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ADDITIVE VOLUMETRIC-DEVIATORIC SPLIT OF FINITE STRAIN TENSOR AND ITS IMPLICATION FOR CRACKING MODELS

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Abstract

The decomposition of finite strain into its volumetric and deviatoric parts has been generally thought to require a multiplicative form. The present paper shows that this decomposition can be formulated as additive, for any choice of finite strain tensor. Such additive decomposition facilitates generalization of nonlinear triaxial small-strain constitutive models for concrete cracking or other softening damage to finite strains. The decomposition is needed for the modeling of impact, explosions and severe earthquake damage in concrete structures, in which very large deviatoric deformations can be produced under high confinement.

1 Introduction

Finite element analysis of certain fracture problems of concrete, such as penetration of a missile into a concrete wall, requires a constitutive model for distributed cracking in the fracture process zone that is applicable at very large strains. The compressive volumetric strain of concrete can of course be never very large because very high hydrostatic pressures are generated by even relatively modest strains. For example, according to the tests of Bažant, Bishop and Chang (1986), the volumetric strain of -6% causes a hydrostatic pressure of about $-300,000$ psi (about $-2,000$ MPa). The deviatoric strains however, can be enormous under conditions of large confining pressure without causing continuous fractures. This was discovered already by Ira Woolson (1905), who cast concrete into a thick steel tube, compressed the cylindrical specimen to about a half of its initial length, which caused considerable bulging, and then, after cutting and removing the steel tube, found the concrete to retain its integrity. Under sufficiently high confining pressure, it is certainly possible to achieve shear strains of the order of 100% while the cracking in concrete remains distributed and discontinuous.

To analyze such problems, for example missile impact, explosive events or deformations of concrete in highly confined members under earthquake loading, one needs a nonlinear triaxial constitutive law applicable at finite strains. Because of scarcity of test data for very large strains and the near impossibility of achieving large but uniform strains in concrete specimens at very large deformation, the constitutive relation for finite strain must be obtained by generalizing the known small-strain constitutive equation and then calibrating the additional parameters for large strains by comparisons with structural tests, in which the strain is non-uniform.

The typical feature of all kinds of constitutive models for concrete as well as other quasibrittle materials is that the strain is split into its volumetric and deviatoric parts, for which the constitutive behavior is treated separately. This split is additive. However, it has been generally believed that, at finite strain, the decomposition of deformation into its volumetric and deviatoric parts must be multiplicative, such that the transformation tensor $\boldsymbol{F} = \boldsymbol{F}_D \boldsymbol{F}_V$, were $\boldsymbol{F}_V \boldsymbol{F}_D$ are the volumetric and deviatoric transformation tensors (Flory, 1961; Sidoroff, 1974; Simo, 1988; Lubliner, 1990). The multiplicative form of the volumetric-deviatoric split is an obstacle to the generalization of existing constitutive models for distributed cracking or other softening damage in concrete. The purpose of the present brief conference contribution is to report that the volumetric-deviatoric split can be formulated as additive.

2 Finite strain analysis

As is well known (e.g., Bažant and Cedolin, 1991, chapter 11), there are infinitely many possible finite strain measures to choose. The simplest choice is the Green-Lagrange finite strain tensor,

$$\varepsilon_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{1}$$

where the numerical subscripts refer to Cartesian coordinates X_i of material points in their initial locations, $u_i = x_i - X_i$ = displacements of the material points, x_i = coordinates of material points in the final deformed state, $F_{ij} = \partial x_i/\partial X_j = u_{i,j} + \delta_{ij}$ = components of the transformation tensor $\boldsymbol{F}$, and δ_{ij} = Kronecker delta. The derivatives, denoted by a subscript preceded by a comma, are the derivatives with respect to X_i, e.g., $u_{i,j} = \partial u_i/\partial X_j$.

It is helpful to recall first the derivation of Eq. (1). Consider the initial line segment dX_i transforming to dx_i. Tensor ε_{ij} is defined by setting $dx_k dx_k - dX_k dX_k = 2\varepsilon_{ij} dX_i dX_j$. Substituting $x_i = X_i + u_i$, $dx_k = x_{k,i} dX_i$ (where $x_{k,i} = \partial x_k/\partial X_i$) one gets

$$\begin{aligned} 2\varepsilon_{ij} dX_i dX_j &= x_{k,i} dX_i x_{k,j} dX_j - dX_k dX_k \\ &= \left[\left(X_k + u_k\right)_{,i}\left(X_k + u_k\right)_{,j} - \delta_{ij}\right] dX_i dX_j \end{aligned} \tag{2}$$

in which one may substitute $X_{k,i} = \partial X_k/\partial X_i = \delta_{ki}$. Since this relation must hold for any dX_i, one has $2\varepsilon_{ij}\left(\delta_{ki} + u_{k,i}\right)\left(\delta_{kj} + u_{k,j}\right) - \delta_{ij} = u_{k,i} + u_{k,j} + u_{k,i}u_{k,j}$, which yields Eq. (1).

