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STRAIN LOCALIZATION AS BIFURCATION OF ELASTO-PLASTIC SOFTENING MATERIALS

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Abstract
The present paper describes the mathematical tools for detecting the onset of discontinuous bifurcation and addresses the post-bifurcation analysis of elasto-plastic solids. Under the assumption of C^0-continuity of velocities (displacements), constitutive expressions are developed which relate the stress rate to the strain rate on both sides of a spatial discontinuity which exhibits jumps of the deformations. Introducing the non-associated elasto-plastic damage model of Drucker-Prager type, numerical calculations are presented for the elastic/plastic bifurcation.

1 Introduction

It is widely accepted that when quasi-brittle materials are deformed sufficiently, they exhibit spatial discontinuities in the form of localized deformation. The formation of cracks and shear band observed in experiments in cementitious and granular materials are typical examples of localized failure mechanisms.

In the beginning, the mathematical formulation of localization criteria utilizing characteristic tensor is summarized (Rice (1976), Runesson et al (1991)). The singularity of the localization tensor determines the onset of strain localization as a bifurcation problem. In this paper we attempt to extend this concept to the post bifurcation analysis in order to distinguish between elastic/plastic bifurcation and plastic/plastic bifurcation as possible alternative cases. An elasto-plastic softening model is employed to carry out numerical simulations of the bifurcation behavior at the constitutive level, imposing either bifurcation mode.

2 Onset of strain localization

For the analysis of jump conditions we introduce the concept of a discontinuity surface, across which bifurcation of deformation is permitted. In the bifurcated state, the velocity gradient field $\nabla\dot{\mathbf{u}}$ exhibits a jump across the discontinuity plane. This kinematic assumption maintains C^0-continuity of displacements which imposes Maxwell compatibility such that the jump of the velocity gradient must have the form

$$[[\nabla\dot{\mathbf{u}}]] = \dot{\gamma}\mathbf{M}\otimes\mathbf{N} \neq \mathbf{0}, \quad [[\dot{\epsilon}]] = \frac{1}{2}([[\nabla\dot{\mathbf{u}}]] + [[\nabla\dot{\mathbf{u}}]]^T) \tag{1}$$

The square brackets express the jump of quantity between the plus and minus side of discontinuity surface (see Fig. 1). The unit normal vector $\mathbf{N}$ defines the orientation of the discontinuity surface, and the unit vector $\mathbf{M}$ indicates the polarization direction. The indeterminate scalar $\dot{\gamma}$ denotes the amplitude of the jump.

At the onset of localization, the material in both sides of the discontinuity is assumed to be in the plastic state. Hence, the constitutive behavior for both sides is expressed by the elasto-plastic constitutive representation,

$$\dot{\sigma}^+ = \mathbf{D}_{ep} : \dot{\epsilon}^+, \quad \dot{\sigma}^- = \mathbf{D}_{ep} : \dot{\epsilon}^-, \quad [[\dot{\sigma}]] = \mathbf{D}_{ep} : [[\dot{\epsilon}]] \tag{2}$$

in which $\mathbf{D}_{ep}$ denotes the elastic-plastic tangential operator, which is in this study cast in the form of a non-associated flow rule.

