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A FINITE ELEMENT APPROACH TO 3D CRACK GROWTH IN BRITTLE SOLIDS

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Abstract
While the mechanics of fracture in three dimensional solids presents many unanswered questions, this paper considers the parallel computational issues associated with a finite element analysis of non-planar crack representation and growth.

1 Introduction

Despite the advances in the mechanics of fracture in both two and three dimensional contexts, little work has focussed on the very real computational difficulties associated with an analysis of non-planar crack growth. There are exceptions, of course, beginning with the pioneering work of Ingraffea and coworkers (Sousa et al. (1991), Martha et al. (1992)) on the representation aspects within a boundary element approach (see Carter et al.(1995) for a recent account) and the elegant contribution of Xu & Ortiz(1993) on non-planar growth, also within a boundary element framework; these contributions apply only to infinite media, of course. Beer (1993) presents an analysis of interface fracture using a two-domain boundary element approach, where internal degrees of freedom are condensed out leaving only interface freedoms. The problem is ironically simple, given that a predefined crack path is followed, and the topology requires only that regions be disconnected as a crack propagates.

This paper discusses a finite element based capability being developed by the authors for the analysis and tracking of arbitrary crack growth in three dimensional brittle solids. The objective is to consider the important features of the topological and algorithmic aspects of this approach, pointing also to the computational challenges to be overcome.

The boundary element method (BEM) has an apparent advantage over the finite element method (FEM) in requiring surface mesh (including crack surface) only, though the appeal is somewhat lost in the case of multiple fractures and internal interfaces, and of course the complexities of interpolation along the crack front and remeshing in the crack wake still exist. More recent developments in 'element free' Galerkin methods offer a promising alternative for crack growth studies (Belytschko et al 1994)). The advantages are mainly that remeshing is clearly not required, and that a dense cloud of nodes can easily be added in the vicinity of the crack tip for improved accuracy. However, it seems that considerable effort is required in the satisfaction of boundary conditions, and the boundary weights must be re-evaluated for each geometry i.e. at each iteration of crack advance.

A finite element system was chosen mainly because it seemed more straightforward for finite geometries to implement constitutive laws for nonlinear material behaviour, such as near-tip cohesive effects and plasticity and damage in the continuum, within a finite element framework. The *major* difficulty was firstly to develop a completely robust mesh generator (Grummitt & Baker (1995)) for arbitrary three dimensional domains: as non-planar cracks propagate, remeshing is required and the generator must be fully automatic and capable of discretizing any geometry that arises; in fact the process is not viewed as remeshing, but rather meshing a new geometry, including crack face boundaries, at each stage of crack advance. This is an important contribution since, as Beer (1993) points out, three dimensional remeshing would normally be done with an interactive CAD editor i.e. with user intervention.

The object oriented approach also allows the definition of a standard mesh of crack tip elements. The system currently employs LEFM only, with stress intensities used pointwise on the front for crack growth and direction criteria; stress intensity calculations for complex cases are compared with available benchmark solutions. However, the 'fracture tube' can easily be closed with a cohesive element, transmitting stiffness across the wake to simulate quasi-brittle behaviour.

Crack advance in non-fatigue situations requires consideration, and is implemented through an adaptive step advance: trial pointwise crack advance with a relative check on stress intensity to toughness is acceptable for sub-critical growth, but is obviously susceptible to unstable growth under constant load in other cases. Algorithms to transform process zones of damage or smeared cracking into a discrete crack advance can also be incorporated within the general framework, and are discussed.

2 Topology and meshing

2.1 Meshing algorithms

We use constructive solid geometry to define the solid via a binary tree structure combining primitive objects into a composite solid (Grummitt & Baker (1995)). A boundary mesh is mapped onto the external and crack face(s) by ray casting through primitives to place nodes on the surface, followed by a modified form of the Nelson (1978) algorithm for a Delauney triangulation of the surfaces. This has been found essential to prevent meshes crossing small inclusions, in particular the crack opening itself, if rigorous and expensive (element) face penetration and intersection checks are to be avoided.

