $\boldsymbol{f}_{30}\Delta V_3\Delta V_0$ to $\boldsymbol{x}_3$, add reaction force $-\boldsymbol{f}_{30}\Delta V_3\Delta V_0$ to $\boldsymbol{x}_0$
- In $H_{\boldsymbol{x}_4}$, add $\boldsymbol{f}_{40}\Delta V_4\Delta V_0$ to $\boldsymbol{x}_4$, add reaction force $-\boldsymbol{f}_{40}\Delta V_4\Delta V_0$ to $\boldsymbol{x}_0$

3 OUTLINE OF NONLOCAL OPERATOR METHOD

NOM generalizes DH-PD, allowing nonlocal models to incorporate arbitrary nonlocal derivatives [8]. In NOM, the support domain is used, which has the same meaning of horizon domain or the nonlocal neighborhood of a material point. The nonlocal derivatives of a field u at point $\boldsymbol{x}_i$ based on support domain $\mathcal{S}_i$ in d-dimensions with highest derivative order up to n can be written as

$$\tilde{\partial}_\alpha u_i := \mathbf{K}_i \cdot \int_{\mathcal{S}_i} w(\mathbf{r})\mathbf{p}_j^h u_{ij}\, dV_j \tag{14}$$

where $\tilde{\partial}_\alpha u_i$ is a list of nonlocal derivatives of different orders, $\mathbf{p}_j^h$ is a list of polynomials de-

fined as

$$\tilde{\partial}_\alpha u_i = (u_{i,0\ldots1}, \ldots, u_{i,n_1\ldots n_d}, \ldots, u_{i,n\ldots0})^T \tag{15}$$

$$\mathbf{p}_j^h = \left(\frac{r_d}{h}, \ldots, \frac{r_1^{n_1}\ldots r_d^{n_d}}{h^{n_1+\ldots+n_d}}, \ldots, \frac{r_1^n}{h^n}\right)^T \tag{16}$$

and $\mathbf{K}_i$ is the generalized shape tensor defined by

$$\mathbf{K}_i := \mathbf{H}_i^{-1}\left(\int_{\mathcal{S}_i} w(\mathbf{r})\mathbf{p}_j^h \otimes \left(\mathbf{p}_j^h\right)^T \mathrm{d}V_j\right)^{-1} \tag{17}$$

The additional notations are defined as

$$\mathbf{r} = (r_1, \ldots, r_d) = (x_{j1} - x_{i1}, \ldots, x_{jd} - x_{id})$$
$$u_{i,n_1\ldots n_d} = \frac{\partial^{n_1+\ldots+n_d} u_i}{\partial x_{i1}^{n_1}\ldots\partial x_{id}^{n_d}}, u_{ij} = u_j - u_i,$$
$$\mathbf{H}_i = \mathrm{diag}\left[h_i, \ldots, \frac{h_i^{n_1+\ldots+n_d}}{n_1!\ldots n_d!}, \ldots, \frac{h_i^n}{n!}\right]$$
$$\alpha_d^n = \{(n_1, \ldots, n_d)\,|1 \le \sum_{i=1}^{d} n_i \le n,$$
$$n_i \in \{0, 1, 2, ...\}, 1 \le i \le d\}$$

These high-order nonlocal derivatives are obtained from the Taylor expansion of a function, more details can be found in Ref [7, 8]. Substituting high-order nonlocal derivatives for traditional derivatives in energy functionals and applying variational principles yields various nonlocal models, such as gradient elasticity, nonlocal plate theories, and nonlocal diffusion equations. For more details, refer to Reference [5].

4 NUMERICAL EXAMPLE: FRAGMENTATION SIMULTATION

DH-PD and NOM are compatible with variational principles, enabling the use of explicit or implicit algorithms from traditional numerical methods. Here, we present an example solved by explicit time algorithms.

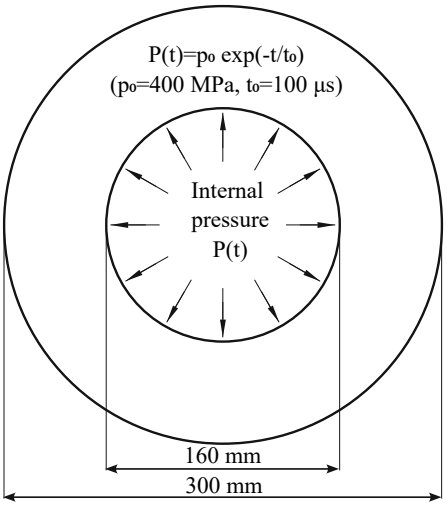


Figure 2: Geometry and boundary condition of the thick cylinder subjected to internal pressure $P(t)$.

The fragmentation simulation examines a complex numerical model of a thick cylinder subjected to internal pressure, as illustrated in Figure 2. The internal surface pressure is applied as $P(t) = p_0 \exp(-\frac{t}{t_0})$, where $t_0 = 100\,\mu s$ and $p_0 = 400\,\mathrm{MPa}$. The material properties used in the simulation are: $E = 210\,\mathrm{GPa}$, $\mu = 0.3$, $G_c = 2 \times 10^4\,\mathrm{J/m^2}$, $\rho = 7850\,\mathrm{kg/m^3}$. To introduce asymmetry in the specimen, the Young's modulus of each material point, E, is perturbed by $\pm 5\%$ [10].

The model consists of 196,556 material points, with a horizon selected as $\ell = 3.01\Delta x$, where $\Delta x = 0.5\,\mathrm{mm}$ represents the spacing between material points. The simulation spans a total duration of $100\,\mu s$ with a fixed time step of $t = 0.1\,\mu s$. The crack propagation at different times is depicted in Figure 3. During loading, at approximately $t \approx 43.6\,\mu s$, cracks begin to initiate from the inner surface of the cylinder. Some of these cracks are arrested, while others propagate and branch outward until they reach the external surface. Additional cracks develop due to the complex interactions of stress waves within the solid.

(a) $t = 43.6\mu s$ (b) $t = 50.7\mu s$

(c) $t = 59.3\mu s$ (d) $t = 87.5\mu s$

Figure 3: Crack propagation in thick cylinder subjected to internal pressure boundary at different steps.

5 CONCLUSION

This paper primarily reviews DH-PD and NOM, introducing their main concepts and features. Due to their compatibility with traditional variational principles, NOM can solve a wide range of non-conventional problems as well as conventional problems. Interested readers can refer to recent relevant literature.

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