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GRADIENT DAMAGE AND RELIABILITY INSTABILITY AS LIMIT STATE FUNCTION

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Abstract

Failure in quasi-brittle materials finally leads to the instability of the structure. A gradient damage model is used to properly regularise the damage localisation process. Unstable structural behaviour of the structure is defined as the loss of the positive definiteness of the system tangent stiffness matrix. Because the damage process can be highly dependent on the heterogeneity of the material, a first-order reliability method for predicting the probability of structural instability is introduced.

1 Introduction

Failure in quasi-brittle materials is characterised by a localisation of deformation. Several continuum models properly describing the damage localisation in strain softening continua, such as non-local and gradient models (Pijaudier-Cabot and Bazant 1987, de Borst et al. 1993) have been developed. These models introduce higher order deformation-gradients, which can be seen as the outcome from a homogenisation of micro-scale phenomena like microcrack initiation, growth and coalescence. At a final stage of the damage process, the structure will lose its load carrying capacity due to instability of the structure. Unstable structural behaviour can be expressed as the loss of the positive definiteness of the system tangent stiffness matrix. However, the construction of a consistent system tangent stiffness matrix for nonlocal damage models is quite complicated. A recent gradient formulation of nonlocal damage (Peerlings et al. 1995) turns out to be a considerable efficient method to calculate the consistent system tangent matrix.

In recent years, the treatment of damage evolution and failure of a structure in a stochastic framework stands as a most promising approach for studying the effects of the complex material heterogeneity on a macroscopic scale (see e.g. Zhang and Kiureghian 1994, Carmeliet and Hens 1994, Carmeliet and de Borst 1995). In this paper a first-order, finite-element based reliability method for estimating the probability of structural instability is presented. A main effort in the application of the above reliability method is the calculation of the gradient of the failure condition with respect to the random variables. An example application on a one-dimensional continuum with a random field elastic modulus demonstrates the methodology.

2 Gradient Damage model

In this section, we present the gradient formulation of the nonlocal damage model as developed by Peerlings et al. (1995). The nonlocal damage model as well as the gradient damage model start from the classical isotropic elasticity-based damage theory:

$$\boldsymbol{\sigma} = (1 - D)\, \mathbf{C}\, \boldsymbol{\epsilon} \tag{1}$$

where $\mathbf{C}$ is the initial elasticity tensor of the virgin material, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the stress and strain tensors and D the damage variable, which grows from zero to one (complete loss of integrity). Damage growth is determined by an evolution law $D = F(\epsilon^{eq})$, in which ϵ^{eq} is an equivalent strain measure defined by Mazars (1984). Damage growth is possible if the damage loading function $f = \epsilon^{eq} - K$ vanishes. The damage parameter K initially equals the damage threshold K_0 and during damage evolution equals the maximum value of ϵ^{eq} ever reached during the loading history. The damage loading function f and the rate of damage growth $\dot{D}$ have to satisfy the discrete Kuhn-Tucker conditions: $f \leq 0$, $\dot{D} \geq 0$, $f\dot{D} = 0$.

In the original nonlocal model (Pijaudier-Cabot and Bazant 1987) the equivalent strain ϵ^{eq} is replaced by a spatially averaged or non-local equivalent strain value $\bar{\epsilon}$, such that:

$$\bar{\epsilon}(\mathbf{x}) = \frac{1}{V_r} \int_V \epsilon^{eq}(\mathbf{x}+\tau)\,\alpha(\tau)\,dV \qquad (2)$$

$$\alpha(\tau) = \exp\left(-|\tau|^2 / 2\,l^2\right)$$

with τ the separation vector, V_r a normalising factor, α a squared exponential weight function and l the so-called internal length scale.

In the gradient damage formulation of the nonlocal concept, the integral equation of eq. 2 is replaced by the partial differential equation:

$$\bar{\epsilon} - c\,\nabla^2 \bar{\epsilon} = \epsilon^{eq} \qquad (3)$$

with $c = l^2/4$. Introducing C^0-continuous quadratic interpolation functions N for the displacement field and the C^0-continuous linear interpolation functions H for the nonlocal equivalent strain field, the coupled incremental finite element formulation of the equilibrium equation and differential equation (3) reads (Peerlings et al. 1995):

