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LOCALISATION AND MESH SENSITIVITY IN GRADIENT DEPENDENT SOFTENING PLASTICITY

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Abstract
A gradient-dependent plasticity theory is applied in finite element solutions of static strain localisation problems. Assuming weak satisfaction of constitutive equations, a layered beam element with a mixed character is developed. The plastic strain field is discretized in addition of the displacement field. Some examples are carried out to verify the performance of both the element and the algorithm.

1 Introduction

Structural concrete exhibits strain softening due to non-homogeneous deformations. Softening behaviour is a precursor to failure and involves localisation of deformation. When it is taken into account in standard continuum theories, the strain softening phenomenon leads to ill-posed boundary value problems, since the governing equations lose ellipticity. Thereafter, numerical simulations suffer from extreme mesh dependence. The localisation zone is completely determined by the discretization and no convergence to unique solution is obtained.

To remedy this improper behaviour, the standard continuum model must be enriched by including either extra or higher order terms (Bazant and Pijaudier-Cabot,1988, de Borst and Muhlhaus,1991 and 1992, Sluys,1992). These terms are used as localisation limiter and allow to keep the problem well-posed. These techniques are commonly referred to as regularisation methods.

In recent works (de Borst and Muhlhaus,1991 and 1992, Muhlhaus and Aifantis,1991, Pamin,1993, Sluys,1992), the use of gradient-dependent plasticity theory has been proved to be successful in preventing the deficiency of the standard continuum. The essential feature of this theory is the dependence of the yield function upon the second order spatial gradient of the plastic strain measure. This gradient dependence makes difficult to determine the increments of the plastic multiplier, as the consistency condition which governs the plastic flow is a partial differential equation. It has been proposed (de Borst and Muhlhaus,1992) to satisfy the yield function in the distributed sense and to discretize the plastic strain field in addition to the usual discretization of the displacement field. As for classical mixed formulations, additional degrees of freedom related to the plastic multiplier are then introduced in the finite elements besides the nodal displacements. In two or three dimensional analyses using continuum elements, this can lead to sizeable problems making the calculations unreasonable.

In this contribution, we scrutinise the possibility of using a layered approach based on gradient plasticity. A layered beam finite element is elaborated. The element has as degrees of freedom the displacement components of the reference axis of the beam and the plastic multiplier components corresponding to each layer. This layered approach allows to perform finite element analyses with a reduced number of degrees of freedom. Application to concrete beams under static loading is presented to validate and illustrate the approach.

2 Incremental formulation

We consider the following set of field equations:

$$\mathbf{L}^{T}\dot{\varepsilon} = 0 \ , \qquad (1)$$

$$\dot{\varepsilon} = \mathbf{L}\dot{\mathbf{u}} \ , \qquad (2)$$

$$\mathrm{F}(\sigma, \kappa, \nabla^{2}\kappa) = 0 \ , \qquad (3)$$

$$\dot{\sigma} = \mathbf{D}(\dot{\varepsilon} - \dot{\lambda}\frac{\partial \mathrm{F}}{\partial \sigma}) \ , \qquad (4)$$

which defines the elasto-plastic rate boundary problem during continued yielding. In the equations mentioned above $\mathbf{L}$ is a differential operator matrix, $\dot{\sigma}$ and $\dot{\varepsilon}$ are the stress and strain rate tensors, respectively, $\dot{\mathbf{u}}$ is a displacement rate vector, $\mathbf{D}$ the elastic stiffness matrix, $\dot{\lambda}$ is a multiplier being a measure of plastic flow intensity, F is the gradient-dependent yield function and κ is the hardening parameter related to the plastic strain tensor $\dot{\varepsilon}^p$ by either the strain-hardening hypothesis.

An incremental-iterative algorithm presented in (de Borst and Muhlhaus,1992) has been derived for gradient plasticity. Unlike in classical plasticity, this algorithm requires a weak satisfaction of the equilibrium condition (1) and the yield condition (3) at the end of iteration j+1 of current loading step, leading to the two following variational equations:

$$\int_V \delta\varepsilon^T \mathbf{D} \left(d\varepsilon - d\lambda\, \mathbf{n}_j\right) dV = \int_S \delta\mathbf{u}^T \mathbf{t}_{j+1}\, dS - \int_V \delta\varepsilon^T \sigma_j\, dV \tag{5}$$

