Fracture Mechanics of Concrete Structures
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NUMERICAL APPROACH BASED ON THE ENERGY CRITERION IN FRACTURE ANALYSIS OF CONCRETE STRUCTURES

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Abstract

A numerical approach for computational crack analysis of concrete structures is presented, which is based on the energy criterion on crack initiation and extension. The new criterion takes the possible energy loss of each individual crack into account, and has its definition built on the theory of homogenization. The energy balance during a fracturing process is also discussed, and the quantitative relation between the dissipated fracture energy and the change in strain energy immediately after cracking is established. By virtue of this concept, numerical approaches using load-control method become a straightforward process even when the snap-back instabilities are involved. The resulting model has been proved its capability and effectiveness for practical use in simulating the general structural behaviors after cracking, and in predicting the failure loads of the structures which include water tunnels, sewers, etc.
Key words: energy criterion, crack, homogenization

1 Introduction

In fracture analysis of concrete structures, the basic problem in implementing the energy principle is the difficulty in determining the available energy to drive the cracks, which is not straightforward because the energy dissipation due to plasticity coexists. The situation is complicated further by the fact that the smeared crack approaches based on the tensile softening law of concrete often result in the divergence of the solutions, because of the nonconsistency of the basic hypothesis and sharp discontinuities in the resulting material behavior. A practical engineering approach to the above problems is proposed in the following. Topics of discussion include the energy criterion for numerical crack analysis, the energy balance in the process of cracking and computational techniques that have been proved successful.

2 Proposed model

2.1 Energy criterion on crack initiation and extension

As a general practice, the tensile strength of concrete is used as a criterion for crack analysis under the strength of material approach. Convenient as it is though, the strength criterion may not suffice as an appropriate criterion for crack analysis of concrete, because it lacks the energy basis which states that crack extension occurs when the energy available for crack growth is sufficient to overcome the resistance of the material.

As mentioned before, determining the available energy to drive the cracks is not straightforward because the energy dissipation due to plasticity coexists. The source of the problem lies in the fact, as Bazant (1996) puts it, the energy dissipated near the cohesive crack due to inelastic behavior of the material cannot be decoupled from the energy dissipated by plasticity in the crack.

In numerical approaches this difficulty can be avoided, however. As it is known, the crack growth resistance in concrete is due to the development of the fracture process zone and the bridging of the crack by aggregate. Therefore instead of directly estimating the energy release rate from potential energy, the overall rate of possible energy loss with crack area $< g'_f >$ is predicted from the strain softening curve of the FPZ and viewed as the effective driving force for fracture, see Shi *et al* (1998).

Fig. 1(a) shows a stable crack occurred when $< g'_f >= g'_{fc}$, with g'_{fc} being the material's resistance to fracture. The stress-displacement relation for the FPZ is expressed in the stress-strain form as the crack is

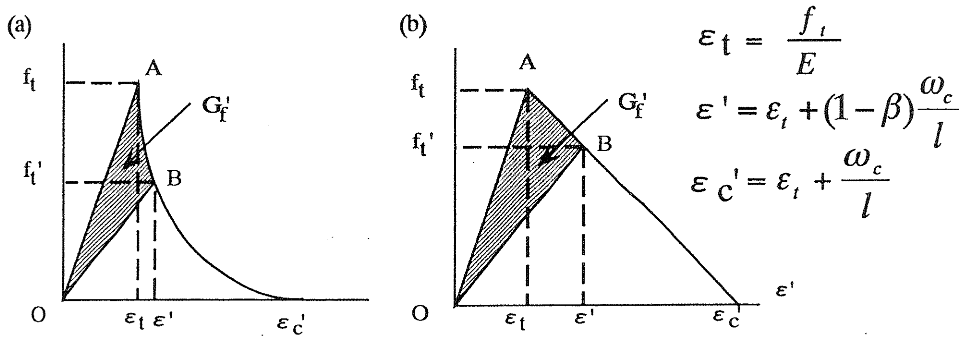

Fig. 1. Energy dissipation due to cracking

smeared inside the effective domain of the stress point along its equivalent length l, which is the length of the domain perpendicular to the crack.

