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TENSILE CRACKING VIEWED AS BIFURCATION AND INSTABILITY IN A DISCRETE INTERFACE MODEL

A. Delaplace
LMT, ENS Cachan / CNRS / Université P. et M. Curie, Cachan, France.
S. Roux
Laboratoire de Physique et Mécanique des Milieux Hétérogènes, Paris, France.
G. Pijaudier-Cabot
LMT, ENS Cachan / CNRS / Université P. et M. Curie, Cachan, France, and Institut Universitaire de France

Abstract
We propose a study of the damage at an interface which is coupled to an elastic homogeneous block. We use two interface representations based on a continuous approach for the first one, and on a discrete approach for the second one. The elastic block is represented by a hierarchical structure, which allows to have a realistic elastic interaction along the interface. After a brief recall on a tool which allows to consider the "bifurcation" modes in a discrete model in the same way as for a continuous one, we show that the localisation in the interface occurs through a cascade of bifurcations, which progressively concentrates damage from the entire interface to a narrow region. For the discrete approach, *i.e.* when random heterogeneities are introduced, the localisation proceeds by a sequence of avalanches. In both cases, the analysis of size effects is also obtained.
Key words: fracture, discrete model, interface, damage, bifurcation

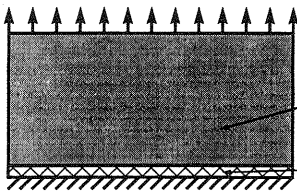

Fig. 1. The model problem.

1 Introduction

During the failure of quasi-brittle materials, the strain field is initially homogeneous, then becomes heterogeneous before the apparition of a macro-crack that propagates up to the rupture. This singular transition is well-described for the class of continuous models. For instance, the loss of uniqueness or the loss of stability are currently used to detect this transition (see *e.g.* Hill (1959), Lemaitre (1992), Bazant and Cedolin (1991)). On the other hand, these criteria could not be employed for rate dependent models, or for discrete models. The solution of the problem is always unique, and no bifurcation occurs. We present in this paper a tool that allows to describe the bifurcation in the discrete model in the same way as for the continuous one. Then, we propose to study a very simple model, that represents a damage interface connected with an elastic bloc (Fig. 1). Two cases are considered for the interface. First, we use a continuous approach, and we propose a complete description of the post-bifurcation branch. We show that damage concentrates from a half-part of the interface to a narrower and narrower region, through a bifurcations cascade. The damage profile at the onset of the first macro-crack is also obtained. Then, we include some heterogeneity in the interface by introducing Daniels' models instead of continuous elements. Again, we show that in this case the localisation occurs through a similar cascade of bifurcations.

2 The stability analysis for the discrete model

First of all, in order to compare the two approaches, we have to write an equivalent of the loss of stability for a discrete model. The main difference between the two apporaches is the response of the considered system. In the continuous case, the response is smooth, but is not unique. Then, an usual stability criterion can be easily computed, from the calculation of the stiffness operator. On the other hand, the response of a discrete model has fluctuations superimposed on the mean behaviour due to the presence of heterogeneities. The solution of the problem is now unique, like in reality, but no tangent stiffness operator could be computed because of these fluctuations. Then, no usual criteria could be used to detect a bifurcation point.

However, by using the concept of avalanches (Hemmer and Hansen 1992), one can elaborate an equivalent of such a criterion. An avalanche of size Δ in the direction $-\kappa$ is defined as the number of fibers that break under a given load (Fig. 2). One can show (Delaplace *et al* 1998) that the divergence of the avalanche mean size, observed in a relevant direction $-\kappa$ that depends on the system, corresponds to the first bifurcation of the equivalent continuous system.

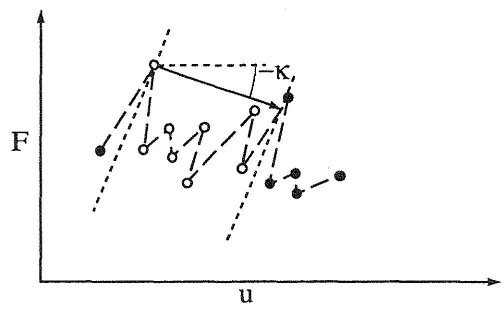

Fig. 2. A 8-size avalanche observed in a direction $-\kappa$.

