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ANALYTICAL SOLUTION OF THE FICTITIOUS CRACK AND EVALUATION OF THE CRACK EXTENSION RESISTANCE FOR A GRIFFITH CRACK

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Abstract
A new approach to determine the crack extension resistance curve had been developed by the authors recently. As a follow-up of the conventional approach by Irwin, the new proposed K_R-curve is directly evaluated according to the cohesive force on the fictitious crack in front of a preformed crack. It issuitable for the crack propagation in quasi-brittle materials with softening stress-separation law like concrete. As one of the sequels of the work, the case of mode I Griffith crack was investigated to derive the analytical expression of the K_R-curve. For achieving the aim, the cohesive force distribution on the fictitious crack at various loading stages is assumed using the softening stress-separation law based on direct tensile tests of concrete. Then, the analytical solution of the fictitious crack model for the mode I Griffith crack is gained. Mathematical expressions of the K_R-curve for mode I Griffith crack are presented. Once the σ_∞-CTOD curve of a material is known, the K_R-curve for mode I Griffith crack can be determined.
Key words: analytical solutions, fictitious crack, K_R-curve, cohesive force, mode I Griffith crack

1 Introduction

During the last two decades, the fictitious crack model proposed by Hillerborg et al. (1976) had been successfully used in the numerical analysis of concrete structures. Later, some researchers attempted to analyse the fictitious crack model, like Reinhardt (1984), Li and Liang (1986), Horii et al. (1987), Li (1990) Planas and Elices (1992, 1993) as well as Lin et al. (1994) for the Griffith crack case. However, it was found that the case studied by Planas and Elices (1992, 1993) and Lin et al. (1994) could be taken as a special case during the post-critical situations.

Furthermore, the conventional approaches, originating from Irwin in the 1950s, to evaluate crack extension resistance in terms of K_R-curve according to the length of a propagating crack and the corresponding load have been applied to concrete. This approach is phenomenological. Opposite to the conventional phenomenological approach proposed by Irwin, the new approach considers the cohesive froces along the fictitious crack. This approach was recently developed by Xu and Reinhardt (1997) for standard three-point bending notched beams. This work is now being extended to the mode I Griffith crack for quasi-brittle materials with strain softening behaviour.

2 Fictitious crack model and softening traction-separation law

A fictitious crack according to Hillerborg et al. (1976) is composed of two parts: one part is the real crack of which the two crack faces are stress free and wholly separated; the other part is the fictitious crack. The fictitious crack is a conceptual crack as mathematical treatment. For simplicity, the original width of the fictitious crack was assumed to be zero.

It was assumed that the fictitious crack can transfer distributed cohesive forces σ resulting from friction and interlock. The cohesive forces are a function of the fictitious crack separation width w. At the fictitious crack tip, the transferred cohesive force is assumed to be the tensile strength, f_t, of concrete. Behind the fictitious crack tip, the transferred cohesive force is assumed to decrease with the increase of the crack width. With the increase of applied load, the fictitious crack will develop continuously. Once the fictitious crack has fully developed, the cohesive force along the trajectory of the developing fictitious crack will be zero. The distribution of the cohesive forces is sketched in Fig. 1 (a) and (b) before and after crack extension.

Outside the fictitious crack the material behaviour is described by a linear stress-strain relation. Within the fictitious crack a bilinear stress distribution is utilized both in numerical studies by finite element code and in analytical investigations. The bilinear traction-separation law is illustrated in Fig. 2.