and

$$\int_V \delta\lambda\left[\mathbf{n}_j^T \mathbf{D} d\varepsilon - \left(h + \mathbf{n}_j^T \mathbf{D}\mathbf{n}_j\right)d\lambda + g\nabla^2(d\lambda)\right]dV = -\int_V \delta\lambda\, f\left(\sigma_j, \kappa_j, \nabla^2\kappa_j\right)dV, \tag{6}$$

in which $\mathbf{d}$ indicates the difference between the values of variables at iteration (j+1) and iteration (j), i.e. $d\lambda = \lambda_{j+1} - \lambda_j$, and $\mathbf{t}$ denotes the boundary force vector. The gradient to the yield surface $\mathbf{n} = \partial F/\partial\sigma$, that indicates the plastic flow direction, is determined for $\sigma = \sigma_j$. For the sake of simplicity, the softening modulus $h = -\partial F/\partial\kappa$ and the additional variable $g = \partial F/\partial\nabla^2\kappa$, which is characteristic of gradient dependent model and related to the internal length scale, are assumed to be constant.

Application of Green's theorem to the last term on the left hand side of equation (6) yields

$$\int_V g\delta\lambda\nabla^2(d\lambda)dV = -\int_V g(\nabla\delta d\lambda)^T(\nabla d\lambda)dV + \int_{S_\lambda} g\delta d\lambda(\nabla d\lambda)^T \upsilon_\lambda \tag{7}$$

with υ_λ the outward normal at the elasto-plastic boundary S_λ. From equation (7) it follows the non-standard boundary conditions that the plastic multiplier field must fulfil

$$\delta d\lambda = 0 \quad \text{or} \quad (\nabla d\lambda)^T \upsilon_\lambda = 0. \tag{8}$$

3 Discretization

The displacement and strain field in eqs. (5) and (6) can be discretized according to the normal finite element procedure

$$\mathbf{u} = \mathbf{N}\mathbf{a} \qquad \varepsilon = \mathbf{B}\mathbf{a} \tag{9}$$

where $\mathbf{N}$ contains interpolation polynomials, $\mathbf{B}$=$\mathbf{LN}$ and $\mathbf{a}$ is a nodal displacement vector. On the other hand C^1-continuous shape functions

$\mathbf{H}$=(H_1,...,H_n) are used for the interpolation of the plastic multiplier since second derivatives of this variable appear in the weak form of the consistency condition (6). Introducing a vector $\mathbf{P} = (\nabla^2 H_1, ..., \nabla^2 H_n)$, the Laplacian of λ can be computed properly. We obtain

$$\lambda = \mathbf{H}^T \Lambda \quad \nabla^2\lambda = \mathbf{P}^T \Lambda \tag{10}$$

where Λ denotes a vector of nodal degrees-of freedom of the λ field.

The Substitution of the above identities in eqs. (5) and (6) and requiring that these equations hold for any admissible variation $\delta\mathbf{a}$ and $\delta\Lambda$, we obtain the following set of algebraic equations :

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{\lambda a}^T \\ \mathbf{K}_{\lambda a} & \mathbf{K}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{da} \\ d\Lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e + \mathbf{f}_i \\ \mathbf{f}_\lambda \end{bmatrix} \tag{11}$$

with the elastic stiffness matrix $\mathbf{K}_{aa}$, the external force vector $\mathbf{f}_e$ and the internal force vector $\mathbf{f}_i$ defined conventionally, and the off-diagonal matrix $\mathbf{K}_{\lambda a}$, the gradient-dependent matrix $\mathbf{K}_{\lambda\lambda}$, the vector $f\lambda$ of non-standard residual forces which emerges from the inexact fulfilment of the yield condition defined in (de Borst and Muhlhaus,1992). The tangent stiffness matrix in the set (11) is non-symmetric due to gradient terms in the submatrix $\mathbf{K}_{\lambda\lambda}$. If the derivation in eq. (8) is considered and assuming that the appropriate boundary conditions are satisfied, $\mathbf{K}_{\lambda\lambda}$ can be symmetrized. However, the symmetrization does not seem to offer much practical advantage, it rather results in a lack of convergence (Meftah,1994 and 1995). For the beam element considered in this paper only a symmetric operator is adopted to validate the layered approach. The implementation of the non-symmetric solver in the finite element code is in final stage and will make possible to use a non-symmetric operator providing better results.