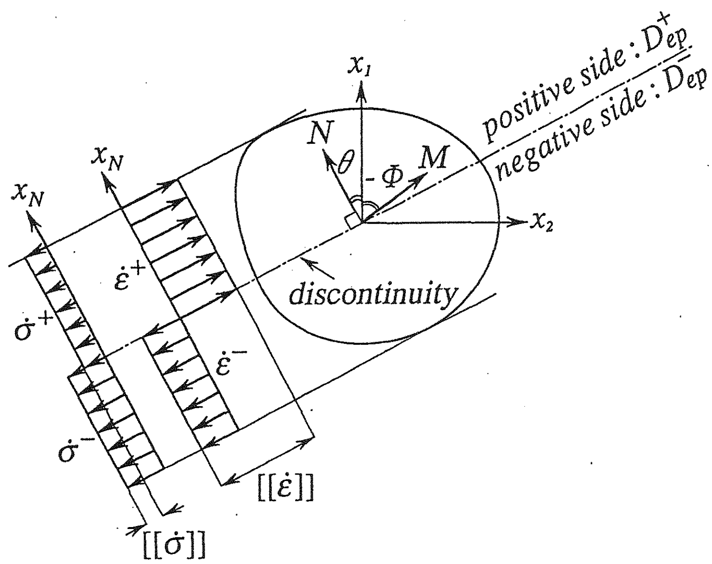

Fig. 1. Formation of discontinuity surface with $\mathbf{N}$, $\mathbf{M}$

Although the stress jump appears across the discontinuity surface, the balance of linear momentum on the surface requires that the surface tractions $\dot{\mathbf{t}}$ remain continuous i.e., $[[\dot{\mathbf{t}}]] = \mathbf{N} \cdot [[\dot{\sigma}]] = \mathbf{0}$. From this balance condition, the well-established localization criterion is obtained in the form

$$det(\mathbf{Q}_{ep}) = 0, \qquad \mathbf{Q}_{ep} = \mathbf{N} \cdot \mathbf{D}_{ep} \cdot \mathbf{N} \tag{3}$$

In analogy to the wave propagation argument, the vector $\mathbf{M}$ denotes the polarization direction, while the amplitude of the jump $\dot{\gamma}$ is related to the wave speed. The corresponding stationary criterion of wave propagation is represented by

$$\rho c^2 = \mathbf{M} \cdot \mathbf{Q}_{ep} \cdot \mathbf{M} = 0 \tag{4}$$

In the case of non-symmetric $\mathbf{Q}_{ep}$ the condition of vanishing wave speed $c \to 0$ may be reached before the characteristic tensor $\mathbf{Q}_{ep}$ experiences a singularity , which is referred to as loss of strong ellipticity in quasi-static localization (e.g. Willam and Sobh (1987)).

It is important to note that the bifurcation statement is so far a pointwise argument at the constitutive level. At the structural or element level, on the other hand, the characteristics of localization must be dealt with as a boundary value problem. This infers that localization features such as shear banding should be searched as one of the possible stationary states under the restriction of kinematic condition as well as the given boundary conditions (see Fig. 2).