Let us now consider the failure behaviour of our problem, an interface coupled with an elastic block.

3 The hierarchical model

In order to study the localisation and the damage at an interface loaded in mode I, we introduce a basic representation of the problem: the interface is a raw of damageable elements, coupled in series with a continuous elastic block. We propose to represent this elastic block by a hierarchical structure: The elastic block is subdi-

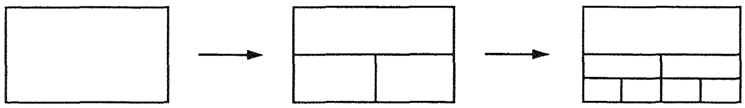

Fig. 3. The two first decompositions of an elastic block.

vided in one upper block, and two lower blocks. This transformation is reiterated for the lower blocks while a sufficient decomposition is not reached. Then, damageable elements are coupled in series with the lower blocks. Hence, for a 10-generation system, the interface is composed of $2^{10-1} = 512$ elements. The blocks have just an elastic behaviour, and can be considered as a discretisation of the block. By choosing a stiffness of $k/2$ for the lower blocks, and a stiffness of k for all the others, the stiffness structure is independent of the chosen generation. This simple structure allows to make calculus on large system sizes, and include a realistic elastic interaction. In order to characterize this interaction, we need to introduce a distance along the interface. The distance d which separates two elements is defined as the generation of the smallest block of size j that separates the two elements:

$$d = 2^{j-1} - 1 \quad (1)$$

Then, we can compute the displacement v of the block i when a force F is applied on the block j. It is just:

$$v(d) = A \log(\frac{B}{d})\frac{F}{k} + v_0 \quad (2)$$

with $A = 1/\log 2$ and $B = 2^n$. This is exactly the form of the Green function for a semi-infinite plan as expected.

Then, we have to write the relations that drive the hierarchical system. We call $F_1^{(n)}(U^{(n)})$ and $F_2^{(n)}(U^{(n)})$ the response force(F)-

displacement(U) of two n-generation system. Their association gives a $n+1$-generation system. The global force of the system is:

$$F^{(n+1)} = F_1^{(n)}(U^{(n)}) + F_2^{(n)}(U^{(n)}) \qquad (3)$$

and the global displacement is:

$$U^{(n+1)} = U^{(n)} + \frac{F_1^{(n)}(U^{(n)}) + F_2^{(n)}(U^{(n)})}{k} \qquad (4)$$

These two equations give a parametric representation of a $n+1$-generation system. We now introduce two representations of the interface.

3.1 The continuous interface representation

In this part, we use a continuous description of the interface. The damageable elements are here continuous elements, with a classical damage law. Note that in this case, no heterogeneity is introduced. Then, two n-generation systems have exactly the same behaviour ($F_1 = F_2$) and the equation 4 is transformed into:

$$U^{(n+1)}(F) = U^{(n)}(F/2) + F/k = U^{(1)}(F/2^n) + (1 - 2^{-n})F/k \qquad (5)$$

This equation relies the global displacement to the interface displacement, $u(F) = U^{(1)}(F)$. We can now compute the tangent (subscript tg) and the secant (subscript sc) stiffness of the entire system at generation n:

$$\begin{cases} K_{tg}^{(n)} = \left(\dfrac{1-2^{1-n}}{k} + \dfrac{2^{1-n}}{1-2u}\right)^{-1} \\ K_{sc}^{(n)} = \left(\dfrac{1-2^{1-n}}{k} + \dfrac{2^{1-n}}{1-u}\right)^{-1} \end{cases} \qquad (6)$$