The above set of equations governs the element behaviour during plastic flow. According to Kuhn-Tucker conditions,

$$\dot{\lambda} \geq 0 \,, \qquad F \leq 0 \,, \qquad \dot{\lambda} F = 0 \,, \tag{12}$$

eq. (11) can be extended to the elastic part of the body. In the elastic elements we set $\mathbf{K}_{\lambda a} = 0$ since the gradient vector $\mathbf{n}$=0. Then the second equation in the set (11) separates from the first one giving the following equation in $d\Lambda$

$$\mathbf{K}_{\lambda\lambda} d\Lambda = \mathbf{f}_\lambda \,. \tag{13}$$

For the elastic state we also set the residual forces $f\lambda$ to zero. We obtain then the desired solution if the global matrix $\mathbf{K}_{\lambda\lambda}$ is non singular after element assembly and introduction of the boundary conditions for the Λ degrees-of-freedom (Pamin,1993). It is therefore not necessary to set the value of the hardening modulus h equal to large number in $\mathbf{K}_{\lambda\lambda}$ to constrain the value of λ to zero for elastic elements (de Borst and Muhlhaus,1992). In this paper h for elastic elements is taken equal to Young's modulus E.

4 Application to beams theory

As first application, we focus our attention on Navier-Bernoulli beam theory (Batoz and Dhatt,1990). Therefore, the stress and strain tensors reduce to their axial components $\sigma = (\sigma_x)$ and $\varepsilon = (\varepsilon_x)$, respectively. Each cross section is devided into *n* layers (fig.1). The nodal displacement and force vectors of the centroidal axis are $\mathbf{a} = (u_i, v_i, \theta_i, u_j, v_j, \theta_j)$ and $\mathbf{f} = (N_i, T_i, M_i, N_j, T_j, M_j)$ where u=axial displacement, v=transverse deflection, θ=rotation of cross section, and subscripts i and j refer to the adjacent cross sections i and j at the ends of the element.

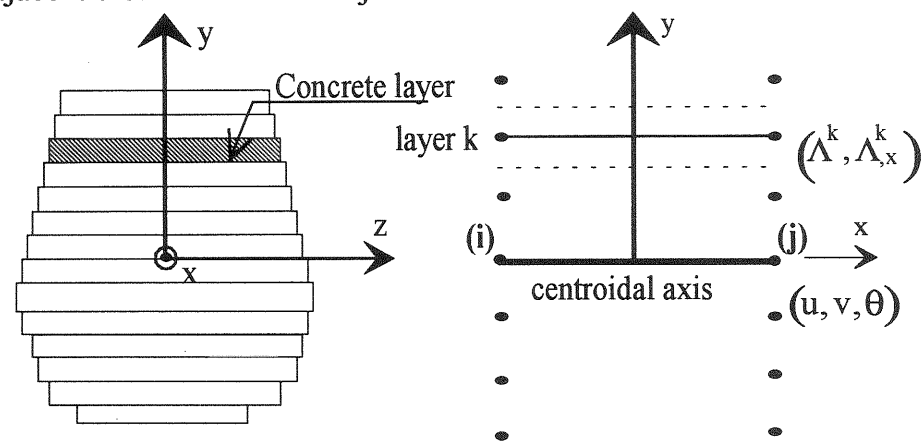


Figure 1 Layered finite element

The hypothesis of plane cross sections remain plane and normal to the centroidal axis allows to determine the displacement in the axial direction at any point x,y (in cartesian coordinates x,y) as $u(x,y) = u(x,0) - y\,\partial v(x,0)/\partial x$, and the strain as $\varepsilon_x(x,y) = \partial u(x,y)/\partial x$. We use finite elements with a linear variation of u and a cubic variation of v. Therefore, the displacement field $\mathbf{u}$ on each layer is related to only the mid-axis displacement components $\mathbf{a}$ by mean of the appropriate interpolation polynomials.

For the plastic multiplier field no assumption can be made on its variation through the cross section, and then independent degrees-of-freedom (d-o-f) Λ per layer corresponding to λ are introduced. At any point of each layer **k** (fig.1), the plastic multiplier is interpolated using two d-o-f per node $\Lambda^k = \left(\Lambda^k, \Lambda^k_{,x}\right)$. Here Hermitian interpolation is considered and the same shape functions as for the transverse displacement are used. The obtained beam element has a mixed character and presents as nodal d-o-f $\mathbf{a} = \left(u\ , v\ , v_{,x} = \theta\ , \Lambda^1\ , \Lambda^1_{,x}\ , \ldots\ , \Lambda^i\ , \Lambda^i_{,x}\ , \ldots\ , \Lambda^n\ , \Lambda^n_{,x}\right)$ where *n* is the layers number.