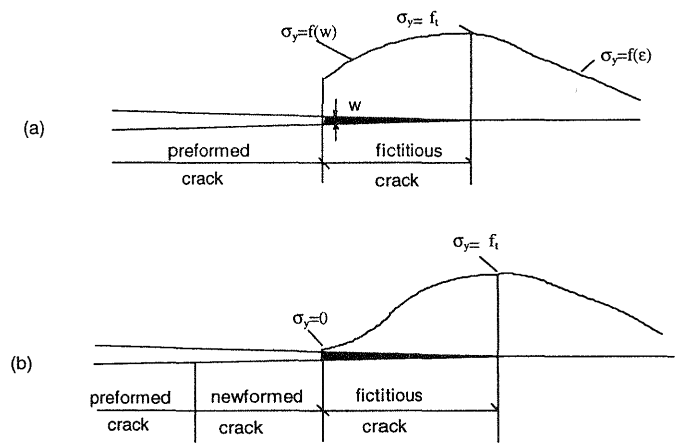

Fig. 1. Cohesive force distribution behind the fictitious crack tip before (a) and after (b) a real crack extension

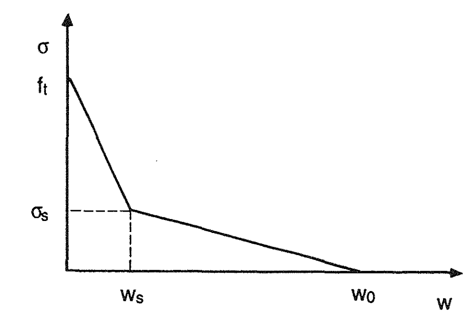

Fig. 2. Illustration of the bilinear softening traction-separation law

The area under the σ-w curve in Fig. 2 is defined as fracture energy G_F of concrete material by Hillerborg et al. (1976). It can be expressed as follows:

$$G_F = (f_t w_s + \sigma_s w_0)/2 \qquad (1)$$

The bilinear softening traction-separation law can be generalized as follows:

$$\begin{aligned} &\sigma = f_t - (f_t - \sigma_s) w / w_s && 0 \leq w \leq w_s \\ &\sigma = \sigma_s (w_0 - w)/(w_0 - w_s) && w_s \leq w \leq w_0 \\ &\sigma = 0 && w \geq w_0 \end{aligned} \qquad (2)$$

Several researchers proposed different values of the break point σ_s and w_s and the crack width w_0 at the stress-free point (Petersson, 1986).

For convenience of analytical analysis of the fictitious crack model, we propose a modified approach to determine the values of σ_s , w_s and w_0 based on investigations in extensive experiments which is stated as following expressions:

$$\begin{aligned} &\sigma_s = f_t (2 - f_t CTOD_c / G_F) / \alpha_F \\ &w_s = CTOD_c \\ &w_0 = \alpha_F G_F / f_t \\ &\alpha_F = \lambda - d_{\max} / 8 \end{aligned} \qquad (3)$$

where $CTOD_c$ is the critical crack tip opening displacement of concrete; d_{max} is the maximum aggregate size in mm and λ is a calibration factor which depends on the deformation property of concrete. The values of λ are 4 to 10. When $\lambda = 9$, α_F leads to the same values as proposed in CEB-FIP Model Code 1990.

3 The analytical solution of the fictitious crack model in case of a Griffith crack

A typical Griffith crack with symmetric fictitious crack zones at the two ends is shown in Fig. 3 (a). In such a crack problem, an infinite plate of unit thickness with a mode I central transverse crack with a preformed length $2a_0$

is subjected to an externally applied tensile stress σ_∞ at the remote boundary and to distributed cohesive forces along the fictitious crack zones. An effective crack "a" consists of an equivalent-elastic stress-free crack and an equivalent-elastic fictitious crack extension. According to the superposition principle, the crack problem shown in Fig. 3 (a) can be taken as a superposition of the two cases shown in Fig. 3 (b) and (c). The stress intensity factor at the effective crack tip shown in Fig. 3 (a) is equal to the superposition of those in Fig. 3 (b) and (c).