The interior is meshed with 4 node tetrahedra (later converted to 10 node tetrahedra through midside node placement) using a modified 'advancing front' technique. Here, one works from a list of working triangular faces (initially the boundary meshes) creating tethedra from the faces: recursively an apex node is placed out from the face, with a check that no existing node can be used instead of creating a new one, completion of connectivity, and updating of the list of working (or unconnected) faces. It has been found that because intractable problems with degenerate formation, and the difficulties in dealing with these, that an initial 'invisible' suite of nodes has to be added to the solid. In severe cases of degeneracy, one of these is incorporated into the mesh, every element whose circumsphere encloses that node is removed, and a Delauney-type process is used to mesh the empty polyhedron.

2.2 Crack tip mesh

In order to avoid problems with tip meshing, transition elements, and poor quality tip elements, we have chosen to encase the crack front in a tube of singular elements centred on the front. This standard mesh of 8 degenerate 27 node Lagrangian brick elements (figure 1), considered as a topological object, travels with the front as it propagates, and is distorted from a perfect cylinder by a set of transformation relations. A regular tip mesh is thus easy to form. As an object, the tube in addition to the external and crack faces form the boundaries to the polyhedron to be meshed with tethedra.

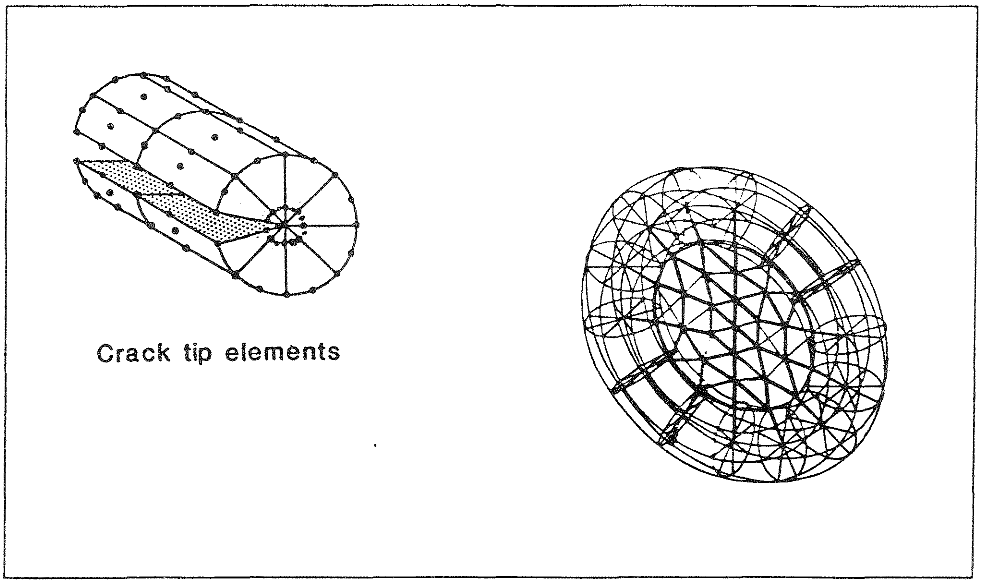

Figure 1: Crack tip elements - 'fracture tube'

2.3 Crack face tracking

The crack face mesh is left unaltered through the propagation phase, unless there is interference from an advancing adjacent crack or initiation from the face; even then the face mesh is stored as a topological description of the crack. The crack face representation therefore grows recursively as the crack propagates. That is, the algorithm for meshing the wake is repeated recursively to generate the face.

When the crack advances, the surface between the trailing nodes of the fracture tube and the two faces of the crack are separately meshed with a Delauney algorithm to ensure an optimal triangulation; this is essential since the mesh will not be altered. The algorithm is a simple implementation of the fundamental notion that the best triangles are formed by minimizing the length of possible diagonals (figure 2). Moreover, since we control the size of crack step, this can be related to the size of tube elements so that in most cases, the algorithm essentially meshes between two rows of nodes on a distorted surface.