The area G_f' represents the energy dissipated during the current fracturing process at a stress point. Suppose that the extension of the crack with an opening displacement ω stops as the cohesive stress σ drops to f_t', with $f_t' = \beta f_t$ where f_t is the material's original tensile strength and $0 < \beta < 1$. β reflects how fast a crack extends at a certain load level. Numerically, for the first iteration while cracking is taking place, sufficiently small α is given prescriptively and $f_t' = (1-\alpha)f_t$ is defined as a transient tensile strength. Then after stress redistribution, if $\sigma \geq f_t'$, $f_t' = \{(1-\alpha)f_t\}(1-\alpha)$ would be defined again. This process is continued until the calculation comes to converge at the given load level. At the convergence, $\beta = (1-\alpha)^m$, where m may vary among cracked points. Note that the degraded tensile strength due to cracking is irreversible.

Analogous to the definition of the energy release rate, we define the rate of the energy loss g_f' as

$$g_f' = \frac{G_f'}{\Delta\omega} \tag{1}$$

where G_f' is the dissipated energy, $\Delta\omega$ is the incremental crack opening displacement. The overall rate of the energy loss with crack area over V_c, which is the potential region of cracks, is then obtained by the volume average

$$< g_f' > = \frac{1}{V_c}\int_{V_c} g_f' dV = \sum_{\alpha=1}^{N} g_f'^{\alpha} \phi^{\alpha} \quad \text{with} \quad \sum_{\alpha=1}^{N} \phi^{\alpha} = 1 \tag{2}$$

where $g_f'^{\alpha}$ is the rate of energy loss for crack α, and $\phi^{\alpha}(=\frac{V^{\alpha}}{V_c})$ represents its volume fraction with V^{α} being the domain in which $g_f'^{\alpha}$ is assumed constant, and N is the total number of potential cracks. Note that a crack will develop when the tensile stress computed in the local Cartesian coordinate system reaches the corresponding local tensile strength f_t' of the material, and is termed as a potential crack before an actual crack analysis is carried out. Here $< g_f' >$ becomes a kind of index which reflects material and structural resistance as well as mesh characteristics against cracking.

For simplicity, a linear relation between the cohesive stress σ and the apparent strain ε' is assumed (Fig. 1(b)). The energy dissipated during the current fracturing process G_f' is given by

$$G_f' = \Delta OAB = \frac{1}{2}(1-\beta)f_t(\varepsilon_t + \frac{\omega_c}{l}) \tag{3}$$

where ω_c is the limit crack opening displacement, $\varepsilon_t = f_t/E$ and E=Young's modulus. Substituting Eqn. 3 into Eqn. 1 leads to

$$g_f' = \frac{G_f'}{\Delta\omega} = \frac{\Delta OAB}{(1-\beta)\omega_c} = \frac{1}{2}f_t(\frac{\varepsilon_t}{\omega_c} + \frac{1}{l}) \tag{4}$$

Eqn. 4 shows that besides the material properties f_t,E and ω_c, g_f' is dependent upon the element's size, shape and the crack orientation through l, but independent of the current material's tensile strength $f_t'(=\beta f_t)$ as long as a linear softening law for the FPZ is assumed.

As a simple fact, the energy loss defined in Eqn. 3 during a crack extension is purely the energy required for the extension of the crack, excluding the energy dissipation due to plasticity. Thus in the numerical approach, a virtual crack analysis is performed at each load increment so that the overall rate of the energy loss $< g_f' >$ can be predicted and compared with the structural resistance to fracture g_{fc}'. At the moment of fracture, $< g_f' > = g_{fc}'$, and the actual crack analysis is then executed.