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{ue} \\ \mathbf{K}_{eu} & \mathbf{K}_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{du} \\ \mathbf{d\bar{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{ext} + \mathbf{F}_u \\ \mathbf{F}_{\bar{e}} \end{bmatrix} \qquad (4)$$

with $\mathbf{du}$ and $\mathbf{d\bar{e}}$ the incremental nodal displacements and nodal nonlocal equivalent strains between iteration steps $k+1$ and k respectively. The partial matrices $\mathbf{K}_{uu}$, $\mathbf{K}_{ue}$, $\mathbf{K}_{eu}$ and $\mathbf{K}_{ee}$ are given by (Peerlings et al. 1995):

$$\mathbf{K}_{uu} = \int_V \mathbf{B}^T (1 - D_k) \mathbf{C} \mathbf{B} \, dV$$

$$\mathbf{K}_{ue} = - \int_V \mathbf{B}^T \mathbf{C} \, \epsilon_k \left[\frac{\partial D}{\partial \bar{\epsilon}} \right]_k \mathbf{H} \, dV \tag{5}$$

$$\mathbf{K}_{ee} = \int_V \left(\mathbf{H}^T \mathbf{H} + \mathbf{P}^T c \, \mathbf{P} \right) dV$$

$$\mathbf{K}_{eu} = - \int_V \mathbf{H}^T \left[\frac{\partial \epsilon^{eq}}{\partial \epsilon} \right]_k \mathbf{B} \, dV$$

The matrices B and P contain the derivatives of the shape functions N and H respectively. The internal forces $\mathbf{F}_u$, $\mathbf{F}_{\bar{e}}$ and the external force $\mathbf{F}_{ext}$ are defined by:

$$\mathbf{F}_u = - \int_V \mathbf{B}^T \sigma_k \, dV \quad , \quad \mathbf{F}_{\bar{e}} = \int_V \mathbf{H}^T \epsilon_k^{eq} \, dV - \mathbf{K}_{ee} \, \bar{\mathbf{e}}_k \tag{6}$$

$$\mathbf{F}_{ext} = \int_S \mathbf{B}^T t_{k+1} \, dS \tag{7}$$

with $\mathbf{t}_{k+1}$ the boundary traction vector.

To demonstrate the essential features of the gradient damage model, we solve the two-dimensional plane stress problem of a double edge notched direct tensile test (Hordijk 1991). The material data are: Young's modulus E = 40000 MPa, tensile strength f_t = 3.7 MPa, l=10mm, the initial damage threshold $K_0 = f_t/E$ and an exponential damage function $D = 1 - (K_0/\bar{\epsilon})[(1 - A) + A \exp(B(\bar{\epsilon} - K_0))]$, with A=0.95 and B=800 has been used. The specimen has a length of 125 mm, a width of 60 mm and notch depths of 5 mm. The specimen is fixed at the bottom and prevented from rotating at the top. Fig. 1a shows the load-displacement diagram, with the displacement measured over a base of 35 mm. Fig. 1b shows the damage distribution at peak loading and at final loading. The resemblence of the strain field with that obtained by Sluys (1991) on a similar specimen, but subjected to an impact load, is striking.

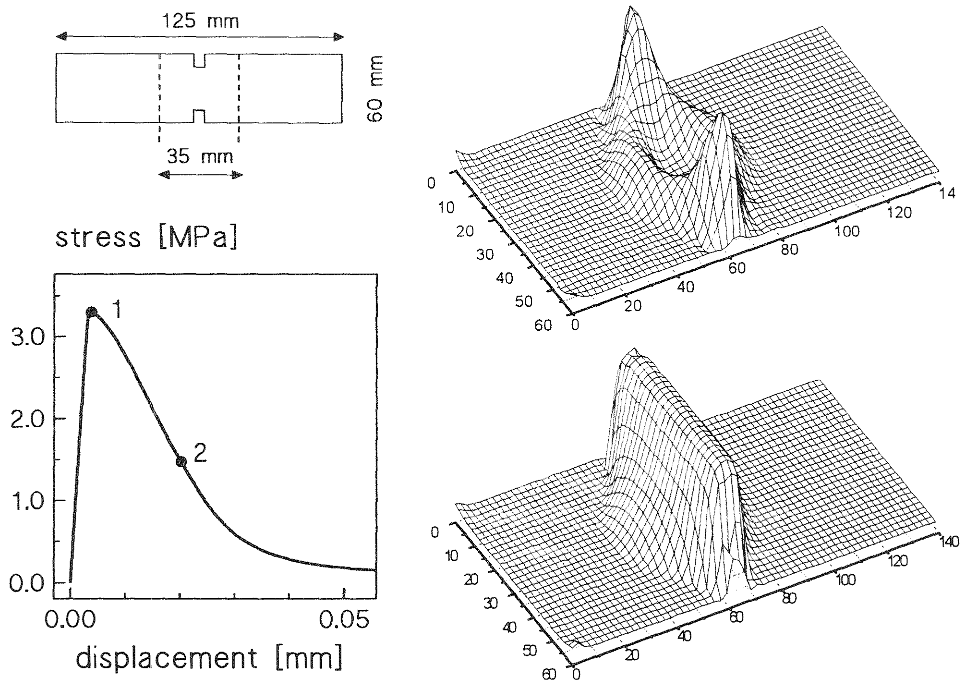

Fig. 1. Load-displacement curve and damage distribution for tensile test

3 Stability analysis

A system is said to be in a state of stable equilibrium if the response on a vanishingly small disturbance also remains vanishingly small. At a state of equilibrium under dead load, the stability condition becomes for a incrementally linear system (de Borst 1986):

