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STRESS-STRAIN RELATION FOR ELASTIC MATERIAL WITH MANY GROWING MICROCRACKS

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Abstract

The studies of the elastic properties of materials with many cracks utilizing theories of composites such as the self-consistent scheme or differential scheme have been confined to the case when the cracks neither grow nor shorten during loading. The present paper, extending the recent studies of Prat and Bažant adapts the theory of stiffness of composites to the case of incremental deformations during which the cracks can grow or shorten. Several families of cracks of different sizes, densities and orientations, characterized by the crack density tensor, are considered. The equality of the energy release rate to the fracture energy of the material is enforced only globally for all the cracks in each family. The theory yields a macroscopic stress-strain relation that appears to give qualitatively good predictions for the response of concrete in basic types of tests. Calculation results are not yet available but are planned for presentation at the conference.
Key Words: Cracked solid, crack propagation, crack density, fracture mechanics, strain-softening, tangent stiffness of composites.

1 Introduction

To achieve and improvement in the constitutive modeling of concrete, the microstructural physical processes of inelastic behavior will have to be reflected in the model. A step in this direction has been the microplane model (Bažant and Oh 1985, Bažant et al. 1996), which allows separating the inelastic phenomena on planes of various orientation in the material. However, the necessity of a kinematic or static constraint in the microplane model is limiting, and fracture mechanics of microcracks (henceforth called cracks) cannot be incorporated into the model.

In a recent study, Prat and Bažant (1997) attempted to use theories of composite materials such as the self-consistent scheme of Hill (1965) and Budianski (1965) or the differential scheme (Hashin 1988) which were applied to materials weakened by cracks by Budianski and O'Connell (1976), Hoenig (1979), Kachanov (1992), Sayers and Kachanov (1995) and others. These studies have led to good models for predicting the compliance or stiffness tensor of the cracked materials from the crack sizes and density. However, these models implied the cracks not to propagate during loading, which meant the stiffness tensor obtained was a secant stiffness tensor (Fig. 1).

Prediction of the tangent stiffness tensor (Fig. 1) requires that the cracks be allowed to propagate during loading, and so the energy criterion of crack propagation must be incorporated into the model. Based on an idea sketched earlier by Bažant and Prat (1995), this has been done by Prat and Bažant (1995) who also devised a method to match the formulation to predictions of the composite material theory and introduced a computational scheme utilizing some convenient features of the microplane model. However, their energy criterion of crack propagation as well as some other features were inconsistent or oversimplified, which was pointed out in their Addendum (Prat and Bažant 1997) and a revised improved formulation was briefly outlined. The purpose of this paper is to present this revised formulation in detail. The numerical computations with this formulation are still in progress at the time of writing but are expected to be presented at the conference and are planned for a subsequent journal article.

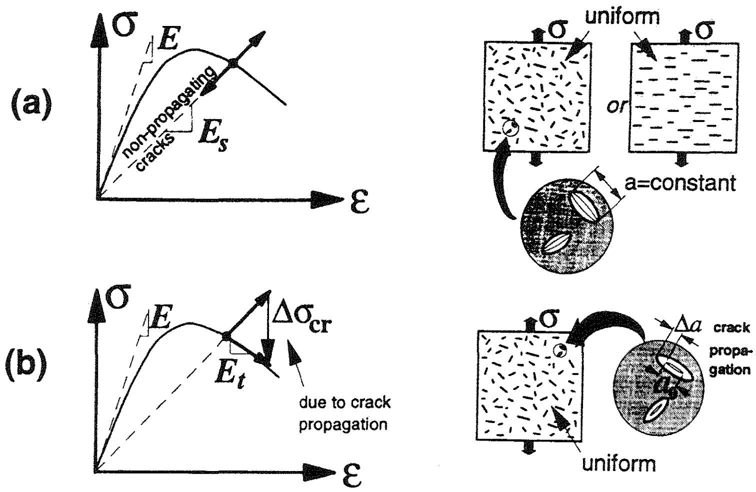

Fig. 1 (a) Secant moduli predicted by classical theories and (b) inelastic stress decrements due to crack growth, leading to tangent moduli.