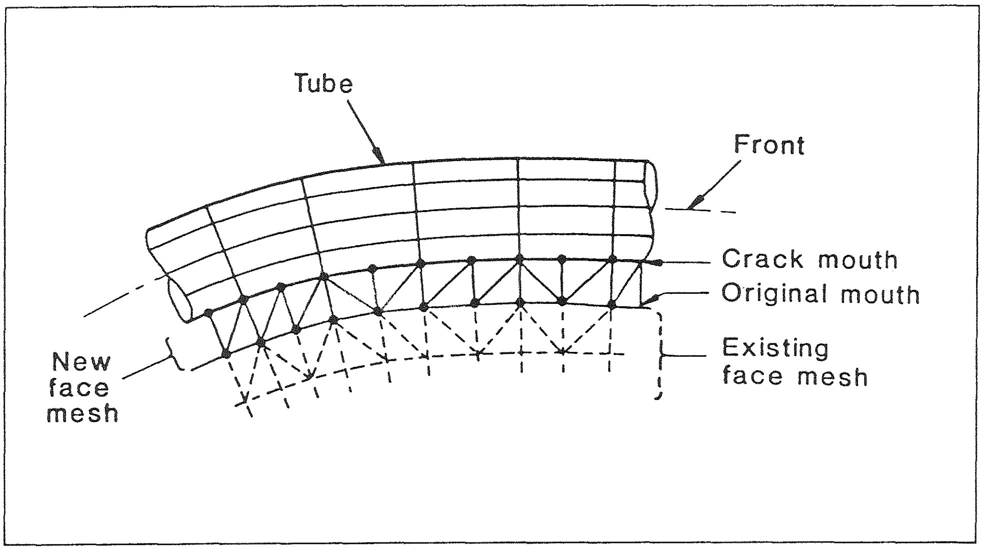

Figure 2: Crack face Delauney meshing

3 Crack tip elements

3.1 Form of singularity

Banks-Sills (1991) has shown that for a suite of 20 node quarter-point elements, the singularity along any ray from the tip varies like $O(1/\sqrt{r})$, but states that results for the 27 node (not collpased) parent should be best with side nodes moved to the 11/32 point. For the collapsed 27 Lagrangian element, we have found two forms of singularity that can be captured: the $O(1/\sqrt{r})$ singularity required for linear elastic fracture mechanics, and also an $O(1/r)$ form suitable for ductile fracture, through an alternate compatibility on the degenerate faces.

Firstly, when the degenerate face is collapsed, and the corresponding nodes on opposite edges are mapped into single nodes (with one set of displacements), it can be shown explicitly that the displacement from the degenerate edge also varies like $O(1/\sqrt{r})$, where r is the distance along a ray from the edge, at any angle within the element. However, for this to be true, it has be found that the mid-side nodes must be placed at the quarter-points, in contrast to the finding in Banks-Sills(1991).

If nodes on the degenerate face are given the same initial coordinates, but retain their identity, then the singularity is now $O(1/r)$. Both results can be shown explicitly by writing shape functions in a polar system (r,θ), centred on the degenerate face, and then evaluating the strains algebraically using the relevant compatibility conditions on nodal degrees of freedom.

3.2 Stress intensity calculations

We now demonstrate the accuracy of linear elastic stress intensity factor calculations in comparison with available benchmark computations for a variety of three dimensional cases. To do so, we adopt the usual displacement extraction techniques (Lim et al. (1992), Sousa et al.(1989)) to find local stress intensity factors for plane strain:

$$\begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} = \frac{E}{4(1-\nu^2)}\sqrt{\frac{\pi}{2r}}\begin{Bmatrix} \Delta v \\ \Delta u \\ (1-\nu)\Delta w \end{Bmatrix} \qquad (1)$$

where (Δu, Δv, Δw) are the crack opening, shearing and tearing displacements in local coordinates, a distance r from the tip.

A J-integral would be feasible since there is a natural path around our standard tip mesh. On the other hand, a stiffness derivative technique, despite its proven accuracy (Banks-Sills (1991)), would be too expensive to apply pointwise at each node on the front, particularly when the appropriate crack direction is not known until circumferential stress or SIFs are examined.

We now present results of stress intensity factors for the case of an inclined elliptical crack in a homogeneous cylinder (figure 3). These were found from (1) using quarter-point displcaments. In this case all three SIFs vary around the crack front, and so figure 4 shows the variation of SIF with angle from the global x-axis, for one quarter of the cylinder (since the problem is doubly symmetric). It can be seen that the SIFs all fluctuate slightly exactly as found in Banks-Sill(1991) as a consequence of numerical accuracy. It can also be seen that the mode I factor increases only very slightly over θ=0 to 90°. This too is reasonable since the ratio of major to minor axis lengths in the ellipse is such that the crack is almost circular in plan.