Let us now proceed similarly, imagining that a small material element is deformed in two steps (Fig. 1) rather than one. In the first step, the element is subjected to pure volumetric (isotropic) expansion (i.e., same expansion in all directions), without any rotation. During this expansion, the point of initial coordinates X_i moves to a point of intermediate coordinates $\xi_i = X_i + u_i'$, and line segment dX_i transforms to line segment $d\xi_i$. In the second step, the material element is transformed by deformation at no change of volume and then is subjected to rigid body rotation (in which the volume change is also zero). In this transformation, the point at coordinates ξ_i moves to $x_i = X_i + u_i$, and segment $d\xi_i$ transforms to dx_i. Let ε_0 be the engineering strain (or Biot strain) giving the relative volume change, that is, $d\xi_i = \left(1 + \varepsilon_0\right) dX_i$. Then $\left(V_0 + \Delta V\right)/V_0 = \left(1 + \varepsilon_0\right)^3 = \det F_{ij} = J$, where J = Jacobian of the transformation, V_0 = initial volume of material element, and ΔV = volume increment. So

$$\varepsilon_0 = \left(\det F_{ij}\right)^{1/3} - 1 \qquad \left(F_{ij} = \delta_{ij} + u_{i,j}\right) \tag{3}$$

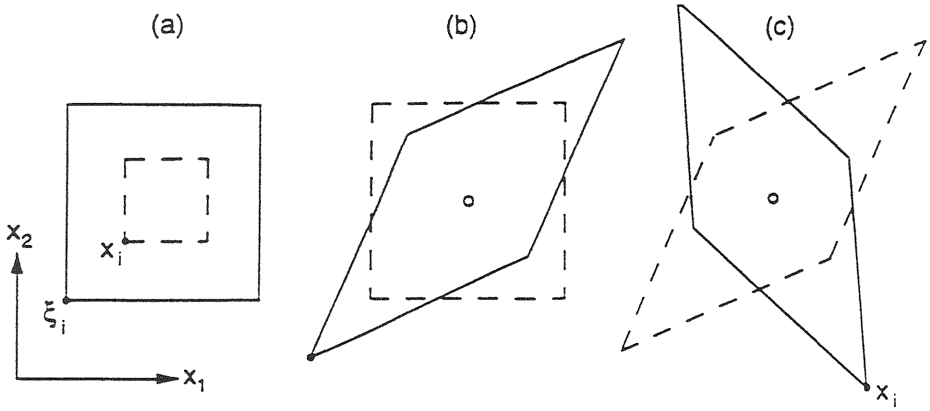

Figure 1: Volume expansion of an elementary cube of material, its subsequent deviatoric deformation, followed by rotation.

Denoting $u''_i = x_i - \xi_i$, and substituting $d\xi_i = (1+\varepsilon_0)\, dX_i$, we have

$$\begin{aligned} dx_k dx_k &= \left(\delta_{ki} + u''_{k,i}\right) d\xi_i \left(\delta_{kj} + u''_{k,j}\right) d\xi_j \\ &= \left(\delta_{ij} + u''_{i,j} + u''_{j,i} + u''_{k,i}u''_{k,j}\right)(1+\varepsilon_0)^2\, dX_i dX_j \end{aligned} \tag{4}$$

and, since $\partial X_i / \partial X_j = \delta_{ij}$,

$$u''_{i,j} = x_{i,j} - \xi_{i,j} = F_{ij} - (1+\varepsilon_0)\,\delta_{ij} = u_{i,j} - \varepsilon_0 \delta_{ij} \tag{5}$$

Denoting also $\varepsilon_{D_{ij}} = \left(u''_{i,j} + u''_{j,i} + u''_{k,i}u''_{k,j}\right)(1+\varepsilon_0)^2/2 = \left(F_{ik}F_{kj}J^{-2/3} - \delta_{ij}\right)/2$ we get

$$\begin{aligned} dx_k dx_k - dX_k dX_k &= (1+\varepsilon_0)^2\, dX_k dX_k + 2\varepsilon_{D_{ij}} dX_i dX_j - dX_k dX_k \\ &= \left(2\varepsilon_0 + \varepsilon_0^2\right) dX_k dX_k + 2\varepsilon_{D_{ij}} dX_i X_j \end{aligned} \tag{6}$$

Now, by definition, we must have $dx_k dx_k - dX_k dX_k = 2\varepsilon_{ij} dX_i dX_j$, and so $\varepsilon_{ij} = \delta_{ij}\varepsilon_V + \varepsilon_{D_{ij}}$ or

$$\varepsilon_{D_{ij}} = \varepsilon_{ij} - \delta_{ij}\varepsilon_V, \quad \varepsilon_V = \varepsilon_0 + \frac{1}{2}\varepsilon_0^2 \tag{7}$$

in which $\delta_{ij}\varepsilon_V = \varepsilon_{V_{ij}}$ = volumetric finite strain tensor, ε_V = Green-Lagrange volumetric finite strain, and $\varepsilon_{D_{ij}}$ = Green-Lagrange deviatoric finite strain tensor.