Regarding to the structural resistance to fracture g_{fc}', a logical approach is $g_{fc}' = g_f'^{initial} \cdot \kappa$ with $g_f'^{initial}$ being the rate of the energy loss when the initial crack of the structure develops. The reason for this is that the creation of the initial crack does not involve the interactions of cracks, therefore $g_f'^{initial}$ should reflect purely the structural resistance to cracking.

And κ is a structural coefficient which reflects the degradation of the structure during fracture. Therefore the crack growth resistance g'_{fc} is not a constant during the entire process of fracture analysis. In this study, $\kappa = |K|/|K^0|$, $|K|$ and $|K^0|$ are determinants of the present stiffness and the initial stiffness matrices of the structure respectively.

2.2 Balance between fracture energy and strain energy

Abrupt load drop simultaneous to the occurrence of stable cracks is a well-observed phenomenon of various kinds of fracture tests on concrete specimens. This is, of course, due to the dissipation of fracture energy by each individual crack which in turn precipitates the loss of the structure's strain energy. According to the principle of energy conservation, the total loss of fracture energy must be balanced by the same amount of strain energy loss in the system. Use of this fact can make the numerical approach much easier.

For simplicity, consider only a single concentrated load acting on a concrete structure. Fig. 2 shows the load-deflection curve. The shaded area *OABC* represents the total strain energy loss when stable cracks occur or extend at the load level P. Of which a small fraction is caused by the material's plasticity. Subtracting this portion from *OABC*, the strain energy loss caused solely by cracking ΔU_f is then given by

$$\Delta U_f = \frac{1}{2}(u_B - u^p)\Delta P \qquad (5)$$

Here u^p is the residual displacement due to plasticity, and ΔP is the load decrease caused by cracking. The energy dissipation W_f during the

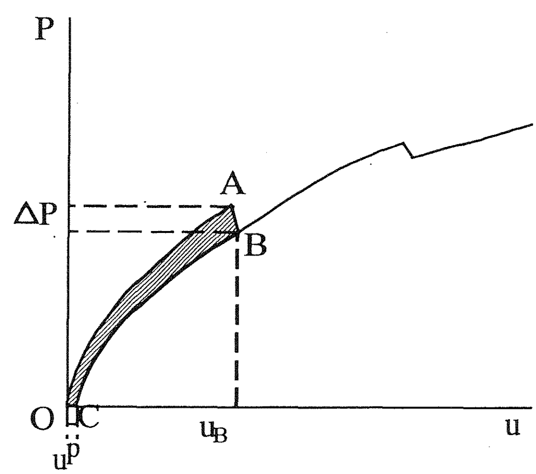

Fig. 2. Loss of strain energy due to cracking

current process of fracturing is given by integrating G_f' of Eqn. 3 over the entire fractured area V_c:

$$W_f = \int_{V_c} G_f' dV = \sum_{\alpha=1}^{N} G_f'^{\alpha} V^{\alpha} \text{ with } \sum_{\alpha=1}^{N} V^{\alpha} = V_c \qquad (6)$$

Equalizing the strain energy loss ΔU_f with the energy dissipation W_f leads to Eqn. 7.

$$\Delta U_f = W_f \qquad (7)$$

The new unstable equilibrium position B after cracking (see Fig. 2) is obtained numerically by computing Eqn. 5 for a number of small negative load increments until Eqn. 7 is satisfied. Sum of these small negative load increments yields the load decrease ΔP. By virtue of Eqn. 7 numerical approaches using the load-control method become a straightforward process even when the snap-back instabilities are involved, whereas more complicated methods such as the arc-length control method usually have to be employed in these situations.

2.3 General framework of the present approach

The elasto-plastic behavior of concrete is analyzed using the classical theory of plasticity, and the Drucker-Prager yield criterion is adopted. The uniaxial stress-strain relation of concrete in compression proposed by the Design Standard for Concrete of JSCE is employed as the plastic deformation. The incremental elasto-plastic stress-strain relation is then built upon the work or strain hardening rule and the normality rule, which postulates the direction of plastic flow.