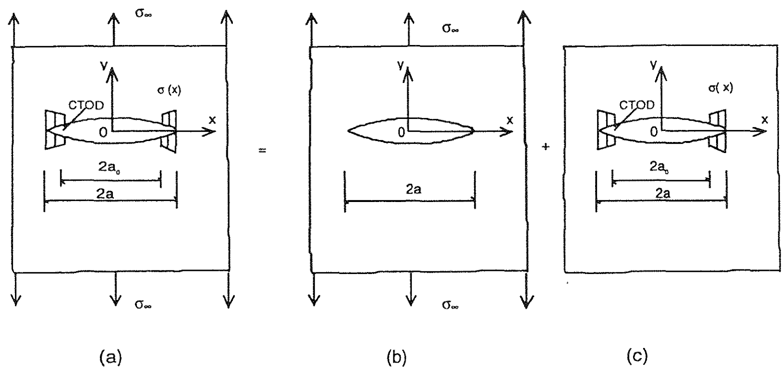

Fig. 3. A Griffith crack with symmetric fictitious crack zones and its superposition

If the stress intensity factor at the effective crack tip due to the external tensile force σ_∞ is denoted by K_I^e in the case shown in Fig. 3 (b) and that due to the distributed cohesive force σ (x) along the fictitious crack zones by K_I^c in Fig. 3 (c), the overall stress intensity factor in the case shown in Fig. 3 (a) can be stated as following equation (see Tada, Paris and Irwin (1985)):

$$K_I = K_I^e + K_I^c$$
$$= \sigma_\infty \sqrt{2\pi a} - \int_{a_0}^{a} 2\sqrt{\frac{a}{\pi}} \frac{\sigma(x)dx}{\sqrt{a^2 - x^2}} \tag{4}$$

During the complete fracture process the cohesive force distribution $\sigma(x)$ is different accordingly to the various loading stages. It starts with crack a_0

only, then the crack extends that $CTOD_c$ and $\sigma_{\infty,\ max}$ are reached at the same time, finally the crack propagates further. The distribution of the cohesive forces are assumed according to Figs. 4 to 7. The subsequent steps are described by:

a) loading stage 1: CTOD = 0.

$$K_I = \sigma_\infty \sqrt{2\pi\, a_0} \tag{5}$$

b) loading stage 2: CTOD ≤ $CTOD_c$.

$$\sigma(x) = \sigma(CTOD) + [f_t - \sigma(CTOD)](x - a_0)/(a - a_0) \qquad a_0 \le x \le a \tag{6}$$

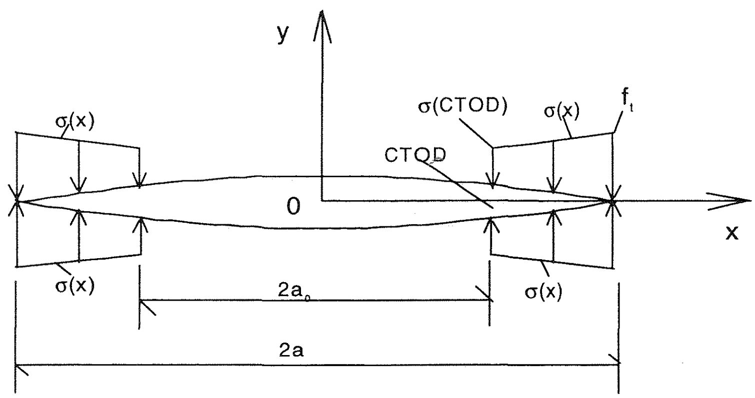

Fig. 4. The cohesive force distribution on the fictitious crack zones when CTOD < $CTOC_c$

At the critical state when $\sigma_\infty = \sigma_{\infty,max}$ and CTOD = $CTOD_c$, the corresponding solution is the following:

$$K_I = \sigma_{\infty,max}\sqrt{2\pi\, a_c} - \frac{2}{1-a_0/a_c}\sqrt{\frac{a_c}{\pi}}\left\{\left[\sigma_s(CTOD_c) - \frac{a_0}{a_c} f_t\right]\arccos\frac{a_o}{a_c} + [f_t - \sigma_s(CTOD_c)]\sqrt{1-(\frac{a_0}{a_c})^2}\right\} \tag{7}$$

c) loading stage 3: $CTOD_c < CTOD < w_0$.