We proceed in the same way as in the preceding section by considering, on one hand, the above discretizations, and on the other hand, the two variational equations (5) and (6) with eq. (7). We then obtain the algebraic equations, in compact fashion, to solve for the layered beam element in gradient plasticity

$$\begin{bmatrix} [K_{aa}] & [K^1_{\lambda a}]^T & \cdots & [K^k_{\lambda a}]^T & \cdots & [K^n_{\lambda a}]^T \\ [K^1_{\lambda a}] & [K^1_{\lambda\lambda}] & \cdots & [0] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [K^k_{\lambda a}] & [0] & \cdots & [K^k_{\lambda\lambda}] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [K^n_{\lambda a}] & [0] & \cdots & [0] & \cdots & [K^n_{\lambda\lambda}] \end{bmatrix} \begin{bmatrix} \mathbf{da} \\ d\Lambda^1 \\ \vdots \\ d\Lambda^k \\ \vdots \\ d\Lambda^n \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{f}^1_\lambda \\ \vdots \\ \mathbf{f}^k_\lambda \\ \vdots \\ \mathbf{f}^n_\lambda \end{bmatrix} \tag{14}$$

where the subscript k indicates that quantities are computed with respect of the considered layer. For the details of these derivations the reader is referred to (meftah,1994).

5 Material model

Since we restrict ourselves to one-dimensional plastic flow problems, a second order gradient yield function can summarised by the set of equations

$$F = \sigma - \bar{\sigma}\left(\varepsilon^p, d^2\varepsilon^p/dx^2\right) \quad , \quad \bar{\sigma} = \sigma_y + h\varepsilon^p - g\,d^2\varepsilon^p/dx^2 \tag{15}$$

with $\varepsilon^p \equiv \lambda \equiv \kappa$. When the maximum tensile strength σ_y is attained softening occurs with h constant. Also an internal length scale l which governs the localisation band width is introduced in the model (de Borst and Muhlhaus,1992), giving the gradient constant $g = -hl^2$.

It is emphasised that the presence of gradient terms in the yield strength is the algorithmic essence of gradient regularisation. The case of negative contribution of this terms occurs on the elasto-plastic boundary, making possible for the localisation zone to spread since the yield strength reduces.

6 Validation of the layered element

6.1 Softening bar in tension

To show the regularisation introduced by gradient dependence, the layered element is used to solve the one dimensional problem of an imperfect bar in tension (Fig.2). For this case an analytical solution exists (de Borst and Muhlhaus,1992) that can be used to verify the width of the localisation zone and inclination of load-displacement diagram obtained numerically.

In the calculations the length of the bar is L=100 mm , the Young's modulus E=20000 N/mm^2 , the tensile strength σ_y =2 N/mm^2 and the softening modulus h=-0.1E. The internal length scale is l =5 mm , giving the gradient constant g = 50000 N and the width of the localisation zone w=2πl=31.4 mm. At the centre of the bar (d=10 mm) an imperfect zone with a 10% smaller value of σ_y is assumed. The additional boundary conditions $A^i = 0$ and $A^i_{,x} = 0$ are introduced for each layer (i) at both edges of the bar.

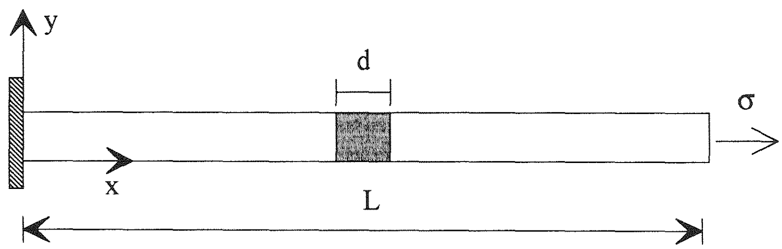
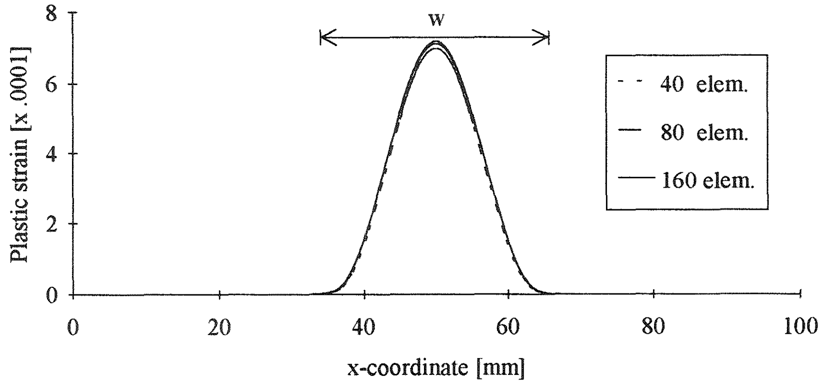