The stability analysis of the system can be done easily. The bifurcation condition is written (Bazant and Cedolin 1991):

$$\frac{dU^{(n)}}{dF} = 0 \qquad (7)$$

or:

$$k + K_{sc}^{(n-1)} + K_{tg}^{(n-1)} = 0 \qquad (8)$$

For a large generation system, and by using equation 5, we can compute the interface displacement at this point:

$$u_1^* = \frac{1}{2} + \frac{2}{3}\frac{k}{\ell_1} + (\frac{10}{9k} - \frac{4}{9})\frac{k^2}{\ell_1^2} + \mathcal{O}(\ell_1^{-3}) \qquad (9)$$

where damage localizes over the length $\ell_1 = 2^{n-1} = L$ on the interface. L refers here to the number of damageable elements. Note that u_1 corresponds effectively to the first bifurcation point. The other displacements correspond to a lower length than ℓ_1, and then occur much later during the failure. At this point, a half part of the interface starts to unload, as the other one is still loaded. Then, the same processus is reiterated in this loaded part, and damage localizes progressively, through a cascade of bifurcation points.

3.1.1 Post-bifurcation response

Knowing the first bifurcation point, and by using the recurrence relations of the hierarchical structure, we can compute the post-bifurcation response. Note that we use here directly the interface behaviour, chosen to be the asymptotic response of a Daniels' model with a uniform distribution between 0 and 1 for the thresholds. Any other distribution could be used in the same way. When the first bifurcation point is reached in a $n+1$-generation system, a n-generation subpart system is elastically unloaded, as the other one is still loaded. The response of the global system is the addition of the two responses. We call $(1/2 + x_1^{(n)}, 1/4 - y_1^{(n)})$ the displacement-force coordinates of the first bifurcation point. From equation 5, we know that the $x_1^{(n)}$ form a geometric sequence of ratio 1/2:

$$x_1^{(n)} = x_1^{(1)} 2^{-n} \qquad (10)$$

By using the interface law, we can compute the $y_1^{(n)}$:

$$y_1^{(n)} = (x_1^{(1)})^2 4^{-n} \qquad (11)$$

Any point $(1/2 + x, 1/4 - y)$ of the $(n-1)$-generation equivalent homogeneous interface law is transformed into $(1/2 + x', 1/4 - y')$

such that:

$$\begin{cases} x' - x &= \dfrac{2^n}{k}\,(y' - y) \\ y' - y &= \dfrac{1}{2}\,(\dfrac{1}{4} - y) - \dfrac{1}{2}(\dfrac{1}{4} - y_1^{(n)})\dfrac{\frac{1}{2} + x + \frac{2^n}{k}(\frac{1}{4} - y)}{\frac{1}{2} + x_1^{(n)} + \frac{2^n}{k}(\frac{1}{4} - y_1^{(n)})} \end{cases} \tag{12}$$

The computation of the successive bifurcation points gives for large system size:

$$\begin{cases} x_j^{(n)} &= x_1^{(1)} 2^{-n} j \\ y_j^{(n)} &= (x_1^{(1)})^2 4^{-n} \left((\dfrac{1}{4} x_1^{(1)} - \dfrac{1}{18} k) 4^j + \dfrac{2}{3} kj - \dfrac{4}{9} k \right) \end{cases} \tag{13}$$

Hence, the bifurcation branch is perfectly described through the expression of the bifurcation points.

3.1.2 The damage profile

With the previous expression, we have access to any variables of the system. For instance, we can introduce and compute a damage variable D in the interface. D is chosen here to vary between 0 and 1, and to be linear with displacement. By using the previous defined distance d, and after the bifurcation branch (that corresponds to the creation of the first macro-crack), the damage is:

$$D(d) = \frac{1}{2} + \frac{k}{2d} \tag{14}$$

This expression is quite unusual, with a very slow decay of the damage with the distance, and gives an "infinite length" of the damage zone ahead of the crack.

3.2 The discrete interface representation

We now use a discrete representation of the interface: the damageable elements are represented by Daniels' models. A Daniels' model is just a set of parallel fibers, clamped between two rigid bars. The fibers have an elastic fragile behaviour. The stiffness of the fibers is the same, chosen to be 1 for simplicity. The only random variable is the threshold t, where the fibers break irreversibly. For each fiber, the threshold is chosen through a probability distribution function $p(t)$, or its cumulative distribution $P(t) = \int_0^t p(t')\, dt'$. This model could be solved analytically. For instance, the mean force F, divided by the number of fibers, is:

$$F(u) = (1 - P(u))u \tag{15}$$

Remember here that the stiffness of the fibers is 1, then a fiber i breaks for a displacement $u = t_i$. The system is now a set of Daniels'

