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A CLASS OF GRADIENT-DEPENDENT DAMAGE MODELS FOR CONCRETE CRACKING

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Abstract

Isotropic and anisotropic damage formulations for concrete fracture are reviewed, including the classical fixed and rotating smeared crack models and more refined approaches based on the microplane concept. Higher-order gradients are introduced to avoid the boundary value problem becoming ill-posed at the onset of softening. For an infinite one-dimensional bar dispersion analyses are carried out to examine the effect of using gradients of different strain measures or internal variables.
Key words: Smeared-crack models, microplane models, localisation, dispersion analysis, gradients.

1 Introduction

Smeared crack models have proven to be flexible in the sense that, in principle, arbitrary crack propagation can be simulated, since no topological constraints exists. Nevertheless, experience has shown that this advantage is less rigorous than it would seem, since it appeared rather difficult to simulate curved crack paths by smeared representations (de Borst 1986, Rots 1991, Feenstra 1993, Feenstra and de Borst 1995). In the early 1990s it became apparent that the failure of the smeared approach to properly predict curved crack paths is rooted in the fact that a smeared concept inevitably introduces *strain* softening into the constitutive model, which at a certain level of loading causes a loss of well-posedness of the incremental boundary value problem. This ill-posedness creates an infinite number of solutions (Benallal *et al.* 1988, de Borst *et al.* 1993), from which a numerical method selects the solution with the smallest energy dissipation that is available in the finite dimensional solution space. In the limit of an infinitely dense mesh, solutions are computed which predict failure without energy dissipation, thus rendering the solution physically meaningless. Early solutions as fracture-energy models in their various forms (Pietruszczak and Mróz 1981, Bažant and Oh 1983), provide a partial solution for the mesh densification problem, but fail to repair the mesh bias issue, i.e. they still predict crack propagation along the direction of the grid lines.

A rigorous solution is the introduction of higher-order continuum models. The first models that were applied to fracture in concrete were nonlocal damage models (Pijaudier-Cabot and Bažant 1987, Bažant and Pijaudier-Cabot 1988) and gradient plasticity models (de Borst and Mühlhaus 1992, de Borst and Pamin 1996). Fully nonlocal approaches, in which spatially averaged quantities are employed in the constitutive models, are computationally unwieldy and are not believed to have potential for large-scale computations of concrete structures. The gradient approaches are more promising, but the gradient plasticity model suffers from the drawback that there is an internal boundary between the elastic and plastic domain, which necessitates a smooth interpolation of the plastic strain field. The required C^1-continuity is believed to reduce the ability of the gradient plasticity model to simulate curved crack propagation accurately, although globally proper directions of crack propagation were computed, independent of the discretisation. Gradient damage approaches, first introduced in a computationally feasible format by Peerlings *et al.* (1996a), do not necessarily require a higher-order continuity of the interpolants of the strain field or the damage field and, as was recently shown by Peerlings *et al.* (1998), are capable of simulating curved cracks.

This contribution will review recent developments of gradient damage approaches. We shall start by local, isotropic damage formulations, and then extend the formulation to anisotropic models, including various forms of smeared crack concepts, such as the fixed crack model and the rotating crack model (Cope *et al.* 1980), and the microplane models (Bažant and Gambarova, 1984). Then, we will develop isotropic and anisotropic gradient damage models. Specifically, we will derive a gradient smeared crack model, and we will indicate how gradient microplane models can be developed (Kuhl *et al.* 1998). Some theoretical considerations using dispersion analyses for one-dimensional infinite media follow to bring out some of the salient differences between the various types of gradient enhancements.

2 Standard damage models

The basic structure of constitutive models that are set up in the spirit of damage mechanics is simple. We have a total stress-strain relation, which for the case of isotropic damage evolution, specialises as

$$\sigma_{ij}=[(1-\omega_1)G(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-{}^2\!/\!_3\delta_{ij}\delta_{kl})+(1-\omega_2)K\delta_{ij}\delta_{kl}]\varepsilon_{kl} \tag{1}$$

with G the virgin shear modulus and K the virgin bulk modulus, which are degraded by the scalar damage variables ω_1 and ω_2, respectively. A simplification can be achieved if it is assumed that the secant shear stiffness and bulk moduli, $(1-\omega_1)G$ and $(1-\omega_2)K$, degrade in the same manner during damage growth. Essentially, this means that Poisson's ratio ν remains constant throughout the damage process and we have

$$\sigma_{ij}=(1-\omega)D^{\mathrm{e}}_{ijkl}\varepsilon_{kl} \tag{2}$$

with ω the damage variable which grows from zero to one (at complete loss of integrity) and D^{e}_{ijkl} the fourth-order elastic stiffness tensor. The total stress-strain relation (2) is complemented by a damage loading function $f=f(\tilde{\varepsilon},\tilde{\sigma},\kappa)$, with $\tilde{\varepsilon}$ and $\tilde{\sigma}$ scalar-valued functions of the strain and stress tensors, respectively, and κ a scalar-valued history variable. The damage loading function f and the rate of the history variable, $\dot{\kappa}$, have to satisfy the discrete Kuhn-Tucker loading-unloading conditions

$$f\leq 0 \quad , \quad \dot{\kappa}\geq 0 \quad , \quad f\dot{\kappa}=0 \tag{3}$$