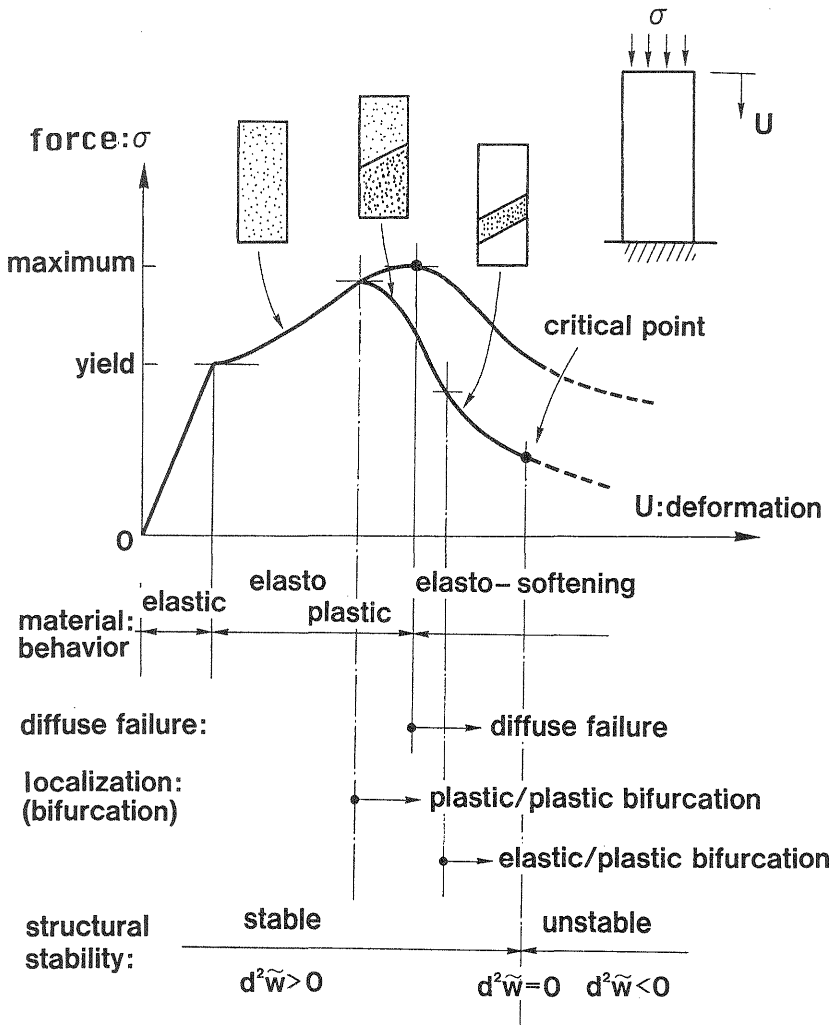

Fig. 2. Inelastic behavior at material and element level: Fundamental criteria during deformation history

3 Extension to post-bifurcation analysis

3.1 Different constitutive assumptions on each side:
Here, we expand the mathematical description of bifurcation beyond the onset of strain localization. To this end, the argument starts with the elasto-plastic constitutive expressions which differ now on each side of the discontinuity (Fig. 3) in contrast to Eq. (2).

$$\begin{aligned} \text{positive side}: \quad & \dot{\sigma}^+ = \mathbf{D}^+_{ep} : \dot{\epsilon}^+, \quad \mathbf{D}^+_{ep} = \mathbf{D}_{ep}(\sigma^+) \\ \text{negative side}: \quad & \dot{\sigma}^- = \mathbf{D}^-_{ep} : \dot{\epsilon}^-, \quad \mathbf{D}^-_{ep} = \mathbf{D}_{ep}(\sigma^-) \end{aligned} \tag{5}$$

in which $\mathbf{D}^+_{ep}$ and $\mathbf{D}^-_{ep}$ denote the elasto-plastic tangential operators on the positive and negative side of the discontinuity surface. This condition reduces to the case of elastic/plastic bifurcation when we consider the special case $\mathbf{D}^+_{ep} \rightarrow \mathbf{D}_{ep} = \mathbf{E}^-_e - \mathbf{E}^-_p$ and $\mathbf{D}^-_{ep} \rightarrow \mathbf{E}_e$.

3.2 Analytical formulation analogous to plasticity:
Now, let us consider the development of a constitutive representation of localization in the post-bifurcation regime. The derivation for the bifurcated material response is made in analogy to the three postulates of the well-known flow theory for rate-independent elasto-plasticity (Yoshikawa and Willam (1995)).

1. Additive decomposition of strain rate: $\dot{\epsilon}^+ = \dot{\epsilon}^- + [[\dot{\epsilon}]]$ (6)
2. Flow rule for localized strain jump: $[[\dot{\epsilon}]] = \dot{\gamma}[\mathbf{M} \otimes \mathbf{N}]^s$ (7)
3. Yield condition for stress jump:

$$F = \mathbf{M} \cdot [[\dot{\mathbf{t}}]] = [\mathbf{M} \otimes \mathbf{N}]^s : [[\dot{\sigma}]] = 0 \tag{8}$$

Introducing the relation of stress rate jump and strain rate jump $[[\dot{\sigma}]] = \dot{\sigma}^+ - \dot{\sigma}^- = \mathbf{D}^+_{ep} : [[\dot{\epsilon}]] - [[\mathbf{E}_p]] : \dot{\epsilon}^-$ into the yield condition, the localization multiplier $\dot{\gamma}$ is readily determined in terms of the strain rate on either side of the discontinuity, yielding

$$\begin{aligned} \dot{\gamma} &= \frac{[\mathbf{N} \otimes \mathbf{M}]^s : [[\mathbf{E}_p]] : \dot{\epsilon}^-}{\mathbf{M} \cdot \mathbf{Q}^+_{ep} \cdot \mathbf{M}} = \frac{[\mathbf{N} \otimes \mathbf{M}]^s : [[\mathbf{E}_p]] : \dot{\epsilon}^+}{\mathbf{M} \cdot \mathbf{Q}^-_{ep} \cdot \mathbf{M}} \\ & [[\mathbf{E}_p]] \equiv \mathbf{E}^+_p - \mathbf{E}^-_p = \mathbf{D}^-_{ep} - \mathbf{D}^+_{ep} \\ & \mathbf{M} \cdot \mathbf{Q}^+_{ep} \cdot \mathbf{M} = [\mathbf{M} \otimes \mathbf{N}]^s : \mathbf{D}^+_{ep} : [\mathbf{N} \otimes \mathbf{M}]^s \end{aligned} \tag{9}$$