2 Effect of Crack Growth on Compliance Tensor

We consider an elastic material intersected by N families of random circular microcracks (henceforth called cracks) of different orientations. The radius a_μ of the cracks of each family ν ($\nu = 1, 2, ...N$) is negligible compared to structure dimensions, and so the cracks on the macroscale may be considered as smeared, their only effect being inelastic strains and changes of the secant compliance tensor $\boldsymbol{C}$ of the macroscopic continuum. This tensor is a function of the crack radii and the number of cracks n_μ in each family, i.e., $\boldsymbol{C} = \boldsymbol{C}(a_1, a_2, ...a_N; n_1, n_2, ...n_N)$. Approximate expressions for this tensor can be obtained by various method for composites; see Kachanov (1992, 1993), and Sayers and Kachanov (1991). By differentiating the secant elastic relation $\boldsymbol{\epsilon} = \boldsymbol{C} : \boldsymbol{\sigma}$ where *eeps* and $\boldsymbol{\sigma}$ are the strain and stress tensors of Cartesian components ϵ_{ij} and σ_{ij}. By differentiation, the strain increment tensor may be expressed as

$$\Delta\boldsymbol{\epsilon} = \boldsymbol{C} : \Delta\boldsymbol{\sigma} + \sum_\mu \frac{\partial \boldsymbol{C}}{\partial a_\mu} : \boldsymbol{\sigma}\ \Delta a_\mu \tag{1}$$

This may be inverted as

$$\Delta\boldsymbol{\sigma} = \boldsymbol{E} : \left(\Delta\boldsymbol{\epsilon} - \sum_{\mu=1}^{N} \frac{\partial \boldsymbol{C}}{\partial a_\mu} : \boldsymbol{\sigma} \ \Delta a_\mu \right) \tag{2}$$

where $\boldsymbol{E}$ is the fourth-order secant stiffness tensor whose 6×6 matrix is the inverse of the matrix of the secant compliance tensor $\boldsymbol{C}(a_1, ...a_N)$.

The complementary energy per unit volume of the material may be written as

$$\Pi^*(\boldsymbol{\sigma}, a_1, ...a_N) = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{C} : \boldsymbol{\sigma} \tag{3}$$

As is well known from fracture mechanics (e.g. Bažant and Planas 1998), the energy release rate (per unit volume of the material) with respect to the growth of a_μ is

$$\mathcal{G}_\mu = \left[\frac{\partial \Pi^*}{\partial a_\mu} \right]_{\boldsymbol{\sigma}=\text{const.}} \tag{4}$$

The energy balance condition of equilibrium crack propagation requires that

$$\mathcal{G}_\mu = 2\pi a_\mu n_\mu G_f \tag{5}$$

where G_f = fracture energy of the material. If $0 < \mathcal{G}_\mu < 2\pi a_\mu n_\mu G_f$, cracks a_μ cannot propagate, and they cannot shorten either. They can shorten if and only if $\mathcal{G}_\mu = 0$. Thus, for each crack family we may write the following crack propagation condition:

$$\frac{1}{2}\boldsymbol{\sigma} : \frac{\partial \boldsymbol{C}}{\partial a_\mu} : \boldsymbol{\sigma} = 2\pi n_\mu \kappa_\mu G_f a_\mu \tag{6}$$

where we have introduced for convenience the crack growth indicator κ_μ such that

$$\begin{aligned} \kappa_\mu &= 1 \quad \text{for} \quad \Delta a_\mu > 0 \\ 0 \le \kappa_\mu &\le 1 \quad \text{for} \quad \Delta a_\mu = 0 \\ \kappa_\mu &= 0 \quad \text{for} \quad \Delta a_\mu < 0 \end{aligned} \tag{7}$$