We first consider the case that the damage loading function does not depend on $\tilde{\sigma}$. Then,

$$f(\tilde{\varepsilon},\kappa)=\tilde{\varepsilon}-\kappa \tag{4}$$

A definition for $\tilde{\varepsilon}$ that is often used for concrete has been proposed by Mazars (1984), see also Mazars and Pijaudier-Cabot (1989):

$$\tilde{\varepsilon}=\sqrt{\sum_{i=1}^{3}(<\varepsilon_i>)^2} \tag{5}$$

with ε_i the principal strains, and $<\varepsilon_i>=\varepsilon_i$ if $\varepsilon_i>0$ and $<\varepsilon_i>=0$ otherwise. This definition has only a limited capability to represent the difference in magnitude between the compressive strength f_{cc} and the tensile strength f_{ct}, which is so characteristic of concrete. Indeed, Peerlings *et al.* (1998) have found that another definition for $\tilde{\varepsilon}$,

$$\tilde{\varepsilon}=\frac{k-1}{2k(1-\nu)}I_1+\frac{1}{2k}\sqrt{\frac{(k-1)^2}{(1-2\nu)^2}I_1^2+\frac{6k}{(1+\nu)^2}J_2} \tag{6}$$

originally proposed by de Vree *et al.* (1995) in the context of polymers, is more appropriate when analysing curved crack propagation in notched specimens. In eq. (6) $I_1 = \varepsilon_{kk}$ is the first invariant of the strain tensor and $J_2 = e_{ij}e_{ij}$ is the second invariant of the deviatoric strain tensor $e_{ij} = \varepsilon_{ij} - {}^1\!/_3\varepsilon_{kk}$. The parameter k governs the sensitivity to the compressive strain components relative to the tensile strain components. The definition of $\tilde{\varepsilon}$ is such that a compressive uniaxial stress $k\sigma$ has the same effect as a uniaxial tensile stress σ. The parameter k is therefore typically set equal to the ratio of the compressive uniaxial strength and the tensile uniaxial strength: $k = f_{cc}/f_{ct}$.

The history parameter κ starts at a damage threshold level κ_i and is updated by the requirement that during damage growth $f = 0$. Damage growth occurs according to an evolution law such that $\omega = \omega(\kappa)$ which can be determined from a uniaxial test. For instance, the evolution relation,

$$\omega(\kappa) = \frac{\kappa_u(\kappa - \kappa_i)}{\kappa(\kappa_u - \kappa_i)} \tag{7}$$

where the tensile strength $f_{ct} = E\kappa_i$, E being Young's modulus, is followed by a linear descending branch up to an ultimate value κ_u, where the load-carrying capacity is exhausted (and thus $\omega = 1$), has been used in the dispersion analyses at the end of this paper.

Stress-based isotropic damage formulations can be elaborated by omitting $\tilde{\varepsilon}$ from the damage loading function f. Then, the following damage relation ensues

$$f(\tilde{\sigma}, \kappa) = \tilde{\sigma} - \kappa \tag{8}$$

still subjected to the Kuhn-Tucker loading-unloading conditions and equipped with an evolution relation for the damage variable $\omega = \omega(\kappa)$. Stress-based damage formulations bear some resemblance to plasticity approaches, with the notable difference that in plasticity the elastic stiffness moduli remain unchanged, while in an isotropic damage formalism they degrade in an isotropic fashion. However, for monotonic loading conditions and uniaxial stressing both models can be made identical. In a first step a damage strain ε_{ij}^{d} can be defined:

$$\varepsilon_{ij}^{d} = \omega\varepsilon_{ij} \tag{9}$$

so that eq. (2) changes into

$$\sigma_{ij} = D_{ijkl}^{e}(\varepsilon_{kl} - \varepsilon_{kl}^{d}) \tag{10}$$

In a fashion similar to the total equivalent strain $\tilde{\varepsilon}$, scalar measures for the equivalent stress, $\tilde{\sigma}$, and for the damage strain, $\tilde{\varepsilon}^{d}$ can be defined. In fact, if the same definition is applied for $\tilde{\sigma}$, $\tilde{\varepsilon}$ and $\tilde{\varepsilon}^{d}$, these quantities can be uniquely related for given hardening/softening characteristics. Accordingly, the loading function (8) can also be cast in the following format:

$$f(\tilde{\varepsilon}^{d}, \kappa) = \tilde{\varepsilon}^{d} - \kappa \tag{11}$$

from which it becomes apparent that now $\kappa = \kappa(\tilde{\varepsilon}^{\mathrm{d}})$, quite similar to plasticity approaches. In the part where we shall compare the properties of the different damage approaches, this identity will be taken as point of departure for the evolution of the internal variable κ. More precisely, we shall use

$$f(\tilde{\sigma}, \kappa) = \tilde{\sigma} - \kappa(\tilde{\varepsilon}^{\mathrm{d}}) \tag{12}$$