$$\sigma(x) = \begin{cases} \sigma_1(x) = \sigma(CTOD) + \left[\sigma_s(CTOD_c) - \sigma(CTOD)\right]\dfrac{x-a_0}{a_s-a_0} & a_0 \le x \le a_s \\ \sigma_2(x) = \sigma_s(CTOD_c) + \left[f_t - \sigma_s(CTOD_c)\right]\dfrac{x-a_s}{a-a_s} & a_s \le x \le a \end{cases} \quad (8)$$

$$\begin{aligned} K_I = {} & \sigma_\infty\sqrt{2\pi a} \\ & - \frac{2}{1-a_o/a_s}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma(CTOD) - \frac{a_0}{a_s}\sigma_s(CTOD_c)\right]\left[\arccos\frac{a_0}{a} - \arccos\frac{a_s}{a}\right]\right. \\ & \left. - a\left[\sigma_s(CTOD_c) - \sigma(CTOD)\right]\left[\sqrt{1-(a_s/a)^2} - \sqrt{1-(a_0/a)^2}\right]\right\} \\ & - \frac{2}{1-a_s/a}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma_s(CTOD_c) - \frac{a_s}{a}f_t\right]\arccos\frac{a_s}{a} + \left[f_t - \sigma_s(CTOD_c)\right]\sqrt{1-(a_s/a)^2}\right\} \end{aligned} \quad (9)$$

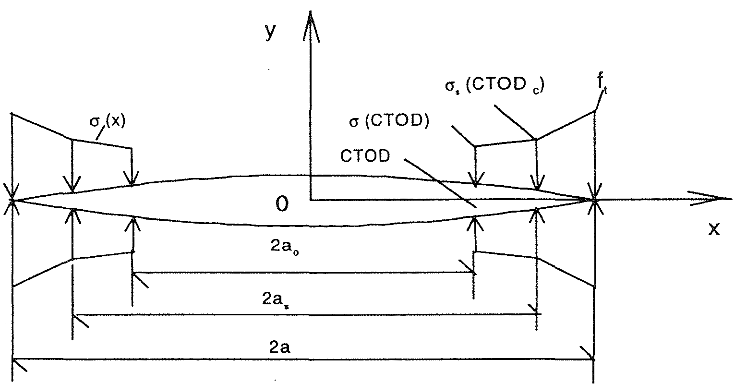

Fig. 5. The bilinear cohesive force distribution on the fictitious crack when the $CTOD_c < CTOD < w_0$

d) a special state: $CTOD = w_0$

$$\sigma(x) = \begin{cases} \sigma_1(x) = \sigma_s(CTOD_c)\dfrac{x-a_0}{a_s-a_0} & a_o \le x \le a_s \\ \sigma_2(x) = \sigma_s(CTOD_c) + \left[f_t - \sigma_s(CTOD_c)\right]\dfrac{x-a_s}{a_{w0}-a_s} & a_s \le x \le a_{w0} \end{cases} \quad (10)$$

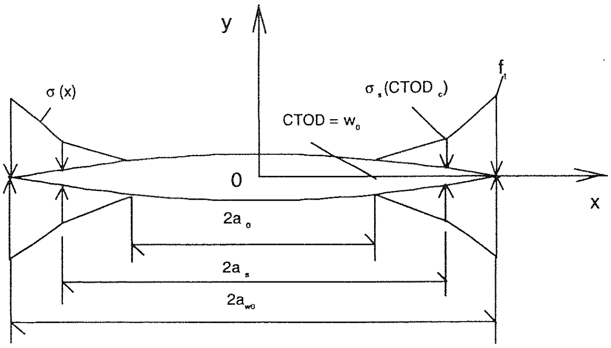

Fig. 6. The fully bilinear cohesive force distribution on the fictitious crack when CTOD = w_0