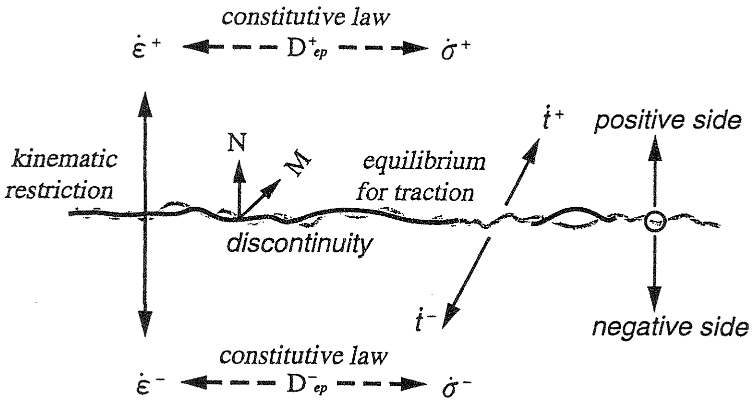

Fig. 3 Relationship between the stress and strain rates across the discontinuity surface

This analytical result for $\dot{\gamma}$ provides the important information about the nature of the jump of strain rate. If the tangential operators in both sides are the same, $\mathbf{D}^-_{ep} = \mathbf{D}^+_{ep}$, then the value of $\dot{\gamma}$ is equal to zero until the localization tensor turns singular. Right at the onset of bifurcation, $\dot{\gamma}$ turns out to be indeterminate. This statement corresponds to the previous equation (3) signaling the onset of strain localization. On the other hand, when discontinuities within the body develop so that the two tangential operators deviate ($\mathbf{D}^-_{ep} \neq \mathbf{D}^+_{ep}$), then $\dot{\gamma}$ becomes a finite value. During this stage, $\mathbf{N}$ is designed the critical direction and $\mathbf{M}$ is the eigenvector of $\mathbf{Q}_{ep}$.

It also follows that the difference of the strain rate tensors on both sides of the discontinuity is expressed as follows:

$$[[\dot{\epsilon}]] = \mathbf{\Phi} : \dot{\epsilon}^-, \qquad [[\dot{\epsilon}]] = \mathbf{\Psi} : \dot{\epsilon}^+ \tag{10}$$

In this notation

$$\begin{aligned} \mathbf{\Phi} &= \frac{[\mathbf{M} \otimes \mathbf{N}]^s \otimes [\mathbf{N} \otimes \mathbf{M}]^s : [[\mathbf{E}_p]]}{\mathbf{M} \cdot \mathbf{Q}^+_{ep} \cdot \mathbf{M}}, \\ \mathbf{\Psi} &= \frac{[\mathbf{M} \otimes \mathbf{N}]^s \otimes [\mathbf{N} \otimes \mathbf{M}]^s : [[\mathbf{E}_p]]}{\mathbf{M} \cdot \mathbf{Q}^-_{ep} \cdot \mathbf{M}} \end{aligned} \tag{11}$$