Eq. (7) shows that an exact additive decomposition into volumetric and deviatoric finite strains is possible.

The preceding derivation relies only on first principles and calls for only minimum familiarity with continuum mechanics. The same result, however, can be obtained and generalized by a shorter argument relying on the polar decomposition theorem $\boldsymbol{F} = \boldsymbol{RU}$ where $\boldsymbol{R}$ = rotation tensor and $\boldsymbol{U} = \left(\boldsymbol{F}^T\boldsymbol{F}\right)^{1/2}$ = right stretch tensor.

Consider a very general class of finite strain tensors, called the Doyle-Ericksen tensors (e.g., Bažant and Cedolin, 1991, Sec. 11.1):

$$\boldsymbol{\varepsilon}^{(m)} = m^{-1}\left(\boldsymbol{U}^m - I\right) \text{ for } m \neq 0, \quad \boldsymbol{\varepsilon}^{(m)} = \ln \boldsymbol{U} \text{ for } m = 0 \qquad \text{(8a, b)}$$

where m is any real number; $\boldsymbol{\varepsilon}^{(2)}$ = Green-Lagrange tensor in Eq. (1), $\boldsymbol{\varepsilon}^{(1)}$ = Biot finite strain tensor, and $\boldsymbol{\varepsilon}^{(0)}$ = logarithmic finite strain tensor. Flory (1961) introduced the multiplicative decomposition

$$\boldsymbol{F} = \boldsymbol{F}_D\boldsymbol{F}_V \qquad (9)$$

(see also Simo, 1978; Sidoroff, 1974) where $\boldsymbol{F}_D$ and $\boldsymbol{F}_V$ are the transformation tensors for the deviatoric and volumetric transformations; $\boldsymbol{F}_V = J^{1/3}\boldsymbol{I}$, $\boldsymbol{F}_D = J^{-1/3}\boldsymbol{F}$. The following tranformation is now possible for $m \neq 0$:

$$\begin{aligned} \boldsymbol{\varepsilon}^{(m)} &= m^{-1}\left[\left(\boldsymbol{F}^T\boldsymbol{F}\right)^{m/2} - \boldsymbol{I}\right] = m^{-1}\left[\left(\boldsymbol{F}_D^T\boldsymbol{F}_D J^{2/3}\right)^{m/2} - \boldsymbol{I}\right] \\ &= m^{-1}\left[\left(\boldsymbol{F}_D^T\boldsymbol{F}_D\right)^{m/2} J^{m/3} - \boldsymbol{I}\right] = \boldsymbol{\varepsilon}_D^{(m)} + \boldsymbol{\varepsilon}_V^{(m)} \qquad (10) \end{aligned}$$

in which

$$\boldsymbol{\varepsilon}_V^{(m)} = m^{-1}\left(J^{m/3} - 1\right)\boldsymbol{I}, \quad \boldsymbol{\varepsilon}_D^{(m)} = m^{-1}\left(\boldsymbol{U}_D^m - \boldsymbol{I}\right)J^{m/3} \qquad (11)$$

where $\boldsymbol{U}_D = \left(\boldsymbol{F}_D^T\boldsymbol{F}_D\right)^{1/2}$ = deviatoric right-stretch tensor. Similarly, for $m = 0$:

$$\boldsymbol{\varepsilon}^{(0)} = \ln\sqrt{\boldsymbol{F}^T\boldsymbol{F}} = \ln\sqrt{\boldsymbol{F}_D^T\boldsymbol{F}_D J^{2/3}} = \boldsymbol{\varepsilon}_D^{(0)} + \boldsymbol{\varepsilon}_V^{(0)} \qquad (12)$$

where

$$\boldsymbol{\varepsilon}_V^{(0)} = \left(\frac{1}{3}\ln J\right)\boldsymbol{I}, \quad \boldsymbol{\varepsilon}_D^{(0)} = \ln \boldsymbol{U}_D \qquad (13)$$

So, the volumetric-deviatoric split of finite strain can be formulated as additive for any choice of the finite strain measure. Note that tensor $\boldsymbol{\varepsilon}_V^{(m)}$ vanishes when the volume change is zero (or $J = 1$), and tensor $\boldsymbol{\varepsilon}_D^{(m)}$ vanishes when the deformation is a pure isotropic expansion ($\boldsymbol{F}_D = \boldsymbol{I}$). This implies that they represent the volumetric and deviatoric deformations.

3 Concluding remarks

It may be concluded that, for any choice of finite strain tensor, the volumetric-deviatoric decomposition can be formulated as additive. This facilitates generalization of the existing nonlinear triaxial constitutive models for concrete cracking and softening damage to finite strains.

An application to the microplane model for concrete will be given in Bažant et al. (1995).

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