3 Anisotropic damage models

While isotropic damage models have been used successfully for describing progressive crack propagation, their disadvantage is that possible compressive strut action is eliminated. This is a drawback especially for the analysis of reinforced concrete members. Directional dependence of damage evolution can be incorporated by degrading the Young's modulus E in a preferential direction. When, for plane-stress conditions, distinction is made between the global x, y-coordinate system and a local n, s-coordinate system a simple loading function in the local coordinate system would be

$$f(\varepsilon_{nn}, \kappa) = \varepsilon_{nn} - \kappa \tag{13}$$

with ε_{nn} the normal strain in the local n, s-coordinate system, subject to the standard Kuhn-Tucker loading-unloading conditions. The secant stiffness relation now reads

$$\boldsymbol{\sigma}_{\mathrm{ns}} = \mathbf{D}^{\mathrm{s}}_{\mathrm{ns}} \boldsymbol{\varepsilon}_{\mathrm{ns}} , \tag{14}$$

with $\boldsymbol{\sigma}_{\mathrm{ns}} = [\sigma_{nn}, \sigma_{ss}, \sigma_{ns}]^{\mathrm{T}}$, $\boldsymbol{\varepsilon}_{\mathrm{ns}} = [\varepsilon_{nn}, \varepsilon_{ss}, \gamma_{ns}]^{\mathrm{T}}$ and $\mathbf{D}^{\mathrm{s}}_{\mathrm{ns}}$ defined as

$$\mathbf{D}^{\mathrm{s}}_{\mathrm{ns}} = \begin{bmatrix} (1-\omega_1)E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & (1-\omega_2)G \end{bmatrix} \tag{15}$$

with $\omega_1 = \omega_1(\kappa)$ and $\omega_2 = \omega_2(\kappa)$. The factor $1 - \omega_2$ represents the degradation of the shear stiffness and can be identified with the traditional shear retention factor β (Suidan and Schnobrich, 1973). It is emphasised that because of the choice of a preferential direction in which damage takes place, the damage variables ω_1 and ω_2 have an entirely different meaning than those that were introduced in the isotropic formulation of eq. (1).

If we introduce ϕ as the angle from the x-axis to the n-axis, we can relate the components of $\boldsymbol{\varepsilon}_{\mathrm{ns}}$ and $\boldsymbol{\sigma}_{\mathrm{ns}}$ to those in the global x, y-coordinate system via the standard tranformation matrices $\mathbf{T}_{\varepsilon}$ and $\mathbf{T}_{\sigma}$:

$$\boldsymbol{\varepsilon}_{\mathrm{ns}} = \mathbf{T}_{\varepsilon}(\phi)\boldsymbol{\varepsilon}_{\mathrm{xy}} \quad \text{and} \quad \boldsymbol{\sigma}_{\mathrm{ns}} = \mathbf{T}_{\sigma}(\phi)\boldsymbol{\sigma}_{\mathrm{xy}} \tag{16}$$

With aid of eqs (16) the damage loading function (13) can be written in terms of the strain components ε_{xx}, ε_{yy} and γ_{xy} of the global x, y-coordinate system:

$$f = \varepsilon_{xx} \cos^2 \phi + \varepsilon_{yy} \sin^2 \phi + \gamma_{xy} \sin \phi \cos \phi - \kappa \tag{17}$$

Similarly, we obtain for the secant stress-strain relation instead of eq. (14):

$$\sigma_{\mathrm{xy}} = \mathbf{T}_\sigma^{-1}(\phi)\,\mathbf{D}_{\mathrm{ns}}^{\mathrm{s}}\mathbf{T}_\varepsilon(\phi)\varepsilon_{\mathrm{xy}} \tag{18}$$

Eqs (17) and (18) incorporate the traditional fixed crack model and the rotating crack model. The only difference is that in the fixed crack model the inclination angle ϕ is fixed when the major principal stress first reaches the tensile strength ($\phi = \phi_0$), while in the rotating crack concept ϕ changes such that the n-axis continues to coincide with the major principal stress direction. This difference has profound consequences when deriving the tangential stiffness, especially with regard to the shear term.

The above framework also allows for incorporation of constitutive models that are based on the microplane concept. As an example we shall consider a microplane model based on the so-called kinematic constraint, which implies that the normal and tangential strains on a microplane that is labelled α, can be derived by a simple projection of the global strains $\varepsilon_{\mathrm{xy}}$ similar to eq. (16):

$$\varepsilon_{\mathrm{ns}}^\alpha = \mathbf{T}_\varepsilon(\phi^\alpha)\varepsilon_{\mathrm{xy}} \tag{19}$$

The stresses on this microplane can be derived in a fashion similar to (14):

$$\sigma_{\mathrm{ns}}^\alpha = \mathbf{D}_{\mathrm{ns}}^\alpha \varepsilon_{\mathrm{ns}}^\alpha \tag{20}$$

with $\mathbf{D}_{\mathrm{ns}}^\alpha$ given by

$$\mathbf{D}_{\mathrm{ns}}^\alpha = \begin{bmatrix} (1-\omega_{\mathrm{N}}^\alpha)E_{\mathrm{N}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1-\omega_{\mathrm{T}}^\alpha)E_{\mathrm{T}} \end{bmatrix} \tag{21}$$

where the initial stiffness moduli E_{N} and E_{T} are functions of the Young's modulus, the Poisson's ratio and a weight parameter, see e.g. Bažant and Prat (1988). The damage parameters $\omega_{\mathrm{N}}^\alpha$ and $\omega_{\mathrm{T}}^\alpha$ for the normal stiffness and the shear stiffness are functions of the history parameters $\kappa_{\mathrm{N}}^\alpha$ and $\kappa_{\mathrm{T}}^\alpha$ in standard fashion: $\omega_{\mathrm{N}}^\alpha = \omega_{\mathrm{N}}^\alpha(\kappa_{\mathrm{N}}^\alpha)$ and $\omega_{\mathrm{T}}^\alpha = \omega_{\mathrm{T}}^\alpha(\kappa_{\mathrm{T}}^\alpha)$. The main departure from the fixed crack model as outlined above is the fact that we need two damage loading functions on each microplane α:

$$f_{\mathrm{N}}^\alpha = \varepsilon_{nn}^\alpha - \kappa_{\mathrm{N}}^\alpha \quad \text{and} \quad f_{\mathrm{T}}^\alpha = \gamma_{ns}^\alpha - \kappa_{\mathrm{T}}^\alpha \tag{22}$$

each subject to the standard Kuhn-Tucker loading-unloading conditions. Finally, the stresses in the global x, y coordinate system are recovered by summing over all the microplanes and by transforming them in a standard fashion according to eq. (16). With eqs (19) and (20) we finally arrive at

$$\sigma_{xy} = \sum_{\alpha=1}^{n} w^{\alpha} \mathbf{T}_{\sigma}^{-1}(\phi^{\alpha}) \mathbf{D}_{ns}^{\alpha} \mathbf{T}_{\varepsilon}(\phi^{\alpha}) \varepsilon_{xy} \tag{23}$$

with n the chosen number of microplanes and w^{α} the weight factors.

Attention is drawn to the fact that the second row of $\mathbf{D}_{ns}^{\alpha}$ consists of zeros. This is because in the microplane concept only the normal stress and the shear stress are resolved on each microplane. The normal stress parallel to this plane therefore becomes irrelevant. Furthermore, attention is drawn to the fact that here we use a relatively simple version of the microplane model, namely one in which no splitting in volumetric and deviatoric components is considered (Bažant and Gambarova 1984). Nevertheless, more sophisticated microplane models, e.g. that by Bažant and Prat (1988), which incorporates such a split, can be captured by the same formalism (Kuhl and Ramm 1998). It is finally noted that the microplane is very similar to the multiple fixed-crack model (de Borst and Nauta 1985, de Borst 1987), except for the fact that the multiple fixed-crack model has been formulated in terms of a strain decomposition in the sense of the stress-based damage model of eq. (12).

For the fixed crack model differentiation of eq. (18) yields the tangential stress-strain relation needed in an incremental-iterative procedure which utilises the Newton-Raphson method:

$$\dot{\sigma}_{xy} = \mathbf{T}_{\sigma}^{-1}(\phi_0)(\mathbf{D}_{ns}^{s} - \Delta\mathbf{D}_{ns})\mathbf{T}_{\varepsilon}(\phi_0)\dot{\varepsilon}_{xy} \tag{24}$$

with $\mathbf{D}_{ns}^{s}$ given by eq. (15) and

$$\Delta\mathbf{D}_{ns} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ d_{31} & 0 & 0 \end{bmatrix} \tag{25}$$

with

$$d_{11} = \frac{\partial \omega_1}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon_{nn}} E \varepsilon_{nn} \quad \text{and} \quad d_{31} = \frac{\partial \omega_2}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon_{nn}} G \gamma_{ns}$$

$\partial\kappa/\partial\varepsilon_{nn} = 1$ upon loading and zero otherwise. We observe that the local material tangential stiffness matrix $\mathbf{D}_{ns}^{s} - \Delta\mathbf{D}_{ns}$ generally becomes non-symmetric.

The fact that in the rotating smeared crack model the local coordinate system of the crack and the principal axes of stress and strain coincide throughout the entire deformation process implies that the secant stiffness matrix $\mathbf{D}_{ns}^{s}$ relates principal stresses to principal strains, and that a secant shear stiffness becomes superfluous. Consequently, there is only one remaining damage parameter, $\omega_1 = \omega$, and we have

$$\mathbf{D}^{s}_{ns} = \begin{bmatrix} (1-\omega)E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

instead of expression (15). With eq. (26) differentiation of the secant stiffness relation in global coordinates yields (Bažant 1983, Willam *et al.* 1986, Feenstra 1993):

$$\dot{\sigma}_{xy} = \mathbf{T}_{\sigma}^{-1}(\phi)(\mathbf{D}^{s}_{ns} - \Delta\mathbf{D}_{ns})\mathbf{T}_{\varepsilon}(\phi)\dot{\varepsilon}_{xy} \tag{27}$$

with $\Delta\mathbf{D}_{ns}$ now given by

$$\Delta\mathbf{D}_{ns} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \tag{28}$$

with

$$d_{11} = \frac{\partial\omega}{\partial\kappa}\frac{\partial\kappa}{\partial\varepsilon_{nn}} E\varepsilon_{nn} \qquad \text{and} \qquad d_{33} = -\frac{\sigma_{nn} - \sigma_{ss}}{2(\varepsilon_{nn} - \varepsilon_{ss})}$$

The tangential stress-strain relation of the microplane model can be cast in the same formalism as that of the rotating and the fixed crack models. Indeed, upon linearisation of eq. (23) we obtain

$$\dot{\sigma}_{xy} = \sum_{\alpha=1}^{n} w^{\alpha}\mathbf{T}_{\sigma}^{-1}(\phi^{\alpha})(\mathbf{D}^{\alpha}_{ns} - \Delta\mathbf{D}^{\alpha}_{ns})\mathbf{T}_{\varepsilon}(\phi^{\alpha})\dot{\varepsilon}_{xy} \tag{29}$$

with $\mathbf{D}^{\alpha}_{ns}$ given by eq. (21) and $\Delta\mathbf{D}^{\alpha}_{ns}$ defined as

$$\Delta\mathbf{D}^{\alpha}_{ns} = \begin{bmatrix} d^{\alpha}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^{\alpha}_{33} \end{bmatrix} \tag{30}$$