The energy balance criterion expressed by (6) in enforced only globally for all the cracks in each family rather than for each crack individually. Using this criterion, we may bring the incremental energy balance condition for each of $\mu = 1, 2, ...N$ to the form:

$$\boldsymbol{\sigma} : \frac{\partial \boldsymbol{C}}{\partial a_\mu} : \Delta\boldsymbol{\sigma} + \sum_{\nu=1}^{N} \left(\frac{1}{2}\boldsymbol{\sigma} : \frac{\partial^2 \boldsymbol{C}}{\partial a_\mu \partial a_\nu} : \boldsymbol{\sigma} - 2\pi n_\mu \kappa_\mu G_f \delta_{\mu\nu} \right) \Delta a_\nu$$
$$= 2\pi n_\mu a_\mu G_f \Delta\kappa_\mu \tag{8}$$

where the crack indicator jump $\Delta\kappa_\mu = 0$ except when the cracks are stationary, i.e., when the growth of cracks a_μ is changing to crack closing or vice versa. For a finite jump $\Delta\kappa_\mu$, Eq. (8) is exact if $\kappa_\mu = \kappa_\mu^{new}$ and $a_\mu = a_\mu^{old}$. This is clear from the fact that

$$\begin{aligned} \Delta(\kappa_\mu a_\mu) &= (\kappa_\mu a_\mu)^{new} - (\kappa_\mu a_\mu)^{old} \\ &= \kappa_\mu^{new} \Delta a_\mu + \Delta\kappa_\mu a_\mu^{old} \end{aligned} \tag{9}$$

Upon substituting (2) into (8), we obtain a system of N equations for N unknowns $\Delta a_1, ...\Delta a_N$:

$$\sum_{\nu=1}^{N} A_{\mu\nu} \Delta a_\nu = B_\mu \qquad (\mu = 1, ...N) \tag{10}$$

where

$$A_{\mu\nu} = \boldsymbol{\sigma} : \left(\frac{\partial \boldsymbol{C}}{\partial a_\mu} : \boldsymbol{E} : \frac{\partial \boldsymbol{C}}{\partial a_\nu} - \frac{1}{2} \frac{\partial^2 \boldsymbol{C}}{\partial a_\mu \partial a_\nu} \right) : \boldsymbol{\sigma} + 2\pi G_f n_\mu \kappa_\mu^{new} \delta_{\mu\nu} \tag{11}$$

$$B_\mu = \boldsymbol{\sigma} : \frac{\partial \boldsymbol{C}}{\partial a_\mu} : \boldsymbol{E} : \Delta\boldsymbol{\epsilon} - 2\pi n_\mu a_\mu^{old} G_f \Delta\kappa_\mu \tag{12}$$

These equations need to be further modified by introducing checking whether the crack might be under compression.

A check for compression is necessary because the energy expression is quadratic, which means that Eq. (6) is invariant when $\boldsymbol{\sigma}$ is replaced by $-\boldsymbol{\sigma}$. So, the stress intensity factor $K_{I\mu}$ can have a negative value even when (2) is satisfied. Hence the sign of $K_{I\mu}$ must be checked for each crack family μ. The case $K_{I\mu} < 0$ is inadmissible.

Our formulation involves only the values of $(K_{I\mu})^2 = E\mathcal{G}_\mu$, but not the values of $K_{I\mu}$. Consequently, the sign of $K_{I\mu}$ must be inferred approximately. There are two possibilities:

- As one approximation, one can consider this sign to be the same as the sign of the stress component in the direction normal to the cracks of the μ-th family, i.e. of

$$\sigma_{N\mu} = \boldsymbol{\nu}_\mu \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_\mu \tag{13}$$

where $\boldsymbol{\nu}_\mu$ = unit vector normal to the cracks. This normal component can be easily determined.

- As an alternative approximation with a clearer physical justification, one can infer the sign of $K_{I\mu}$ from the sign of the normal component of the cracking strain tensor $\boldsymbol{\epsilon}^{cr}$, i.e.

$$\epsilon^{cr}_{N\mu} = \boldsymbol{\nu}_\mu \cdot \boldsymbol{\epsilon}^{cr} \cdot \boldsymbol{\nu}_\mu, \qquad \boldsymbol{\epsilon}^{cr} = \boldsymbol{C}^{cr} : \boldsymbol{\sigma} \tag{14}$$

  An expression for the cracking compliance tensor $\boldsymbol{C}^{cr}$ was given in equation (28) of Bažant and Prat (1997).