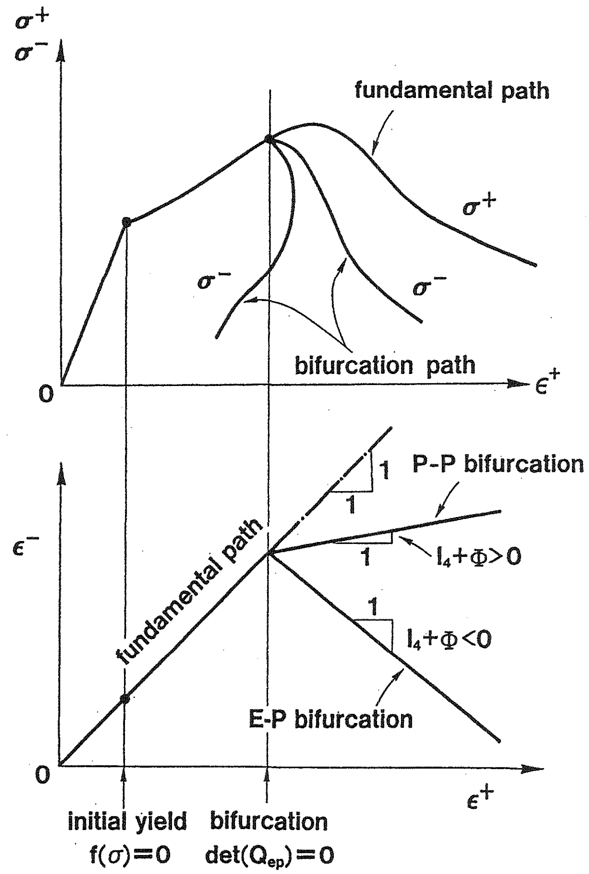

Fig. 4 Sketch of fundamental/bifurcation paths (upper); Two strain tensors across discontinuity (lower).

Both of Φ and Ψ are non-dimensional fourth-order tensors which provide the fundamental characteristics of discontinuous bifurcation. Finally, the constitutive representations are

$$\dot{\sigma}^+ = \begin{cases} \mathbf{D}^+_{ep} : \dot{\epsilon}^+ \\ \mathbf{D}^+_{ep} : (\mathbf{I}_4 + \Phi) : \dot{\epsilon}^- \end{cases} \tag{12}$$

$$\dot{\sigma}^- = \begin{cases} \mathbf{D}^-_{ep} : \dot{\epsilon}^- \\ \mathbf{D}^-_{ep} : (\mathbf{I}_4 - \Psi) : \dot{\epsilon}^+ \end{cases} \tag{13}$$

In view of each pair of these equations, it is obvious that two different strain fields are provided for the single stress rate when the tensors Φ and Ψ are non-zero. This is a fundamental aspect for the loss of uniqueness and post-bifurcation state due to the existence of a spatial discontinuity in the deformations. Fig. 4 schematically depicts the stress-strain behavior involving the bifurcation path and the relationship between the two strain tensors.

4 Elasto-plastic damage model

Here in this study, we employed the non-associated elasto-plastic constitutive model for the tangential operator in the form

$$\mathbf{D}_{ep} = \mathbf{E}_e - \mathbf{E}_p, \quad \mathbf{E}_p = \frac{\mathbf{E}_e : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}_e}{H_p + \mathbf{n} : \mathbf{E}_e : \mathbf{m}} \tag{14}$$

where the scalar quantity H_p defines the plastic modulus, and the Drucker-Prager model is used for both the yield criterion f and the plastic potential q in the form

$$f = \alpha I_1 + \sqrt{J_2} - k, \qquad q = \beta I_1 + \sqrt{J_2} - k' \tag{15}$$

For $\alpha \neq \beta$ we have a non-associated flow rule and for $\alpha = \beta = 0$ the Drucker-Prager model reduces to the von-Mises criterion.

In order to represent the hardening and softening behavior of quasi-brittle materials like concrete, we introduce the damage-based model proposed by Wu and Tanabe(1990). This model resorts to converting the Drucker-Prager expression to the Mohr-Coulomb surface by the compressive meridian matching. The material parameters such as the cohesion c, the internal friction angle ϕ and the dilation angle ψ evolve, depending on the damage accumulated with the plastic work. These are expressed

$$\begin{aligned} c &= c_0 exp[-(m\omega)^2] \\ \phi &= \begin{cases} \phi_0\sqrt{2\omega - \omega^2} & \omega \leq 1 \\ \phi_0 & \omega > 1 \end{cases}, \quad \psi = \begin{cases} \psi_0\sqrt{2\omega - \omega^2} & \omega \leq 1 \\ \psi_0 & \omega > 1 \end{cases} \end{aligned} \tag{16}$$

where m is a material parameter, and c_0, ϕ_0 and ψ_0 mean the initial values in the undamaged state. The values of ϕ and ψ increase while c decreases according to the increase of the damage parameter ω , expressing the progressive growth of microcracks.