with

$$d^{\alpha}_{11} = \frac{\partial\omega^{\alpha}_{\mathrm{N}}}{\partial\kappa^{\alpha}_{\mathrm{N}}}\frac{\partial\kappa^{\alpha}_{\mathrm{N}}}{\partial\varepsilon^{\alpha}_{nn}} E_{\mathrm{N}}\varepsilon^{\alpha}_{nn} \qquad \text{and} \qquad d^{\alpha}_{33} = \frac{\partial\omega^{\alpha}_{\mathrm{T}}}{\partial\kappa^{\alpha}_{\mathrm{T}}}\frac{\partial\kappa^{\alpha}_{\mathrm{T}}}{\partial\gamma^{\alpha}_{ns}} E_{\mathrm{T}}\gamma^{\alpha}_{ns}$$

where $\partial\kappa^{\alpha}_{\mathrm{N}}/\partial\varepsilon^{\alpha}_{nn} = 1$ if $f^{\alpha}_{\mathrm{N}} = 0$ and zero otherwise, and $\partial\kappa^{\alpha}_{\mathrm{T}}/\partial\gamma^{\alpha}_{ns} = 1$ if $f^{\alpha}_{\mathrm{T}} = 0$ and zero otherwise.

4 Isotropic gradient damage models

Standard damage models suffer from the disadvantage that the rate boundary value problem becomes ill-posed at a generic stage of the loading process (Benallal *et al.* 1988, de Borst *et al.* 1993). To remedy this shortcoming, it has been proposed to enrich the loading function with gradients of the equivalent strain (Peerlings *et al.* 1996a, 1996b), or of the history parameter (de Borst *et al.* 1996). Since there is a direct relation between the history parameter and the damage parameter, higher-order gradients of the damage can also be introduced for this purpose (Mühlhaus *et al.* 1994, de Borst *et al.* 1996, Comi 1998). When we first consider the enhancement of the loading function f with gradients of the equivalent strain, a simple extension would be to replace eq. (4) by

$$f(\tilde{\varepsilon}, \nabla^2\tilde{\varepsilon}, \kappa) = \tilde{\varepsilon} + g\nabla^2\tilde{\varepsilon} - \kappa \tag{31}$$

where g is a material parameter of the dimension length squared. We adopt the phenomenological view that $\sqrt{g}$ reflects the length scale of the failure process which we wish to describe macroscopically.

Formulation (31) has a severe disadvantage when applied in a finite element context, namely that it requires computation of second-order gradients of the local equivalent strain $\tilde{\varepsilon}$. Since this quantity is a function of the strain tensor, and since the strain tensor involves first-order derivatives of the displacements, third-order derivatives of the displacements have to be computed, which would necessitate C^1-continuity of the shape functions. To obviate this problem, eq. (31) is replaced by

$$f(\bar{\varepsilon}, \kappa) = \bar{\varepsilon} - \kappa \tag{32}$$

where $\bar{\varepsilon}$ follows from:

$$\bar{\varepsilon} - g\nabla^2\bar{\varepsilon} = \tilde{\varepsilon} \tag{33}$$

which implies essentially that the assumption $\nabla^2\bar{\varepsilon} \approx \nabla^2\tilde{\varepsilon}$ is made, which can be shown to involve neglecting gradients of the fourth-order (Peerlings *et al.* 1996a, 1996b). When $\bar{\varepsilon}$ is discretised independently and use is made of the divergence theorem, a C^0-interpolation for $\bar{\varepsilon}$ suffices.

In a fashion similar to the derivation of the gradient damage models based on the introduction of a Laplacian of the equivalent strain $\tilde{\varepsilon}$, we can elaborate a gradient formulation by introducing a Laplacian of κ in the damage loading function, so that (de Borst *et al.* 1996):

$$f(\tilde{\varepsilon}, \kappa, \nabla^2\kappa) = \tilde{\varepsilon} - \kappa - g\nabla^2\kappa \tag{34}$$

Obviously, the gradient parameter g will have different values for the various formulations.

For stress-based isotropic damage models, we replace the loading function (12) by

$$f(\tilde{\sigma}, \kappa) = \tilde{\sigma} - \kappa(\bar{\varepsilon}^{\mathrm{d}}) \tag{35}$$

with

$$\bar{\varepsilon}^{\mathrm{d}} = \tilde{\varepsilon}^{\mathrm{d}} + g\nabla^2\tilde{\varepsilon}^{\mathrm{d}} \tag{36}$$

It is noted that the set of equations (35)-(36) formally becomes identical to the gradient-enhanced plasticity model of de Borst and Mühlhaus (1992). Of course, the internal variable $\bar{\varepsilon}^{\mathrm{d}}$ has a completely different meaning than the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{p}}$ in plasticity approaches in the sense that $\bar{\varepsilon}^{\mathrm{d}}$ does not correspond to a permanent strain and that, accordingly, all strain is recoverable.

