Fracture Mechanics of Concrete Structures
Proceedings FRAMCOS-3
AEDIFICATIO Publishers, D-79104 Freiburg, Germany

HIGHER ORDER BEAM THEORY IN GRADIENT PLASTICITY: DESCRIPTION OF FAILURE MODES WITH WARPING

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Abstract
Refined higher-order displacement model for the behavior of concrete beams incorporating gradient dependent plasticity regularization is presented. This theory incorporates a more realistic nonlinear variation of longitudinal displacements through the beam thickness. The discrete problem size is significantly reduce due to the use of an original multilayered approach based on simplified kinematics.
Key words: Gradient plasticity, failure, strain localization, finite element

1 Introduction

Localization of deformation into narrow bands of intense straining caused by strain softening is a characteristic feature of plastic deformation. It has been experimentally observed in many engineering materials, such as concrete, rocks and soils. Localization phenomena are often associated with a significant reduction of the load-carrying capacity of the structures, hence, the onset of localization is naturally considered as the inception of failure of engineering structures. Considerable efforts have been devoted over the last decade to obtain a comprehensive understanding of the problem and to describe this behavior quantitatively. However, numerous attempts to simulate the behavior with softening plasticity or damage theories failed in the sense that the solution appeared to be fully determined by fineness and the direction of the finite element discretization. The underlying reason of this pathologically mesh dependency is the a local change of character of the governing equations, which results in a loss of well-posedness of the boundary value problem. More details and references on these aspects can be found in the papers by Pijaudier-Cabot et *al.* (1988) and Lasry and Belytschko (1988).

To remedy the situation, generalized continuum theories should be adopted. These models incorporate an *internal length parameter* which plays the role of localization limiter, that is, a parameter that allows to control the localization zone size by preventing loss of ellipticity of governing equations. This internal length parameter can be introduced through the incorporation of higher order derivatives leading to the so-called gradient continuum theories. The capabilities of gradient dependence have been investigated for both softening plasticity and damage models, de Borst and Muhlhaus (1992), Muhlhaus and Aifantis (1991) and Peerlings et *al.* (1995). Recently, formulations and algorithms for gradient dependent models have been presented in a finite element context, Benallal and al. (1997), de Borst et al. (1993), Comi and Perego (1996), Meftah (1997a,b), Pamin (1994) and Peerlings et *al.* (1995).

In a gradient plasticity model the yield strength depends not only on the effective plastic strain but also on its Laplacian. Therefore, even if gradient dependent models bear the significant advantage of being local in a mathematical sense, the increment of the plastic strain can not be obtained at a local level since the consistency condition which governs the plastic flow becomes a second order partial differential equation. One possibility is to use a finite difference method. The algorithm is then a sequence of separate approximate solutions of the equilibrium problem using finite elements and the plastic yielding problem using finite differences, Belytschko and Lasry (1989). A more general approach is to use only finite elements and to solve the two coupled problems simultaneously, de Borst and Muhlhaus (1992). For this purpose, it was required to satisfy weakly the yield condition and todiscretize the plastic strain field in addition to the standard discretization of the displacement field. Therefore, C^1 continuous shape functions, for the plastic strain field interpolation, are needed in order to compute properly second gradients of the equivalent plastic strain. As a consequence, this mixed formulation leads to an oversized discretized problems making the calculations unreasonable. Therefore, a nonlinear layered finite element approach is developed, Meftah (1997a). It consists in a quasi-bidimensional method which allows to perform finite element analyses with a reduced number of degrees of freedom and, thus, does not require much calculation cost and large amount of memory work-load.

The paper will focus on the use of gradient plasticity theory, as a localization limiter to predict objectively failure, incorporated in a beam theory to describe Mode-I failure analyses showing a warping of the cross section of the beam. The multilayered model is presented and illustrated with an example. It seems to be a promising approach allowing to extend the applicability of the gradient plasticity regularization.

2 Finite element formulation

The essential feature of the gradient plasticity theory is the dependence of the yield function

$$f(\sigma,\kappa,\nabla^2\kappa)=0 \tag{1}$$

upon the second order spatial gradient of the plastic strain measure κ, σ being the stress tensor. This gradient dependence makes difficult to determine the increments of the plastic multiplier $\dot{\lambda}$, which represents a measure of plastic flow intensity, Meftah (1997b). The main reason is that the consistency condition which governs the plastic flow is a partial differential equation. Therefore, It has been proposed, de Borst and Muhlhaus (1992) to satisfy the yield function together with the equilibrium equation in a distributed sense, de Borst and Muhlhaus (1992),