$$K_I = \sigma_\infty \sqrt{2\pi a_{w_0}} - \frac{2\sigma_s(CTOD_c)}{a_s - a_o}\sqrt{\frac{a_{w_0}}{\pi}}\left\{a_o\left[\arccos\frac{a_s}{a_{w_0}} - \arccos\frac{a_o}{a_{w_0}}\right] - a_{w_0}\left[\sqrt{1-(a_s/a_{w_0})^2} - \sqrt{1-(a_0/a_{w_0})^2}\right]\right\} \quad (11)$$
$$- \frac{2}{1 - a_s/a_{w_0}}\sqrt{\frac{a_{w_0}}{\pi}}\left\{\left[\sigma_s(CTOD_c) - \frac{a_s}{a_{w_0}} f_t\right]\arccos\frac{a_s}{a_{w_0}} + \left[f_t - \sigma_s(CTOD_c)\right]\sqrt{1-(a_s/a_{w_0})^2}\right\}$$

e) loading stage 4: CTOD > w_0.

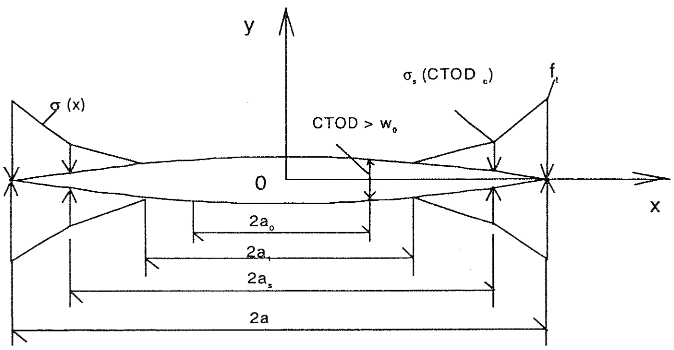

Fig. 7. After new formed stress free crack appeared the bilinear cohesive force distribution on the fictitious crack when CTOD > w_0

$$\sigma(x) = \begin{cases} \sigma_1(x) = \sigma_s(CTOD_c)\frac{x-a_1}{a_s-a_1} & a_1 \le x \le a_s \\ \sigma_2(x) = \sigma_s(CTOD_c) + [f_t - \sigma_s(CTOD_c)]\frac{x-a_s}{a-a_s} & a_s \le x \le a \end{cases} \quad (12)$$

$$K_I = \sigma_\infty\sqrt{2\pi a} - \frac{2\sigma_s(CTOD_c)}{a_s - a_1}\sqrt{\frac{a}{\pi}}\left\{a_1\left[\arccos\frac{a_s}{a} - \arccos\frac{a_1}{a}\right] - a\left[\sqrt{1-(a_s/a)^2} - \sqrt{1-(a_1/a)^2}\right]\right\} - \frac{2}{1-a_s/a}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma_s(CTOD_c) - \frac{a_s}{a}f_t\right]\arccos\frac{a_s}{a} + [f_t - \sigma_s(CTOD_c)]\sqrt{1-(a_s/a)^2}\right\} \quad (13)$$

4 Crack extension resistance associated with the cohesive force for the Griffith crack in concretes

The conventional approach of crack extension resistance curve was proposed by Irwin in the 1950s and early 1960s which can be expressed in terms of the K_R-curve:

$$K = K_R(\Delta a) \quad (14)$$

Where K is the stress intensity factor at the propagating crack tip and $K_R(\Delta a)$ is the crack extension resistance during crack propagation which is a function of the extension of crack propagation $\Delta a = a - a_0$. In the conventional approach to determine the crack extension resistance, $K_R(\Delta a)$ is evaluated by the external applied load and the corresponding length of propagating crack measured in tests. This approach was widely applied in practice.

Later, through highly refined finite element calculations, Tvergaard and Hutchinson (1992) studied the crack extension resistance of strain hardening metal materials for mode I Griffith crack problem using a traction-separation law.