The mode II factor is seen to decrease from its maximum at the highest point on the crack front to almost zero at the sides: this too seems correct given that shearing displacements would be zero at these points. The mode II follows the opposite trend, as expected, with maximum tearing displacement at the sides, and zero at the high point.

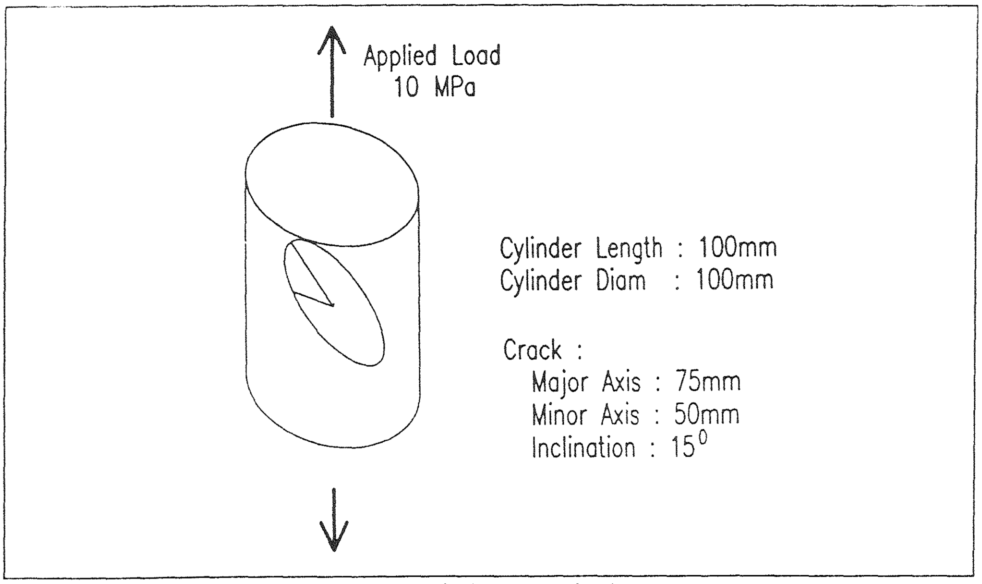

Figure 3: Inclined elliptical crack in a cylinder

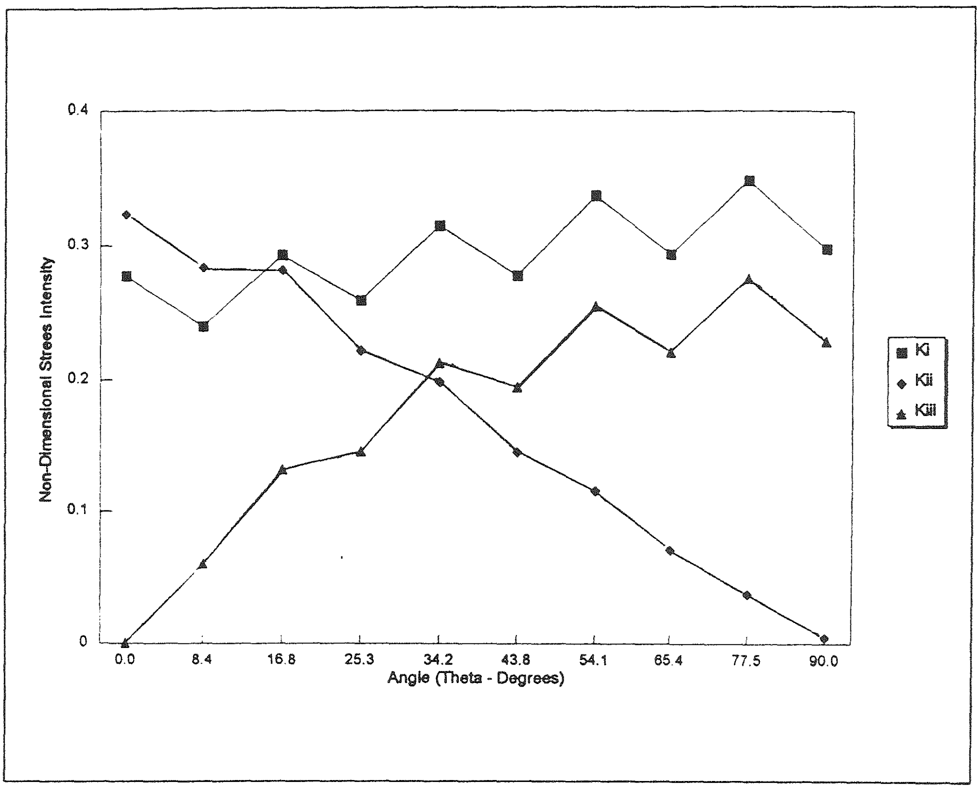

Figure 4: Variation in stress intensity factors

4 Crack advance

Crack advance and remeshing present three fundamental computational issues to be addressed: (i) crack criterion and a mechanism to allow growth; (ii) mesh smoothing and interpolation on the front; and, (iii) information mapping to a new mesh. Each of these is disussed in the following.

4.1 Propagation control

While there may be cases when cracks are essentially cohesive throughout their paths, there are many problems where cracks may initiate from a zone of damage, but propagate as linear elastic cracks when viewed at the structural scale. One case in point is the chip forming process in rock cutting, where elastic cracks propagate rapidly from a zone of very high compression under the cutting tool. Hence, we now discuss modes of control within the LEFM formulation discretized on a finite element mesh, but end the section with a discussion of the distributed to discrete representations.

Gerstle et al. (1988) use a crack criterion for fatigue situations. A natural crack length advance arises from the use of a fatigue power law, for example, which relates maximum and minimum SIFs to the rate change of crack length with cycles (see e.g. Suresh (1991)).

For sub-critical growth, where crack propagation is quite stable, a variety of growth mechanisms could be adopted. We suggest a simple adaptive pointwise trial step: if the new SIFs exceed toughness, we allow a crack advance of the order of the trailing mesh (figure 2) at the point of the largest K values. Other points on the front advance in proportion with their stress intensities. This is similar to the apporach of Xu & Ortiz (1993) who pose the problem in terms of crack velocity, allowing the fastest point to move furthest.