For concrete in tension, the model is based on the smeared crack approach proposed by Dahlblom *et al* (1990). This approach is attractive, especially among practical engineers because only the constitutive relation in the element region of interest needs to be updated after cracking. To avoid the divergence of the solutions in practical applications, we propose the $E' - \omega$ relation to be used directly in the constitutive relation along with the tensile-softening law, so that the decreasing of concrete rigidity due to cracking can be expressed as an irreversible process, as it is in reality.

The specific formulation of the constitutive equation for cracked elements is omitted in this paper, because it is a straightforward process. A special scheme of solution should be elucidated however, because of the inconsistency of theories involved in the formulation of the problem, i.e., the incremental strain theory for elasto-plastic analysis and the total strain theory for crack analysis. In the numerical approach, instead of directly calculating the displacement increment for each load increase, the total displacement is obtained based on the initial stiffness matrix of the system K_0, which is constantly modified through E' whenever a crack develops. The difference between the system's initial position and the previous converged position before load increment then yields the incremental displacement which is required in the subsequent elasto-plastic analysis. Because of this special scheme of solution, the total displacement at a given load increment has to be evaluated in a unique way. As illustrated in Fig. 3, the total displacement at the start of the convergence iteration is evaluated as

$$u_i^{\text{int}} = u_i^{K_0} + (u_{i-1} - u_{i-1}^{K_0}) \tag{8}$$

where u_i^{int} is the initial displacement, $u_i^{K_0}$ is a reference displacement evaluated by the initial stiffness K_0, and u_{i-1} is the total displacement converged at the previous load level. Fig. 4 shows the flow chart for crack analysis, and Fig. 5 illustrates the main steps used in the present approach.

In this study, reinforcing steel is treated as one-dimensional element, neglecting its shear stiffness. The stress-strain curve of reinforcing steel is assumed to be elastic perfectly plastic identical in tension and compression. The bond between steel and concrete is supposed to remain perfect.

3 Comparison with experimental results

The validity of the developed approach was examined from the analyses of available experimental data on flexural tests of both plain concrete and reinforced concrete beams.