$$\dot{\mathbf{u}}^T(\mathbf{K}_t + \mathbf{K}_t^T)\,\dot{\mathbf{u}} > 0 \tag{8}$$

for all kinematically admissible velocity vectors $\dot{\mathbf{u}}$. The matrix $\mathbf{K}_t$ is the system tangent stiffness matrix, which for a gradient damage model is given according to eq.4 by:

$$\mathbf{K}_t = \mathbf{K}_{uu} - \mathbf{K}_{ue}\,\mathbf{K}_{ee}^{-1}\,\mathbf{K}_{eu} \tag{9}$$

The structure is said to be in a critical state of neutral equilibrium if:

$$\det(\mathbf{K}_t + \mathbf{K}_t^T) = \det(\mathbf{K}_s) = 0 \tag{10}$$

which according to Vieta's rule,

$$\det(\mathbf{K}_s) = \prod_{i=1}^{n} \lambda_i$$

with λ_i the eigenvalues of $\mathbf{K}_s$, implies that at least one eigenvalue vanishes. The vanishing of the minimal eigenvalue of K_s, in which the rows and columns corresponding to fixed displacements have been removed, is therefore identical to eq. 10.

To illustrate the condition of loss of structural stability, we consider the example of an axially loaded tensile bar of linear softening material. The length of the bar is L=100 mm, Young's modulus E=20000 MPa, the tensile strength f_t=2 MPa, a softening modulus h=-0.01E and l=4mm. To initiate and promote localisation of damage in the middle of the specimen a 10% reduction of the cross section of the bar over a length of 10 mm is assumed. Figure 2a shows the load-elongation diagram for three meshes of 80, 160 and 320 elements. A snap-back response is observed, which is due progressive damage growth in a continuously decreasing damage zone. Figure 2b shows the evolution of the minimal eigenvalue as function of the elongation. We see that in this case the condition of loss of structural stability is met at the peak load.

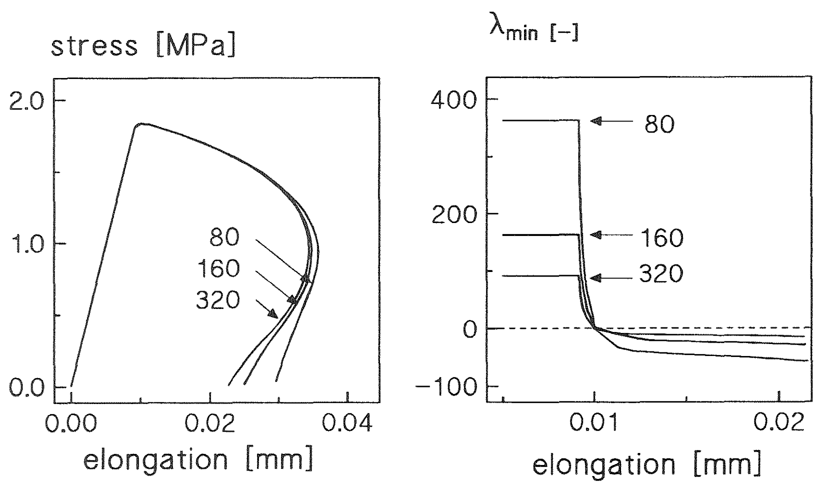

Fig. 2. Load-elongation and minimal eigenvalue-elongation curve.

4 Reliability analysis

Let **X** denote the basic random variables, representing the uncertainties in the structure and its environment, and g a function of **X** such that $g(\mathbf{x}) \leq 0$ denotes failure and $g(\mathbf{x}) > 0$ denotes the safe state of the structure. The limit state function describing the structural failure mode of instability is according to eq.10 :

$$g(\mathbf{x}) = \lambda_{min}(\mathbf{x}) \tag{11}$$

with λ_{min} the minimal eigenvalue of the tangent matrix $\mathbf{K}_s$.