5 Anisotropic gradient damage models

Anisotropic gradient damage models can be developed in a manner similar to isotropic gradient damage models. We take as point of departure the damage loading function (13) in the local n,s-coordinate system, where we replace the normal strain ε_{nn} by its nonstandard equivalent:

$$f(\bar{\varepsilon}_{nn},\kappa)=\bar{\varepsilon}_{nn}-\kappa \tag{37}$$

or, identically

$$f=\bar{\varepsilon}_{xx}\cos^2\phi+\bar{\varepsilon}_{yy}\sin^2\phi+\bar{\gamma}_{xy}\sin\phi\cos\phi-\kappa \tag{38}$$

instead of eq. (17). We now apply the averaging process to each of the nonstandard strain components $\bar{\varepsilon}_{xx}$, $\bar{\varepsilon}_{yy}$ and $\bar{\gamma}_{xy}$. In the spirit of eq. (33) they have to satisfy

$$\begin{aligned}
&\bar{\varepsilon}_{xx}-g\nabla^2\bar{\varepsilon}_{xx}=\varepsilon_{xx}\\
&\bar{\varepsilon}_{yy}-g\nabla^2\bar{\varepsilon}_{yy}=\varepsilon_{yy}\\
&\bar{\gamma}_{xy}-g\nabla^2\bar{\gamma}_{xy}=\gamma_{xy}
\end{aligned} \tag{39}$$

or written in a vector format

$$\bar{\boldsymbol{\varepsilon}}_{\mathrm{xy}}-g\nabla^2\bar{\boldsymbol{\varepsilon}}_{\mathrm{xy}}=\boldsymbol{\varepsilon}_{\mathrm{xy}} \tag{40}$$

After solution of the set of Helmholtz equations (40), $\bar{\boldsymbol{\varepsilon}}_{\mathrm{xy}}$ is substituted into the damage loading function (38) to determine whether loading occurs and next into the secant stress-strain relation to solve for the stresses $\boldsymbol{\sigma}_{\mathrm{xy}}$.

The tangential stress-strain relation attains a slightly different format due to the fact that the nonstandard strain components must be discretised independently. For a gradient-enhanced version of the fixed crack model, we obtain upon linearisation of eq. (18)

$$\dot{\boldsymbol{\sigma}}_{\mathrm{xy}}=\mathbf{T}_{\sigma}^{-1}(\phi_0)\mathbf{D}_{\mathrm{ns}}^{\mathrm{s}}\mathbf{T}_{\varepsilon}(\phi_0)\dot{\boldsymbol{\varepsilon}}_{\mathrm{xy}}-\mathbf{T}_{\sigma}^{-1}(\phi_0)\Delta\mathbf{D}_{\mathrm{ns}}\mathbf{T}_{\varepsilon}(\phi_0)\dot{\bar{\boldsymbol{\varepsilon}}}_{\mathrm{xy}} \tag{41}$$

with

$$\Delta\mathbf{D}_{\text{ns}} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ d_{31} & 0 & 0 \end{bmatrix} \tag{42}$$

where

$$d_{11} = \frac{\partial \omega_1}{\partial \kappa} \frac{\partial \kappa}{\partial \bar{\varepsilon}_{nn}} E\varepsilon_{nn} \quad \text{and} \quad d_{31} = \frac{\partial \omega_2}{\partial \kappa} \frac{\partial \kappa}{\partial \bar{\varepsilon}_{nn}} G\gamma_{ns}$$

$\partial\kappa/\partial\bar{\varepsilon}_{nn} = 1$ upon loading and zero otherwise.

For the gradient-enhanced rotating crack model, we obtain upon linearisation of eq. (18)

$$\dot{\sigma}_{\text{xy}} = \mathbf{T}_\sigma^{-1}(\phi)\mathbf{D}_{\text{ns}}^{\text{s}}\mathbf{T}_\varepsilon(\phi)\dot{\varepsilon}_{\text{xy}} - \mathbf{T}_\sigma^{-1}(\phi)\Delta\mathbf{D}_{\text{ns}}\mathbf{T}_\varepsilon(\phi)\dot{\bar{\varepsilon}}_{\text{xy}} \tag{43}$$

but now $\Delta\mathbf{D}_{\text{ns}}$ is given by

$$\Delta\mathbf{D}_{\text{ns}} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \tag{44}$$

with

$$d_{11} = \frac{\partial \omega}{\partial \kappa} \frac{\partial \kappa}{\partial \bar{\varepsilon}_{nn}} E\varepsilon_{nn} \quad \text{and} \quad d_{33} = -\frac{\sigma_{nn} - \sigma_{ss}}{2(\varepsilon_{nn} - \varepsilon_{ss})}$$

In a gradient-enhanced microplane model (Kuhl *et al.* 1998) the single damage loading condition (37), or equivalently eq. (38), must be replaced by two loading functions for each microplane:

$$f_{\text{N}} = \bar{\varepsilon}_{nn}^\alpha - \kappa_{\text{N}}^\alpha \quad \text{and} \quad f_{\text{T}} = \bar{\gamma}_{ns}^\alpha - \kappa_{\text{T}}^\alpha \tag{45}$$

where $\bar{\varepsilon}_{nn}^\alpha$ and $\bar{\gamma}_{ns}^\alpha$ are obtained by solving eq. (40) for $\bar{\varepsilon}_{xx}$, $\bar{\varepsilon}_{yy}$ and $\bar{\varepsilon}_{xy}$ followed by a transformation to the coordinate system of microplane α via the matrix $\mathbf{T}_\varepsilon(\phi^\alpha)$. The tangential stiffness relation can then be derived to be of a form similar to the fixed and rotating crack models. Indeed, upon linearisation of eq. (23) we obtain