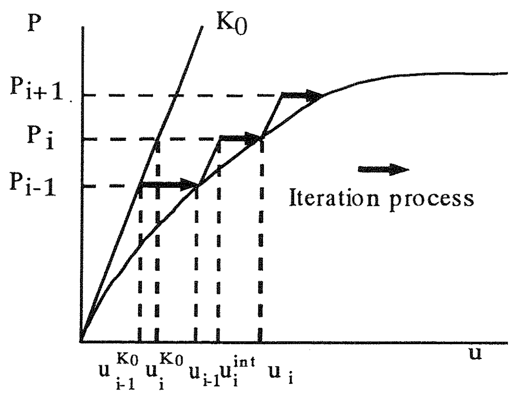

Fig. 3. Total displacement at a given load

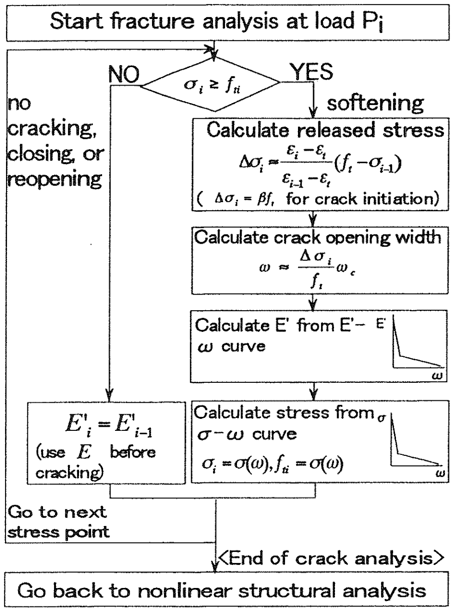

Fig. 4. Flow for crack analysis

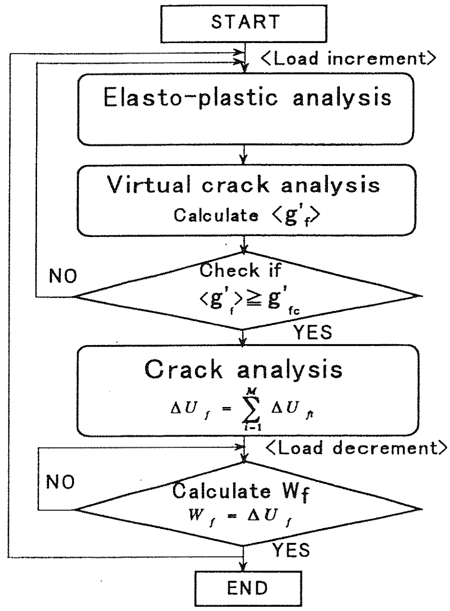

Fig. 5. Flow for nonlinear analysis based on energy approach

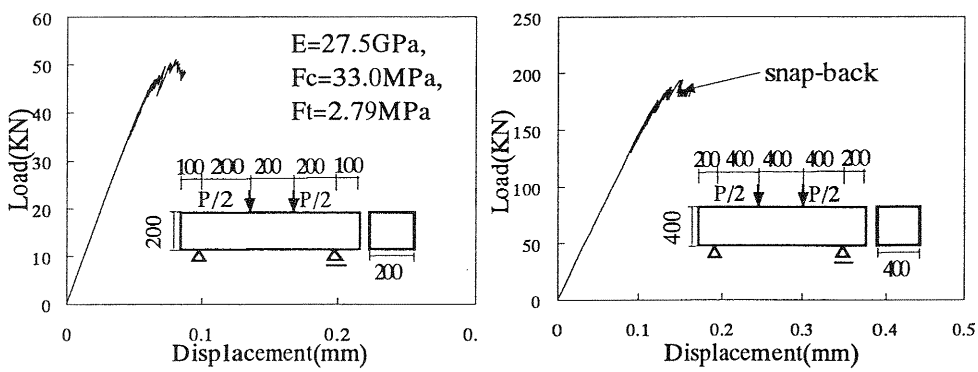

Fig. 6. Load-displacement relation

3.1 Fracture test of concrete beam

Experimental results for plain concrete beams involved data in reference to the results of round robin flexural tests of concrete beams, see Report of Committee of Application of Fracture Mechanics to Concrete Structures (1993). Fig. 6 shows the load-displacement relations. The loads at crack initiation and failure are reasonably agreed with the experimental results. Though the structural response of plain concrete beams is rather unstable after the critical load, the proposed approach using the load-control method can still trace this unstable behavior up to a certain point.

3.2 Fracture test of reinforced concrete beams

For reinforced concrete beams, the study involved a series of experiments, which were designed to investigate the load carrying capacity of aging sewers and the effectiveness of several rehabilitation methods, see Haibara *et al* (1997). The results of numerical analyses on two test cases are outlined below.

The analytical models are shown in Fig. 7 and the material properties are listed in Table 1. While case 1 represents a sound beam, case 2 shows a beam whose tensile steels are exposed to simulate the situation when the concrete cover is eroded due to the hostile environment of sewers. The load-displacement relations for case 1 and case 2 are shown respectively in Fig. 8 and Fig. 9. Agreement between the numerical solutions and the experimental results is deemed reasonable, especially regarding to the initial cracking load, the maximum load carrying capacity and the progressively weakened structural stiffness in the process of fracturing. As regards to the reinforcement, yielding started with tensile bars at the center and at the loading point. After that, a slight increase of load caused the concrete in compression reaching its strain limit, symbolizing the compressive failure of concrete. Due to the use of multidirectional fixed crack approach, the crack patterns of numerical solution show certain discrepancies with those observed in the tests. This situation might be improved if the rotating crack approach is adopted.