Choosing the direction of crack growth is also non-trivial. The principle hypothesis used Sousa et al. (1991) is that non-planar cracks advance in the direction given by the pointwise maximization of potential energy release rate along the crack front. This would seem the most rational criterion available, since it embodies the Irwin relationship between energy release rate and stress intensity. For the general case involving plane strain and anti-plane strain loading (i.e. non-planar):

$$G = \frac{(1-\nu^2)}{E}(K_I^2 + K_{II}^2) + \frac{(1+\nu)}{E}K_{III}^2 \qquad (2)$$

In practice, stress intensity relations are used to define the direction of growth. Difficulties can arise in cases of crack branching and/or kinking (Baker & Grummitt (1991): a check on circumferential stress is better.

Furthermore, when we do not have a sub-critical or fatigue regime, care must be taken that the computational mechanism does not lead to spurious results. We outline two possible approaches. One can either use a load parameter to control crack growth at constant toughness, using the trial step advance concept. Crack advance is of course a pointwise concept, and it is not clear what effect will arise using a constant length increment for a single load parameter, since a crack may naturally grow at different rates along the front. Alternatively, one might consider the length of a fictitious crack at constant energy release. That is, evaluate damage and then let crack advance by the equivalent length of the fictitous crack; notwithstanding there are considerable difficulties in information mapping.

4.2 Front node interpolation

In order to avoid excessive distortion as the front grows, and potentially curves back on itself, the front nodes are reinterpolated at each advance. Once new nodal positions are calculated, an interpolation function is determined for that front location; B-splines would appear to be reliable. New nodes are then placed uniformly along the interpolant, substituting for the old. Importantly, the process of generating the new tube, and mapping information on the tube is left until the front has been interpolated.

4.3 Information mapping

A signature of all remeshing schemes is the need to map information from the current mesh to the replacement discretization. Here we outline two generic approaches.