$$\dot{\sigma}_{\text{xy}} = \sum_{\alpha=1}^{n} w^\alpha \left[\mathbf{T}_\sigma^{-1}(\phi^\alpha)\mathbf{D}_{\text{ns}}^\alpha\mathbf{T}_\varepsilon(\phi^\alpha)\dot{\varepsilon}_{\text{xy}} - \mathbf{T}_\sigma^{-1}(\phi^\alpha)\Delta\mathbf{D}_{\text{ns}}^\alpha\mathbf{T}_\varepsilon(\phi^\alpha)\dot{\bar{\varepsilon}}_{\text{xy}} \right] \tag{46}$$

with $\mathbf{D}_{\text{ns}}^\alpha$ given by eq. (21) and $\Delta\mathbf{D}_{\text{ns}}^\alpha$ defined as

$$\Delta\mathbf{D}_{\text{ns}}^\alpha = \begin{bmatrix} d_{11}^\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_{33}^\alpha \end{bmatrix} \tag{47}$$

with

$$d_{11}^{\alpha} = \frac{\partial \omega_{\mathrm{N}}^{\alpha}}{\partial \kappa_{\mathrm{N}}^{\alpha}} \frac{\partial \kappa_{\mathrm{N}}^{\alpha}}{\partial \bar{\varepsilon}_{nn}^{\alpha}} E_{\mathrm{N}} \varepsilon_{nn}^{\alpha} \quad \text{and} \quad d_{33}^{\alpha} = \frac{\partial \omega_{\mathrm{T}}^{\alpha}}{\partial \kappa_{\mathrm{T}}^{\alpha}} \frac{\partial \kappa_{\mathrm{T}}^{\alpha}}{\partial \bar{\gamma}_{ns}^{\alpha}} E_{\mathrm{T}} \gamma_{ns}^{\alpha}$$

with $\partial \kappa_{\mathrm{N}}^{\alpha} / \partial \bar{\varepsilon}_{nn}^{\alpha} = 1$ if $f_{\mathrm{N}}^{\alpha} = 0$ and zero otherwise, and $\partial \kappa_{\mathrm{T}}^{\alpha} / \partial \bar{\gamma}_{ns}^{\alpha} = 1$ if $f_{\mathrm{T}}^{\alpha} = 0$ and zero otherwise.

6 Dispersion analyses

In the isotropic gradient-enhanced damage formulations different variables have been considered for regularising the governing set of field equations. In the section on anisotropic gradient-enhanced damage models the regularisation by means of the nonstandard strain $\bar{\varepsilon}$ (Peerlings *et al.* 1996a) has been used exclusively. Indeed, this regularisation has proven to be theoretically sound (Peerlings *et al.* 1996b), computationally robust, and versatile in the sense that curved crack propagation can be simulated (Peerlings *et al.* 1998). Below we shall carry out dispersion analyses for the various gradient models advocated so far and use these results to bring out their typical properties.

We consider an infinite one-dimensional medium, so that the equations of motion, the kinematic equations and the stress-strain relation reduce to

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \quad , \quad \varepsilon = \frac{\partial u}{\partial x} \quad \text{and} \quad \sigma = (1 - \omega) E \varepsilon \tag{48}$$

complemented by a model-dependent definition of the nonstandard quantity. For the loading function we assume that momentarily $f = 0$ throughout the bar, so that we have a linear comparison solid in the sense of Hill (1958). Combination of eqs (48) leads to

$$c_{\mathrm{e}}^{-2} \frac{\partial^2 u}{\partial t^2} = (1 - \omega) \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial \omega}{\partial \kappa} \frac{\partial \kappa}{\partial x} \tag{49}$$

with $c_{\mathrm{e}} = \sqrt{E/\rho}$ the one-dimensional elastic bar velocity.

We first consider the 'explicit' gradient damage model of eq. (31), which for the one-dimensional case reduces to

$$\kappa = \varepsilon + g \frac{\partial^2 \varepsilon}{\partial x^2} \tag{50}$$

Substitution of eq. (50) into eq. (49) yields

$$c_{\mathrm{e}}^{-2} \frac{\partial^2 u}{\partial t^2} = \left(1 - \omega - \varepsilon \frac{\partial \omega}{\partial \kappa} \right) \frac{\partial^2 u}{\partial x^2} - g \varepsilon \frac{\partial \omega}{\partial \kappa} \frac{\partial^4 u}{\partial x^4} \tag{51}$$

We now substitute a small harmonic disturbance

$$\delta u = \hat{u}\, \mathrm{e}^{ik(x-ct)} \tag{52}$$

with $\hat{u}$ the amplitude, k the wave number and c the propagation speed of the disturbance, into eq. (51) with $\varepsilon = \varepsilon_0$ and $\omega = \omega_0$, which characterises a homogeneous state. The result is given by (Peerlings *et al.* 1996b):

$$c = c_\mathrm{e} \sqrt{1 - \omega_0 - \varepsilon_0 \left(\frac{\partial \omega}{\partial \kappa} \right)_0 (1 - g k^2)} \tag{53}$$

For the *linear* softening relation of eq. (7) the above expression reduces to

$$c = c_\mathrm{e} \sqrt{\frac{h}{h+E} \left(1 - \frac{\kappa_\mathrm{u}}{\varepsilon_0}\, g k^2 \right)} \tag{54}$$

with h the softening modulus: $h = -f_\mathrm{ct}/\kappa_\mathrm{u}$. Real wave speeds are obtained if the term under the square root is nonnegative, which implies that there is a critical wave number k_crit where we have a stationary wave. The corresponding critical wave length is given by

$$\lambda_\mathrm{crit} = 2\pi \sqrt{g\, \kappa_\mathrm{u} / \varepsilon_0} \tag{55}$$

which corresponds to the width of the localisation zone. We observe that for progressive damage, λ_crit gradually decreases to a minimum value $\lambda_\mathrm{crit} = 2\pi\sqrt{g}$ for $\varepsilon_0 = \kappa_\mathrm{u}$.