5 Numerical examples

Computational results of uniaxial compression of concrete-like materials are presented in Fig. 5. In the calculation, elastic/plastic bifurcation as a possible bifurcation mode is assumed; whereby the positive side in the plastic state while the negative side in a state of elastic unloading. Under the deformation control of ϵ^+, the response of the other variables such as σ^+, σ^- and ϵ^- are obtained

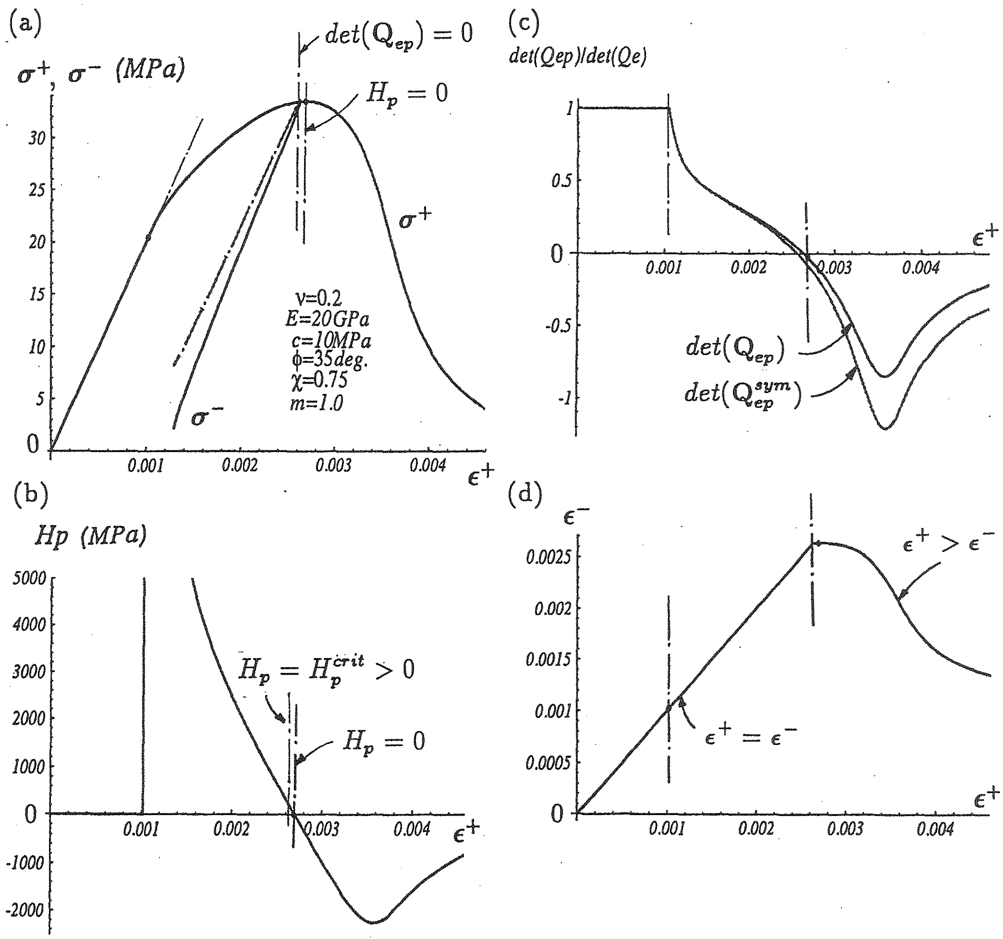

Fig. 5 Numerical results of uniaxial compression:

(a) $\sigma^+, \sigma^- \sim \epsilon^-$ relation, (b) the plastic modulus H_p, (c) $det(Q_{ep})/det(Q_e)$, and (d) $\epsilon^+ \sim \epsilon^-$ relation from incremental calculations.

The numerical results indicate that the bifurcation point by means of the singularity of $det(\mathbf{Q}_{ep})$ is first satisfied prior to the peak point ($H_p = 0$) due to the non-associativeness ($\phi_0 = 35\,\text{deg}, \psi_0 = 0$) of the employed constitutive model. After onset of bifurcation, the stresses σ^+, σ^- as well as the strain ϵ^-, ϵ^- deviate according to the development of the non-dimensional operators $\mathbf{\Phi}$ and $\mathbf{\Psi}$.

6 Concluding remarks

The example problem demonstrates the pointwise bifurcation of stresses and strains across the discontinuity surface. Further work will extend these concepts from the constitutive level to the structural level in boundary value problems, assuming volume fractions of localized fracture regions.

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