Recently, a new approach to evaluate the crack extension resistance of quasi-brittle softening materials was proposed by Xu and Reinhardt (1997) according to the cohesive force on the fictitious crack which is directly described by the softening traction-separation law. The basic principle of the new approach is that the crack extension resistance is composed of two parts. One part is the inherent toughness of a material denoted with symbol K_{Ic}^{ini} which resists the initial propagation of an initial mode I crack. Another part is caused by the cohesive force distributed on the fictitious crack during crack propagation. The crack extension resistance determined by the approach is defined as follows:

$$K_R(\Delta a) = K_{Ic}^{ini} + K^c(f_t, f(\sigma), a) \tag{15}$$

f_t is the tensile strength of a material, a is length of a propagating crack in a loaded body and f (σ) is the distribution function of the cohesive force along the fictitious crack which can be determined by the traction-separation law presented by eqs. (2) and (3).

For the distinguished loading stages for the mode I Griffith crack problem considered in this paper, the expressions of the distribution function of cohesive force f (σ) have been stated in equations (6), (8), (10) and (12). Using the analytical solution of the fictitious crack model for the mode I Griffith crack problem presented in the above section, the mathematical expressions of the crack extension resistance curve associated with cohesive force on the fictitious crack can be stated as follows:

a) loading stage 1: CTOD =0, i.e. $a = a_0$.

$$K_R(\Delta a) = K_{Ic}^{ini} \tag{16}$$

b) loading stage 2: 0< CTOD ≤ $CTOD_c$, i.e. $a_0 < a \leq a_c$

$$K_R(\Delta a) = K_{Ic}^{ini} + \frac{2}{1-\frac{a_o}{a}}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma(CTOD) - \frac{a_0}{a}f_t\right]\arccos\frac{a_o}{a} + \left[f_t - \sigma(CTOD)\right]\sqrt{1-(\frac{a_0}{a})^2}\right\} \tag{17}$$

c) loading stage 3: $CTOD_c < CTOD \le w_0$, i.e. $a_c < a \le a_{w_0}$

$$
\begin{aligned}
K_R(\Delta a) &= K_{Ic}^{ini} \\
&+ \frac{2}{1-a_0/a_s}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma(CTOD) - \frac{a_0}{a_s}\sigma_s(CTOD_c)\right]\left[\arccos\frac{a_0}{a} - \arccos\frac{a_s}{a}\right]\right. \\
&\left. - a\left[\sigma_s(CTOD_c) - \sigma(CTOD)\right]\left[\sqrt{1-(\frac{a_s}{a})^2} - \sqrt{1-(\frac{a_0}{a})^2}\right]\right\} \\
&+ \frac{2}{1-a_s/a}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma_s(CTOD_c) - \frac{a_s}{a}f_t\right]\arccos\frac{a_s}{a}\right. \\
&\left. + \left[f_t - \sigma_s(CTOD_c)\right]\sqrt{1-(\frac{a_s}{a})^2}\right\}
\end{aligned}
\qquad (19)
$$

d) loading stage 4: $CTOD > w_0$, i.e. $a > a_{w_0}$

$$
\begin{aligned}
K_R(\Delta a) &= K_{Ic}^{ini} \\
&+ \frac{2\sigma_s(CTOD_c)}{a_s - a_1}\sqrt{\frac{a}{\pi}}\left\{a_1\left[\arccos\frac{a_s}{a} - \arccos\frac{a_1}{a}\right] - a\left[\sqrt{1-(a_s/a)^2} - \sqrt{1-(a_1/a)^2}\right]\right\} \\
&+ \frac{2}{1-a_s/a}\sqrt{\frac{a}{\pi}}\left\{\left[\sigma_s(CTOD_c) - \frac{a_s}{a}f_t\right]\arccos\frac{a_s}{a} + \left[f_t - \sigma_s(CTOD_c)\right]\sqrt{1-(a_s/a)^2}\right\}
\end{aligned}
\qquad (20)
$$

where the crack length is incrementally increased and cohesive force, stress intensity factor, and crack extension resistance can be computed.

5 Conclusions

The fictitious crack model has been applied to the mode I Griffith crack. Analytical expressions for the K_R-curve during the complete fracture process have been derived which take account of the cohesive force along the fictitous crack. The cohesive forces follow the stress-crack opening relation of a softening quasi-brittle material. The presented formulae are the basis for an iterative computation which is in progress.

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