The latter observation is in contrast with the 'implicit' gradient model of eq. (33). For the one-dimensional case we now have

$$\kappa = \varepsilon + g\, \frac{\partial^2 \kappa}{\partial x^2} \tag{56}$$

instead of eq. (50). We cannot directly substitute eq. (56) into eq. (49), because κ is defined in an implicit sense. For this reason we consider eqs (49) and (56) simultaneously, and substitute the perturbations (52) and

$$\delta \kappa = \hat{\kappa}\, \mathrm{e}^{ik(x-ct)} \tag{57}$$

We obtain for the propagation speed of the perturbation

$$c = c_\mathrm{e} \sqrt{1 - \omega_0 - \varepsilon_0 \left(\frac{\partial \omega}{\partial \kappa} \right)_0 (1 + g k^2)^{-1}} \tag{58}$$

For linear softening eq. (58) reduces to

$$c = c_\mathrm{e} \sqrt{\frac{h}{h+E} \left(1 - \frac{\kappa_\mathrm{u}}{\varepsilon_0}\, \frac{g k^2}{1 + g k^2} \right)} \tag{59}$$

and the critical wave length now reads

$$\lambda_{\mathrm{crit}} = 2\pi\sqrt{g\,(\kappa_{\mathrm{u}}/\varepsilon_0 - 1)} \tag{60}$$

Similar to the 'explicit' gradient model, we observe that for progressive damage, λ_{crit} decreases. A striking difference is that for $\varepsilon_0 = \kappa_{\mathrm{u}}$, λ_{crit} reduces to zero, which implies that in the limiting case of complete stiffness degradation, the width of the localisation zone reduces to zero, i.e. we recover a line crack as one would expect physically.

Next, we consider the gradient damage model that is obtained by replacing the history parameter κ by its nonstandard equivalent $\bar{\kappa}$, cf. eq. (34), which for the one-dimensional case reads:

$$\bar{\kappa} = \kappa + g\,\frac{\partial^2 \kappa}{\partial x^2} \tag{61}$$

Applying the perturbations (52) and (56) to eqs (49) and (61) we obtain

$$c = c_{\mathrm{e}}\sqrt{1 - \omega_0 - \varepsilon_0\left(\frac{\partial \omega}{\partial \kappa}\right)_0 (1 - gk^2)^{-1}} \tag{62}$$

which implies that the gradient constant g should now have a negative sign for expression (62) to be meaningful. Then, the result is fully identical to that of the 'implicit' strain-based gradient model (56), including the expression for the critical wave length.

Finally, we examine the stress-based gradient damage model of eq. (35). In a one-dimensional context eq. (36) is replaced by

$$\bar{\varepsilon}^{\mathrm{d}} = \varepsilon^{\mathrm{d}} + g\,\frac{\partial^2 \varepsilon^{\mathrm{d}}}{\partial x^2} \tag{63}$$

while, considering (10), the one-dimensional stress-strain relation reads

$$\sigma = E(\varepsilon - \varepsilon^{\mathrm{d}}) \tag{64}$$

Combining the equation of motion and the kinematic relation (eqs (48)), the stress-strain relation (64) and definition (63) for the nonstandard damage strain $\bar{\varepsilon}^{\mathrm{d}}$, we obtain after some algebraic manipulations

$$c_{\mathrm{e}}^{-2}\frac{\partial^2 u}{\partial t^2} = \frac{h}{h+E}\frac{\partial^2 u}{\partial x^2} + \frac{gh}{h+E}\left(\frac{\partial^4 u}{\partial x^4} - c_{\mathrm{e}}^{-2}\frac{\partial^4 u}{\partial x^2 \partial t^2}\right) \tag{65}$$

where $h = \partial\kappa/\partial\bar{\varepsilon}^{\mathrm{d}}$, which becomes a (negative) constant for linear softening. Not surprisingly, eq. (65) is formally identical to gradient plasticity (Sluys *et al.* 1993) and so are the expressions for the propagation speed

$$c = c_{\mathrm{e}}\sqrt{\frac{h(1 - gk^2)}{h + E - hgk^2}} \tag{66}$$

and the critical wave length for stationary waves

$$\lambda_{\text{crit}} = 2\pi\sqrt{g} \tag{67}$$

Indeed, since in this one-dimensional dispersion analysis we consider a linear comparison solid, the coincidence is complete. Note, that in contrast to the other three gradient damage approaches, the critical wave length attains a finite value irrespective of the strain level. Apparently, the choice of the regularising field variable can have a major influence in addition to the choice of the formalism, i.e. plasticity or damage (Huerta and Pijaudier-Cabot 1994), or the choice for a differential *vs* an integral type of nonlocal model (Peerlings *et al.* 1996b).

7 Concluding remarks

A unified framework has been set up that encompasses various versions of smeared-crack models as well as microplane models. An enrichment has been proposed using spatial gradients, such that the governing equations remain elliptic under static loading conditions and hyperbolic under dynamic loading conditions for progressive damage evolution. Dispersion analyses have been carried out to underscore this assertion and to highlight the differences between various gradient enrichments.

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