Figure 2 Imperfect bar in tension.

Figures 3 presents the results of the finite element analysis. The obtained results are very closed to those existing in the literature (de Borst and Muhlhaus,1992, Pamin,1993). They show convergence to unique solution upon mesh refinement, i.e. the analytical post-peak branch $\Delta u / \Delta \sigma = (1 - \pi)/E$ is retrieved and a localisation zone with a width close to w=10π is observed. It is emphasised (de Borst and Muhlhaus,1992) that the approach is capable of simulating the size effect, since the ratio l/L governs the response in the post-peak regime. The increase of the structural size L with constant l will result in a more brittle behaviour, which is consistent with experiments.



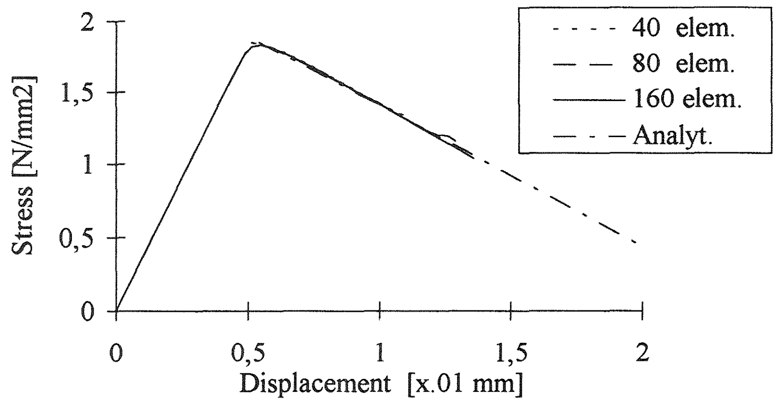


Figure 3 Distribution of plastic strains along the bar (top) and stress vs. displacement (bottom) for different mesh refinements.

6.2 Algorithm behaviour in bending

A gradient-independent perfect plasticity test is performed to verify the algorithm and the behaviour of the new layered beam element in bending. A problem of the plastic flow in a clamped cantilever beam has been chosen. The schematic element layout (40 elts.) and loading are presented in Fig. 4. The geometry and the material data for the concrete beam are based on (Hughes,1987): the length L=400 mm, the height and the width are H=B=70 mm, the Young's modulus E=20000 N/mm^2, the tensile strength σ_y=6 N/mm^2 and the plastic modulus h and gradient constant g are set to zero. The results of finite element analyses are presented in figure 5. It gives the overall load-displacement response of the beam for different number of layers. We observe that increasing number of layers leads to more regular and soft response. It is noted that at certain stage of loading lack of convergence occures. The reason is a return mapping inside of the yield contour. No proper algorithm has yet been elaborated to avoid this undesirable behaviour.

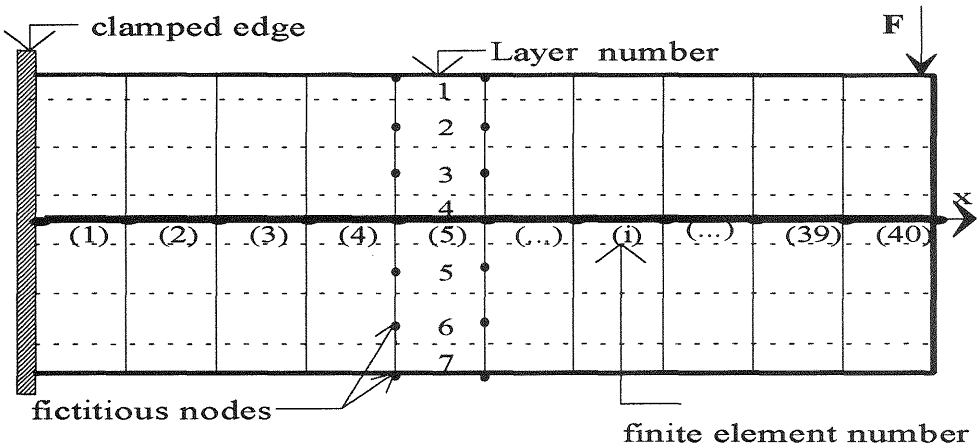


Figure 4 Clamped cantilever beam.