It is convenient in reliability analysis to transform the variables **X** into the standard normal space $\mathbf{Y}=\mathbf{Y}(\mathbf{X})$, where elements of **Y** are statistically independent normal variables. The limit state surface $g(\mathbf{x})$ is then mapped onto the failure surface $G(\mathbf{y})$ in the standard normal space. The probability of instability P_f is defined as:

$$P_f = \int_{g(\mathbf{x}) \leq 0} f(\mathbf{x})\, dx = \int_{G(\mathbf{y}) \leq 0} \phi(\mathbf{y})\, dy \tag{12}$$

with $f(\mathbf{x})$ the probability density function of **X**, and $\phi(\mathbf{y})$ the standard normal density of **Y**.

To evaluate the probability integral, the first-order reliability method is used. In this method, an approximation to the integral is obtained by linearizing the failure surface $G(\mathbf{y})$ at one or more design points $\mathbf{y}_i^*$. If there is only one significant design point, the first order approximation of the probability is given by $P_f = \Phi(-\beta)$, where Φ is the standard normal cumulative distribution and β, commonly called the reliability index, is the euclidian norm of $\mathbf{y}^*$. In the standard normal space, $\mathbf{y}_i^*$ is the point on the limit state surface closest to the origin. This point is found by solving a constrained optimisation problem of minimising $F(\mathbf{y})=\mathbf{y}^T\mathbf{y}$ subject to $G(\mathbf{y})=0$. Many algorithms exist for this problem (see Liu and Der Kiureghian 1991). In this study, we will use the HL-RF method, which is based on the following recursive formula:

$$\mathbf{y}_{k+1} = \frac{1}{\| \nabla_y G(\mathbf{y}_k) \|^2} \left[\nabla_y G(\mathbf{y}_k)\, \mathbf{y}_k - G(\mathbf{y}_k) \right] \nabla_y G(\mathbf{y}_k)^T \tag{14}$$

with $\nabla_y G(\mathbf{y}_k)$ the gradient vector of the limit state function. This gradient vector can be efficiently calculated by (Haug et al. 1986):

$$\nabla_y G = \nabla_y \lambda_{min} = \mathbf{Z}^T (\nabla_y \mathbf{K}_s) \mathbf{Z} \tag{13}$$

with $\mathbf{Z}$ the normalized eigenvector corresponding to the eigenvalue λ_{min}, and $\nabla_y \mathbf{K}_s$ the gradient of the symmetric part of the tangent matrix, which can easily be calculated by the finite element code.

As an example, we consider the axially loaded tensile bar of figure 2. The elastic modulus E is assumed to vary randomly along the bar and to be a homogeneous Gaussian random field, with mean 20000 MPa, standard deviation 1000 MPa, and an autocorrelation function $\rho=\exp(-0.5\ \tau^2/d^2)$, where τ is the distance between any two points along the bar. Using the midpoint method (Der Kiureghian and Ke 1988), the random field is represented by 40 random variables. The tensile loading consists of a fixed elongation of 0.0095 mm. Table 1 gives the reliability index for three correlation parameters d. Fig. 3 shows the profiles of the elastic modulus at the most likely failure point.

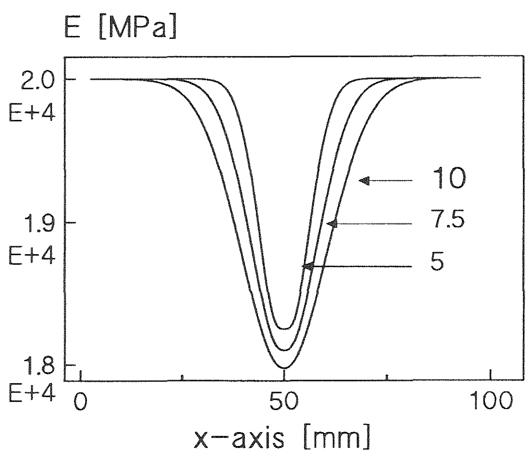

Fig. 3. Profile of elastic modulus E at most likely failure point for three correlation parameters d

Table 1. Reliability index β as a function of the correlation parameter d

d (mm)	β (-)
5	1.76
7.5	1.87
10	2.07

The analysis indicates that instability is sensitive to the elastic modulus only in a zone surrounding the damage localisation zone. The width of this zone of lower elastic modulus increases with increasing correlation parameter d. We also observe that the structure is safer (higher reliability index) for high values of the correlation parameter d.

5 Conclusion

A first-order finite element reliability method has been introduced for the assessment of the probability of structural instability of strain softening continua. The method takes advantage of the recent development of a gradient damage model. Extensions to two-dimensional problems and to other random variables, such as the initial damage threshold, will be developed.

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