4.3.1 'Putting' information

Often schemes require that stress and strain data be mapped. The usual approach would be to store data for each Gauss point on the old mesh, remesh, then find where those points lie in the new mesh, and from the collection of such points in each new element, interpolate to the new Gauss stations. There are two problems: (i) the exercise in finding the local coordinates (in iso-parametric elements) in the new mesh for given global values from the old is time consuming and non-trivial; (ii) once the points have been located, the process of interpolating is liable to error since there is no à-priori gaurantee that the appropriate number of points will be located within each element on the new mesh to give a reasonable stress smoothing (figure 5).

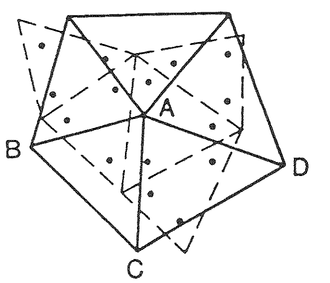

Figure 5: Interpolation from sampling points

4.3.2 'Getting' information

Here we begin with points on the new mesh and bring the relevant data to these. If stress data is required, we again store data against the old Gauss points and then remesh, but then we locate the new Gauss points on the new mesh, and find the simplex of nearest neighbours of the same order for each element. Data from the simplex (whether these belong to one element in the old mesh or not) is then interpolated to the new Gauss points. The interpolation is now done in global coordinates by extending the appropriate order of hyperplane through the simplex points (figure 6).

The task of identifying each simplex is not difficult (nearest neighbour search), but there may easily be cases where the geometric shape described by the simplex would be degenerate, or nearly so, leading to inaccurate interpolations.

An alternative is to begin with the nodal positions on the new mesh, and treat these as arbitrary points within elements on the old mesh, to which displacements can be mapped via the standard element approximations. That is, each new node is located within an element, and for our tetrahedra, Lagrangian coordinates are calculated. New nodal doisplacements are then computed from the element shape functions.

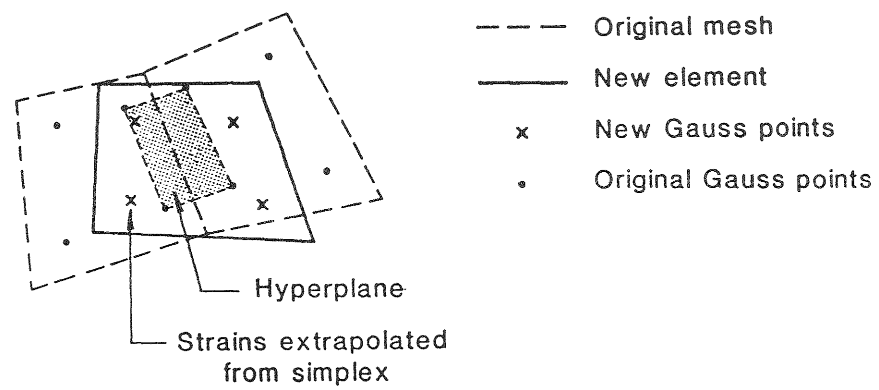

Figure 6: Stress hyperplane through Simplex

The parent element can be located either by testing whether the new node lies within the circumsphere of the element, or through a coarse search to find nearest apex nodes. There will be several possible candidates for the parent element. This is found by calculating volume coordinates, L_i, from the point for each possible element, j. For the point to lie inside element j, we require: $\Sigma L_i^j=1$.

The displacement interpolation then continues as follows. Consider a discretized domain, Ω_i in step i, consisting of an assembly of elementwise coordinates, X_i^e. Writing **a** for the assemblage of nodal displacements, **N** for the element shape functions, and $\forall\{.\}$ for an assembly over all elements, the mapping process above can be written:

$$\begin{aligned} a_i &= \forall\{N(X_i^e)\, a^e_{i-1}\} \\ &= A_{i-1}\, a_{i-1} \end{aligned} \tag{3}$$

Of course, there must be a control on the interpolated displacaments, and given the energy basis of FEA, we use a strain energy control. That is, we require an equivalence of total energies immediately after remeshing, before any further load increment:

$$\Pi(\Omega_i) \equiv \Pi(\Omega_{i-1}) \tag{4}$$

and since the boundary mesh remains unaltered, (4) requires an equivalence on strain energy, U. This can be achieved if we modify the displacements for new internal nodes in (3) by a factor:

$$a_i = \sqrt{s_i A_{i-1} a_{i-1}}; \qquad s_i = \frac{U(\Omega_{i-1})}{U(\Omega_i)} \tag{5}$$

If required, strains can then be found from the mapped displacement field.