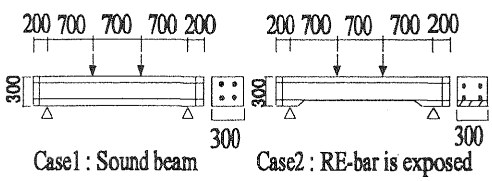

Fig. 7. Analytical models

Table 1. Material properties

CONCRETE			RE-BAR	
E (kgf/cm^2)	ν	fc (kgf/cm^2)	E (kgf/cm^2)	σy (kgf/cm^2)
2.54x10^5	0.186	215.3	2.00x10^6	3570.0
ft (kgf/cm^2)	ωc (mm)	εu		
29.5	0.1	0.0035		

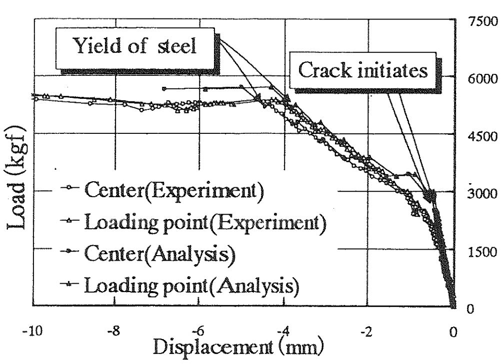

Fig. 8. Load -displacement relation (case 1)

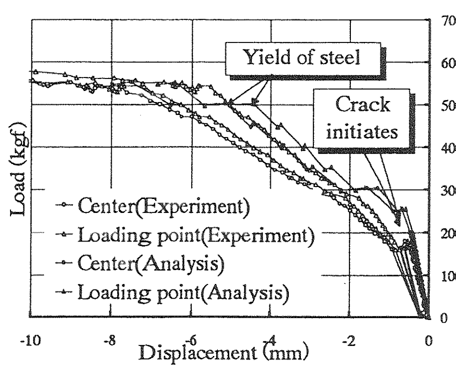

Fig. 9. Load-displacement relation (case 2)

4 Conclusions

A numerical approach for computational crack analysis of concrete structures is proposed, which is based on the overall rate of the energy loss per unit crack area. The validity of the criterion has been proved by early numerical studies, though further verification is still necessary. Next, the energy balance during a fracturing process is discussed and an equation is built for the dissipated fracture energy and the decrease in strain energy after cracking. Based on this relation, numerical approaches using load-control method become a straightforward process even when the snap-back instabilities are involved. The resulting model has been proved successful, in the sense that the finite element analyses do not diverge prematurely, and is capable of predicting plain concrete as well as reinforced concrete behaviors with reasonable accuracy as have been shown by a number of analyses on concrete beams.

5 References

Bazant, Z. P. (1996), Analysis of work-fracture method for measuring fracture energy of concrete, **Journal of Engineering Mechanics**, ASCE, 122(2), 138-144.

Dahlblom, O. and Ottosen, N. S. (1990),Smeared crack analysis using generalized fictitious crack model, **Journal of Engineering Mechanics**, ASCE, 116(1), 55-76.

Haibara, T., Nakatsui, K., Ito, M. and Nakatsubo, Y. (1997), Evaluation of load carrying capacity of aged RC sewers, **Proceedings of the 52th Annual Conference of JSCE,** 6, 576-577.

Shi, Z. and Nakano, M. (1998), Fracture analysis of concrete structures by fracture energy approach**, Proceedings of Structural Engineers World Conference, San Francisco, USA.**

Uchida, Y., Rokugo, K. and Koyanagi, W. (1993), **Report of Committee of Application of Fracture Mechanics to Concrete Structures**, JCI, 346-347.