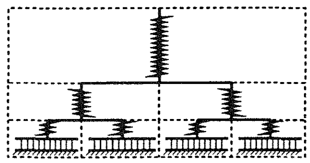

Fig. 4. A 3-generation system, with a discrete interface.

models joined through a hierarchical structure (Fig. 4). Again, we are interested by the bifurcation and post-bifurcation response. For such a discrete representation, the failure of the interface is unique, and no criteria of loss of stability can be computed.

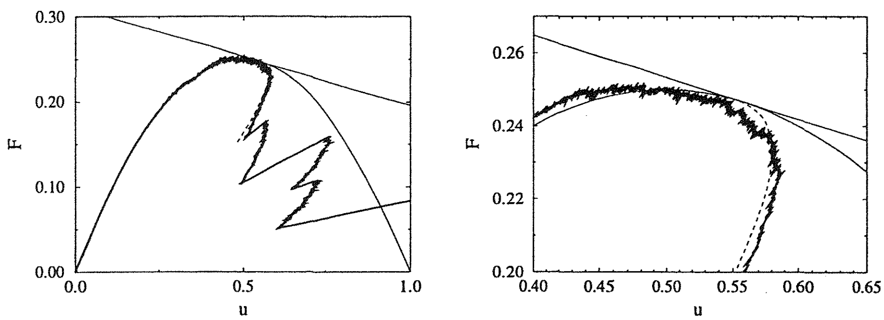

Fig. 5. The response of the interface for a 11-generation system.

However, as mentioned in section 2, one can show that the study of avalanches in a direction k, where k is the stiffness of the elastic block, allows to write an equivalent to a stability criterion for the continuous models. Then, we can compare directly if the discrete model gives the same result than for the continuous one. First of all, we can note the good agreement of the response of such a model with the continuous one: Fig. 5 shows the response of the interface for a 11-generation system, with 10 fibers for each Daniels' model. The continuous response is represented by a dotted curve. The line is the direction of the observed azalanches, that corresponds to the first bifurcation point.

As we saw before, after the first bifurcation point, damage localises through a bifurcation cascade in a narrower and narrower region. This behaviour is exactly the same for the discrete model: on Fig. 6 is represented the location of the fibers for a 12-generation system, with 10 fibers for each Daniels' model. The y-axis is the succession of the broken fibers under the loading.

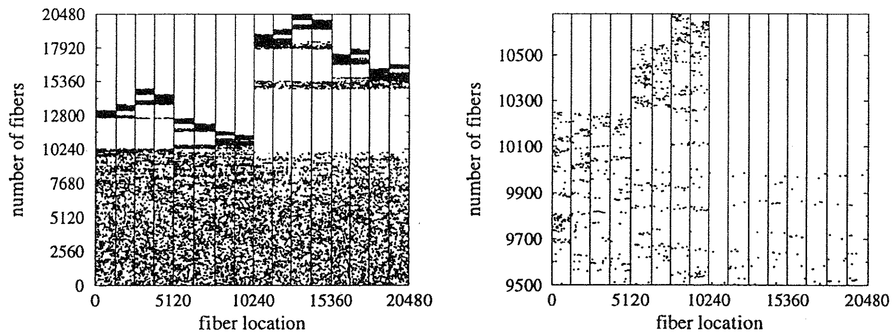

Fig. 6. The representation of the broken fibers.

4 Conclusion

We have used two approaches to study the bifurcation branch in a softening band coupled with an elastic block. After reintroduce a tool that allows to compare the bifurcation behaviour in the continuous approach and the discrete one, the following points are observed:

- The damage in the interface is first homogeneous, and then condensate progressively into a more and more narrow region, until one element is damaged to failure, providing an initiation site for crack propagation.
- The bifurcation points that lead to the crack formation are well defined, and a disordered system follows the global evolution of the homogeneous case, with a similar localisation condensation.

- Just before the apparition of the crack, the damage profile is obtained as an inverse power law, and is spread over all the interface.

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