$$\begin{aligned}&\int_{\Omega_\lambda}\delta\lambda\cdot F\left(\sigma,\kappa,\nabla^2\kappa\right)\,\mathrm{d}\Omega=0\\&\int_{\Omega}\delta\mathbf{u}\cdot div\sigma\,\mathrm{d}\Omega=0\end{aligned} \tag{2}$$

and to discretize the plastic multiplier field λ in addition to the usual discretization of the displacement field $\mathbf{u}$. As for classical mixed formulations, additional degrees of freedom Λ related to the plastic multiplier are then introduced in the finite elements besides the nodal displacements $\mathbf{a}$ such

$$\mathbf{u} = \mathbf{N}\mathbf{a} \quad \varepsilon = \mathbf{B}\mathbf{a} \quad \lambda = \mathbf{H}^{\mathrm{T}}\Lambda \quad \nabla^2\lambda = \mathbf{P}^{\mathrm{T}}\Lambda \tag{3}$$

where $\mathbf{N}$ is the classical shape function matrix for displacement, $\mathbf{B}$ is the matrix that lies nodal displacements to strains, $\mathbf{H} = \left[\mathrm{H}_1,\ldots,\mathrm{H}_n\right]^{\mathrm{T}}$ is a certain C^1 shape functions vector related to the plastic multiplier field and allowing to compute properly its Lapalcean, i. e. $\mathbf{P} = \left[\nabla^2\mathrm{H}_1,\ldots,\nabla^2\mathrm{H}_n\right]^{\mathrm{T}}$.

By considering the above discretizations of the two fields, the variational equations obtained from the weak satisfaction of both equilibrium and yield conditions (Eq.2), and requiring that these identities hold for any admissible variations $\delta\mathbf{a}$ and $\delta\Lambda$, we obtain the set of algebraic equations that govern the incremental equilibrium process in gradient plasticity

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{\lambda a}^{\mathrm{T}} \\ \mathbf{K}_{\lambda a} & \mathbf{K}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathrm{d}\mathbf{a} \\ \mathrm{d}\Lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e - \mathbf{f}_i \\ \mathbf{f}_\lambda \end{bmatrix} \tag{4}$$

with the elastic stiffness matrix $\mathbf{K}_{aa}$, the external force vector $\mathbf{f}_e$ and the internal force vector $\mathbf{f}_i$ defined conventionally, and the off-diagonal matrix $\mathbf{K}_{\lambda a}$ and the gradient-dependent matrix $\mathbf{K}_{\lambda\lambda}$ defined in, Meftah (1997a). Further, $\mathbf{f}_\lambda$ is the vector of non-standard residual forces which emerges from the inexact fulfillment of the yield condition.

In two or three dimensional analyses using continuum elements, the use of gradient plasticity as a localization limiter leads to sizable problems making the calculations unreasonable. Indeed, a large number of nodal parameters related to the plastic multiplier field are to be considered due to the continuity requirements on this field (C^1 continuity). Therefore, a layered approach allows to perform finite element analyses with a reduced number of degrees of freedom.

3 Multilayered finite element model

In a previous multilayered approach based on gradient plasticity, Meftah (1997b), the behavior of beams was satisfactorily approximated by the elementary Euler-Bernoulli theory of bending. The main assumption in this theory is that the transverse normal to the reference middle plane remains so during bending, implying that the transverse shear strain becomes zero. Thus, the bending rotation becomes a first derivative of transverse displacement and, hence, the theory requires the transverse displacement field to be C^1 continuous. The Euler theory may lead to serious discrepancies in the case of deep beams with small aspect ratios where shear effects are significant. Further, the resulting finite element formulation turns out to be computationally inefficient to describe failures with a warping of the cross section of the beam. These drawbacks was illustrated by different examples, Meftah (1997a).

In order to predict a more realistic behaviour for small aspect ratio beams, the transverse shear stress and strain, their parabolic variation across the depth and the warping of the cross section are taken into account. The following kinematics assumptions on the displacement field are then made, Salomon et *al.* (1997),

$$\mathbf{u} = \left| \begin{array}{l} u(x,y) = u(\mathrm{x}) + y\,\beta(x) + \dfrac{4}{3h^2}\left(\beta(\mathrm{x}) + \dfrac{\mathrm{d}v}{\mathrm{d}x}\right) y^3 \\ v(x,y) = v(x) \end{array} \right. \tag{5}$$

where u and v are respectively the axial and transverse displacements, β is the rotation of the normal to the undeformed middle plane to the deformed shape, and d denotes derivation. The last term of the right hand side of the upper part of equation (5) gives the expression of the warping function, which is related to the nodal displacements and then makes need to the use of C^0 shape function only.