4.3.3 Crack front mapping

One should also consider carefully the information that is required of an analysis since this can reduce the efforts significantly during remeshing. For example, if our only concern is crack growth, and a consequent study of structural stability, then we only require displacements in the tip mesh for the purposes of extracting stress intensity factors. That being so, we would simply map the displacement field on the fracture tube during the interpolation phase referred to above, for completeness. Since SIFs are found by relative displacements, mapping is not actually needed, unless we do use a circumferential stress check for crack direction.

The crack face mesh remains unaltered once formed, so that no mapping of face displacements is required.

4.3.4 Rezoning with Plasticity

Standard plasticity formulations assume that a plastic zone forms and spreads over an element. They require the storage of stress and strain information at the sampling stations, with numerical intergation over element domains. However, under remeshing, these stations would no longer correspond to the same locations within elements. One could 'rezone' the areas of plasticity with respect to the new element descriptions, with corresponding translation of stresses onto new sampling stations. In areas near crack fronts, there are likely to be significant element movements, and the process would need to be investigated thoroughly. It is feasible to limit the remeshing to crack front areas but this cannot exclude the possibility of plasticity in the remeshed areas.

An alternative being explored is to implement a version of the *plastic node method* (Yueda & Fujikubo (1991)), a technique which concentrates plasticity at 'checking' points (any point(s) within an element) leaving the remainder of an element elastic. Plastic deformations accumulate at the element nodes once a checking point yields. Incremental plasticity is defined in a similar fashion to the standard, except that one deals with a direct nodal force to nodal displacement 'elasto-plastic' stiffness relations where the yield condition is a function of nodal forces. The advantage is

Damage and/or smeared cracks would be allowed on this object. In the spirit of 'effective stress' concepts in continuum damage, a discrete crack would form when the secant modulus had softened to a given value[1].