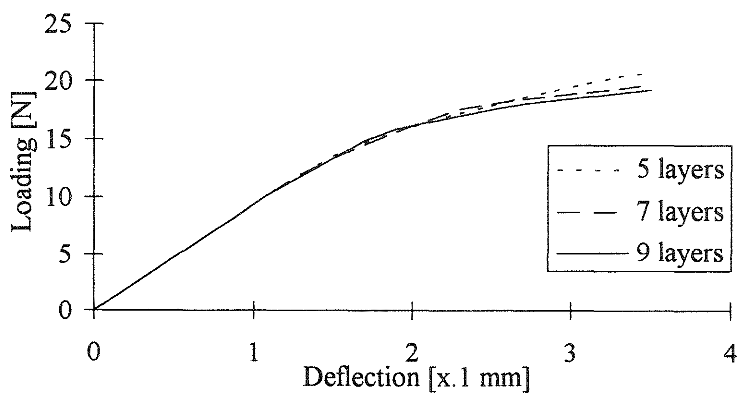


Figure 5 Load-displacement for different layers number.

7 Conclusion

The finite element implementation of gradient-dependent plasticity has been presented. The theory includes a regularising dependence of the yield function on the Laplacian of plastic strain measure. The fundamental feature of the used algorithm is a weak satisfaction of the yield condition which is coupled with equilibrium condition.

A layered beam element with separate interpolation of displacement and plastic strain fields has been developed. It has been validated and applied in calculation of strain localisation. Some obtained results show good agreement with available ones, but both of the algorithm and the element require further study. For this reason different elements with different interpolations of displacement field have been developed and tested. It turned out that the elements should fulfil some additional conditions, namely balance of interpolations for displacement and plastic multiplier and existence of suitable integration quadrature, i.e. a sufficient number of integration points to prevent zero-energy modes for the tow fields and optimal for accuracy sampling positions.

References

Batoz, J.L. and Dhatt, G. (1990) **Modélisation des Structures par Eléments Finis**. Volume 2, Edition Hermès.

Bazant, Z.P. and Pijaudier-Cabot, G. (1988) Nonlocal continuum damage, localisation instability and convergence. **J. Appl. Mech.**, 55, 287-293.

Borst, R. de and Mühlhaus, H.B. (1991) Continuum models for discontinuous media, in **Fracture Processes in Concrete, Rock and Ceramics**. (eds J.G.M. van Mier et al.), E & FN Spon, Chapman & Hall, London, 601-618.

Borst, R. de and Mühlhaus, H.B. (1992) Gradient-dependent plasticity: Formulation and algorithmic aspects. **Int. J. Num. Meth. Eng.**, 35, 521-539.

Hughes, T.J.R. (1987) **The Finite Element Method. Linear Static and Dynamic Analysis**. Prentice-Hall, New Jersey.

Meftah, F. (1994) Modélisation de la localisation dans les géo-matériaux : approche par la plasticité aux gradients. I.Element de poutre multicouches. Rapport d' avancement de travaux de recherche, LBS-INSA-Lyon, France.

Meftah, F., Reynouard, J.M. and Merabet, O. (1995) A layered beam element for gradient plasticity, in **Computational Plasticity : fundamentals and applications**. (eds D.R.J Owen, E. Onate and E. Hinton), Pineridge Press, Swansea, U.K., 2117-2127.

Mühlhaus, H.B. and Aifantis, E.C. (1991) A variational principle for gradient plasticity. **Int. J. Solids Structures**, 28 , 845-857.

Pamin, J. (1993) Mesh sensitivity in gradient-dependent softening plasticity: III Finite element formulations. TNO-BOUW Report No BI-93-NM-R1166, TU-Delft Report No 25-2-93-2-15, Delft, The Netherlands.

Sluys, L.J. (1992) Wave propagation, localisation and dispersion in softening solids. Dissertation, Delft University of Technology, Delft, The Netherlands.