The elastic constitutive relation is then given by

$$\begin{bmatrix} \sigma \\ \tau \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon \\ \gamma \end{bmatrix} \tag{6}$$

with τ and G the shear stress and the shear modulus respectively and E Young's modulus that relates axial stress σ to axial strain ε. The strains, computed from the kinematic assumption made in equation (5) taking into account the warping of the cross-section and allowing to stresses to satisfy locally the equilibrium equations and the shear-free boundary condition, are then given by

$$\varepsilon = \begin{vmatrix} \varepsilon(x,y) = \dfrac{\partial u(x,y)}{\partial x} = \dfrac{\partial u}{\partial x} + y\dfrac{\partial \beta}{\partial x} - \dfrac{4}{3h^2}\left(\dfrac{\partial \beta}{\partial x} + \dfrac{\partial^2 v}{\partial x^2}\right) y^3 \\ \gamma = \dfrac{\partial u(x,y)}{\partial y} + \dfrac{\partial v}{\partial x} = \left(\beta(x) + \dfrac{\mathrm{d}v}{\mathrm{d}x}\right)\left(1 - 4\dfrac{y^2}{h^2}\right) \end{vmatrix} \tag{7}$$

where h is the thickness of the beam. This relation allows then to define the matrix $\mathbf{B}$ that lies nodal displacements to strains.

Concerning the plastic multiplier field, no assumption can be made concerning its variation through the cross section. This one is then divided into superposed layers giving the variation of the plastic multiplier by mean of nodal parameters $\Lambda^{\mathrm{k}} = \left(\Lambda^{\mathrm{k}}_{\mathrm{i}}, \Lambda^{\mathrm{k}}_{\mathrm{j}}\right)$ at each layer k. Here, one dimensional C^1 continuous interpolation polynomials are considered for the plastic multiplier field. In order to avoid stress oscillations, the use of a C^0 quadratic interpolation for the displacements seems advisable (the Babuska-Brezzi conditions for mixed finite elements in compressible solids, Zienkiewicz (1991)).

The obtained beam finite element has a mixed character and presents a variable number of degrees of freedom, that is the displacement components of the reference axis of the beam and the plastic multiplier components corresponding to each layer

$$\mathbf{a}_{\mathrm{g}} = \left(\mathbf{a}\,,\, \Lambda^1,\, \ldots,\, \Lambda^k\,,\, \ldots\,,\, \Lambda^n\right) \tag{8}$$

where n is the number of layers and $\mathbf{a}_{\mathrm{g}}$ is a vector gathering the nodal values of the two fields. We then obtain the algebraic equations, given by relation (9), to solve for the layered beam element in gradient plasticity where the stiffness matrix $[\mathbf{K}_{\mathrm{aa}}]$, the coupling matrix $\left[\mathbf{K}^{\mathrm{k}}_{\lambda\mathrm{a}}\right]$, the gradient-dependent matrix $\left[\mathbf{K}^{\mathrm{k}}_{\lambda\lambda}\right]$ and the non standard residual forces $\mathbf{f}^{\mathrm{k}}_{\lambda}$ are defined in, Salomon et *al.* (1997) for the particular finite element developed. The subscript k indicates that quantities are computed with respect of the considered layer.

$$\begin{bmatrix} [\mathbf{K}_{aa}] & [\mathbf{K}_{\lambda a}^{1}]^T & \cdots & [\mathbf{K}_{\lambda a}^{k}]^T & \cdots & [\mathbf{K}_{\lambda a}^{n}]^T \\ [\mathbf{K}_{\lambda a}^{1}] & [\mathbf{K}_{\lambda\lambda}^{1}] & \cdots & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{K}_{\lambda a}^{k}] & [\mathbf{0}] & \cdots & [\mathbf{K}_{\lambda\lambda}^{k}] & \cdots & [\mathbf{0}] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{K}_{\lambda a}^{n}] & [\mathbf{0}] & \cdots & [\mathbf{0}] & \cdots & [\mathbf{K}_{\lambda\lambda}^{n}] \end{bmatrix} \begin{bmatrix} \mathrm{d}\mathbf{a} \\ \mathrm{d}\Lambda^{1} \\ \vdots \\ \mathrm{d}\Lambda^{k} \\ \vdots \\ \mathrm{d}\Lambda^{n} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e - \mathbf{f}_i \\ \mathbf{f}_{\lambda}^{1} \\ \vdots \\ \mathbf{f}_{\lambda}^{k} \\ \vdots \\ \mathbf{f}_{\lambda}^{n} \end{bmatrix}. \quad (9)$$

Not that only axial stress flow, related to cracking, is affected by gradient regularization. The shear stress is considered here to follow an elastic behavior during loading process. This avoids to introduce a second plastic multiplier field related to shear stress when assigned to have a softening behavior.