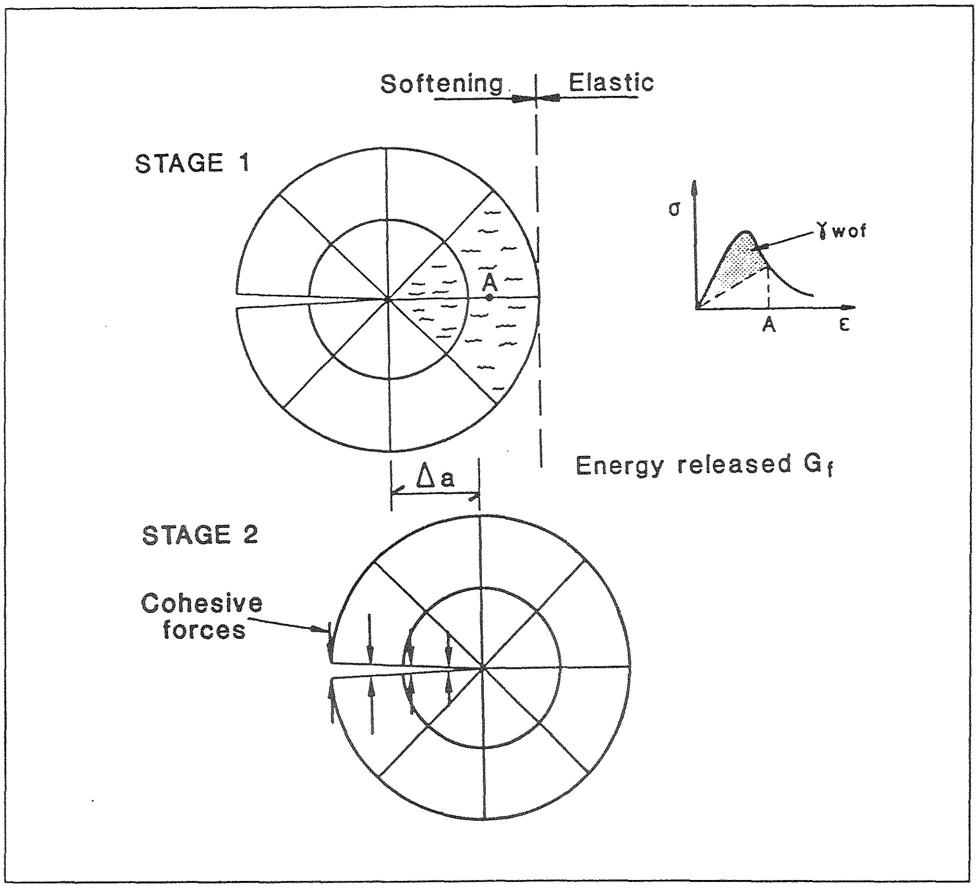

Figure 7: Distributed to discrete representations

It must be recognized that complete softening before crack advance is not realistic with remeshing, because of the demands on information mapping in the remeshed zones. That is, complete softening may have occurred near the tip, but only partial softening ahead. Thus when the discrete crack advances to the boundary of complete softening, the entire stress and strain field ahead would have to be mapped onto the new mesh. This is possible but the effort and inherent and accumulated errors may make the process unattractive. It may be better simply to allow softening

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[1] The basic idea was discussed at a project meeting with G.Baker, A.R.Ingraffea, W. van Kesteren, G.Kusters, W.Lim, J.Rots and P.Wawrzynek, at TNO, Delft, October 1994.

that no numerical integration is carried out over an element. In this context, then, checking points, with stored stress/strain information, can be independent of element domains, with an appropriate change to the elasto-plastic matrices under remeshing; this latter is the computational procedure to study, initially using a von Mises criterion, say. The plastic nodes still need to be located within new elements, but they do not themselves move with remeshing, and no interpolation of stress-strain data is required. Excessive element constraint in plasticity can be removed by increasing element order; 20 node tetrahedra are suitable for incompressible problems, whereas 10 node are marginal (Bell et al. (1993)).

4.4 Crack closure

Crack closure and penetration is detected by the normal element face intersection check carried out by the mesh generator. The penetration displacements, $\Delta\mathbf{u}^*$, could be determined and removed by applying contact forces, which will be found by influence functions for the pointwise mode I, II, and III displacements, iteratively until $\Delta\mathbf{u}^*=\mathbf{0}$.

Mode I displacements are simply found by raising a normal to an element face until it intersects the opposite face. The difficulty lies is estimating the shearing displacements correctly, since corresponding points on opposite crack faces may naturally undergo relative translations when the crack is open. Hence, one might propose to work only with mode I influence forces, but this ignores any friction contact that have occurred in the real specimen. The problem is that it has been shown that shear on an interface or joint (thence also a crack face) has a significant effect on crack growth and direction (Duncan-Fama & Stump (1994)). The effect of friction on the contact problem needs to be investigated carefully.

4.5 Distributed to discrete representations

Rots (1992) transformed smeared crack regions into discrete cracks by removing the damaged zone, though he noted that the process of removing elements can lead to numerical difficulties. Nonetheless, the general concept holds many advantages: (i) for cohesive materials, this provides the most rational mechanism for crack propagation, including cases of unstable growth and structural softening; (ii) the damage/smeared crack zone ahead of the tip gives rise to a natural criterion for crack direction; (iii) energy criteria fit well within the process. The problems in a three dimensional context mainly relate to information mapping.

For the sake of simple house-keeping, it would be sensible to examine damage within the fracture tube. Thence, we would define the tube object to consist of two (or three) layers to capture sufficient detail (figure 7).

to spread to the tube boundary, and then drive the crack through the partially softened zone at the appropriate energy release rate. The energy dissipated could be found by integrating across the tube domain (akin to Mazars & Pijaudier-Cabot (1994)). The residual energy in this domain, above the fracture energy, could be applied (i) as crack face closure forces, iteratively until the energy is consumed, or (ii) through the stiffness of a cohesive element.

The information mapped in the polyhedral zone outside the tube would consist then of elastic stresses only. The crack extension to tube boundaries according to propagation of softening provides a 'cohesive' control on advance along the front.

5 Concluding remarks

In this article, we have outlined the procedures and challenges in a finite element based computational tool for the simulation of crack growth in three-dimensional brittle solids. The main contribution to date lies in the development of a sound toplogical basis, and robust meshing algorithm for arbitrary domains. The task now is to validate the computational strategies in simulation of more complex non-planar crack growth.

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