4 Validation

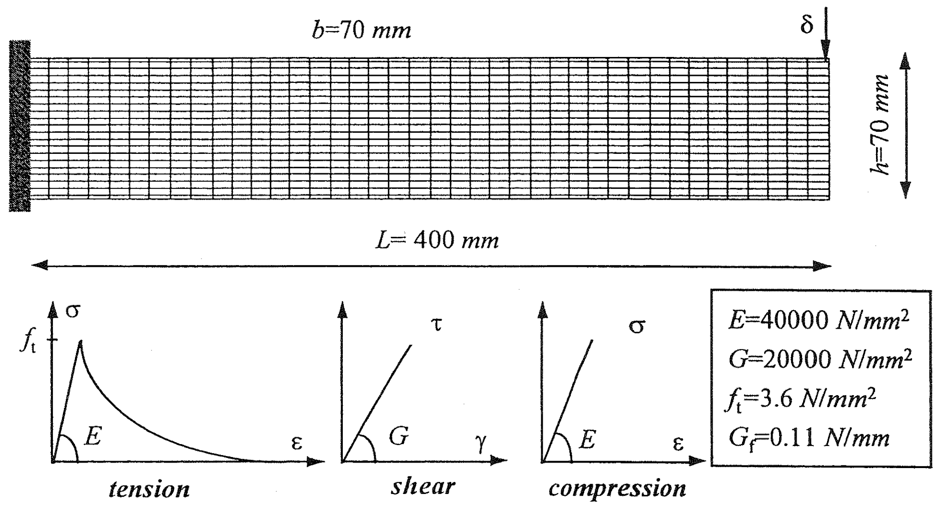

Fig. 1 Cantilever beam : geometry, mesh and material configurations

The elaborated model is applied to a Mode-I concrete fracture problem of a concrete beam under static loading with the use of the tensile fracture energies G_f as a material parameter. The objective of this validation is to show that the model gives a reliable results, though it is based on a simplified kinematic. Note that in all calculations the tangent stiffness operator is employed in full Newton-Raphson scheme. Displacement control have been used. A tolerance of 10^{-6} is assumed in a convergence criterion with an energy norm.

The geometry and the material data for the cantilever concrete beam of Fig.1 are given in, Salomon et *al.* (1997). The beam is discretized using the multilayered beam finite element where the represented layers correspond to only the plastic strain interpolation over the height of the beam. *Note that the displacement degrees of freedom are those of the reference axis of the beam.* Deformation control is used; the vertical displacement δ of the load application point is enforced.

Fig.2 presents the incremental deformations of the beam. It is observed that fracture does not localize in one element-wide vertical band at the fixed edge, which would be the case for a local approach, but is distributed to the neighboring elements due to the gradient regularization. The crack patterns are represented by the isovalues of the plastic/fracture strain. The same localization zone and post peak load-displacement diagram are obtained even if the mesh is refined in the center of the beam made, Salomon et *al.* (1997). Further the deformed shape shows a visible warping of the cross section ensured by the incorporation of a higher order beam theory, notable at the free edge of the beam.

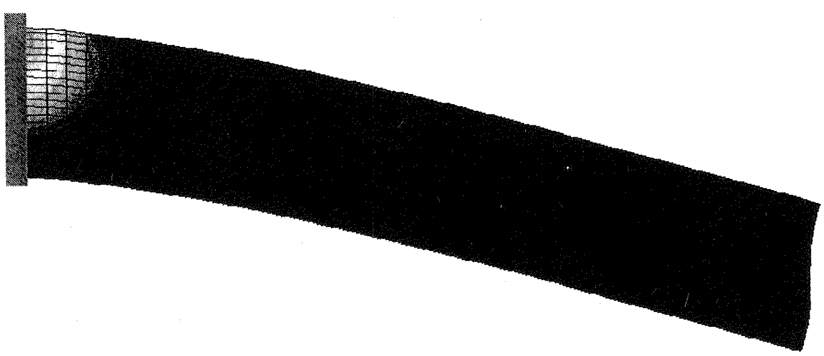

Fig. 2 Cantilever beam : deformed configuration and crack pattern

5 Conclusion

The emphasis here is to establish the credibility of the layered formulation to predict Mode-I failure in deep beams with small aspect ratios, especially when warping is concerned. While the discussion is limited to a particular type of loading and boundary conditions, this theory can be used to tackle any type of loading and boundary conditions making it a promising approach. Compared to the conventional two dimensional finite element model (plane stress and plane strain configurations), the multilayered finite element method, that follows from the beam theory, allows a lower calculation cost and a smaller amount of memory work-load. Further, the uniaxial character of the plastic flow ensures the stability of the iterative process with a quadratic convergence (a maximum of 10 iterations in all the presented examples).

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