Inexact though such estimations of the sign of $K_{I\mu}$ must be (for the microscale), they are nevertheless consistent with the macroscopic approximation of $\boldsymbol{C}$. The reason is that all the composite material models for cracked solids are based on the solution of one crack in an infinite solid, for which the sign of $K_{I\mu}$ coincides with the sign of $\sigma_{N\mu}$ (or $\epsilon^{cr}_{N\mu}$).

In each loading step of an explicit finite element program, the six independent components of $\Delta\epsilon$ are known. Eq. (10) then represents a separate system of only N equations for $\Delta a_1, ...\Delta a_N$. The values of κ_μ are set according to the sign of Δa_μ in the preceding step or preceding iteration. If $\Delta a_\mu = 0$, or if (due to numerical error) $|\Delta a_\mu|$ is nonzero but less than a certain chosen very small positive number δ, the exact implementation gets complicated but, for the sake of simplicity, one may perhaps arbitrarily set $\kappa_\mu = 0.5$ without affecting the solution significantly.

After solving (10), we may evaluate $\Delta\boldsymbol{\sigma}$ from (1). Then, after incrementing $\boldsymbol{\sigma}$, we must check whether the case $\Delta a_\mu > 0$ and $\sigma_{N\mu} < 0$ (or $\epsilon^{cr}_{N\mu} < 0$) occurs for any μ. If it does, the corresponding equation in the equation system (10) must be replaced by the equation $\Delta a_\mu = 0$. The solution of such a modified equation system must be iterated until the case $\Delta a_\mu > 0$ and $\sigma_{N\mu} < 0$ (or $\epsilon^{cr}_{N\mu} < 0$) would no longer occur for any μ (difficulties with the convergence of these iterations have often been encountered).

3 Compliance Tensor

The rest of the present formulation remains the same as described in detail by Prat and Bažant (1997) and summarized in Bažant and

Planas (1998, p. 564). To evaluate the secant compliance $\boldsymbol{C}$ from the number of cracks and their radii, Sayers and Kachanov's (1991) technique is used. In this technique, $\boldsymbol{C}$ is assumed to be derived from an elastic potential F of the cracked material. This potential is assumed to be a quadratic tensor polynomial in $\boldsymbol{\sigma}$ and a linear tensorial function of the crack density tensor $\boldsymbol{\alpha} = \sum_{\mu} n_{\mu} a_{\mu}^{3} \boldsymbol{\nu} \otimes \boldsymbol{\nu}$.

This yields for $\boldsymbol{C}$ a quadratic tensor polynomial with two scalar parameters η_1 and η_2 depending on $\boldsymbol{\alpha}$. Their dependence is calculated by considering the case of isotropic cracking for which the effective secant modulus and effective Poisson ratio must match those calculated for random cracks by some composite material method, for which the self-consistent method (used by Budianski and O'Connell, 1976) and the differential scheme have been considered. The latter performed more reasonably in the softening range and has therefore been adopted for the calibration of these parameters.

4 Computations Using Integration Scheme of Microplane Model

The crack density is described by a continuous function n_{μ} of spherical angles ϕ and θ. In numerical computations, this function is sampled at spherical discrete points $(\phi_{\mu}, \theta_{\mu})$ corresponding to the orientations of microplanes in the microplane model (Bažant and Oh 1985, 1986).

Although at the time of writing the computations are still in progress, the present model appears to give realistic response curves for the uniaxial, biaxial and standard triaxial tests on concrete. The response curves are quite similar to those published by Prat and Bažant (1997) which were obtained by a previous simpler version of the model with an energy inconsistent crack propagation criterion. For reader's convenience some of these curves are reprinted in Fig. 2 (in which f_c' = compression strength of concrete, ϵ_1 = principal strain magnitude, σ_1, σ_2 = principal stress magnitudes, and σ_c = lateral confining stress magnitude). The exact curves corresponding to the present model (which are intended for presentation at the conference) will be published in a forthcoming journal article.

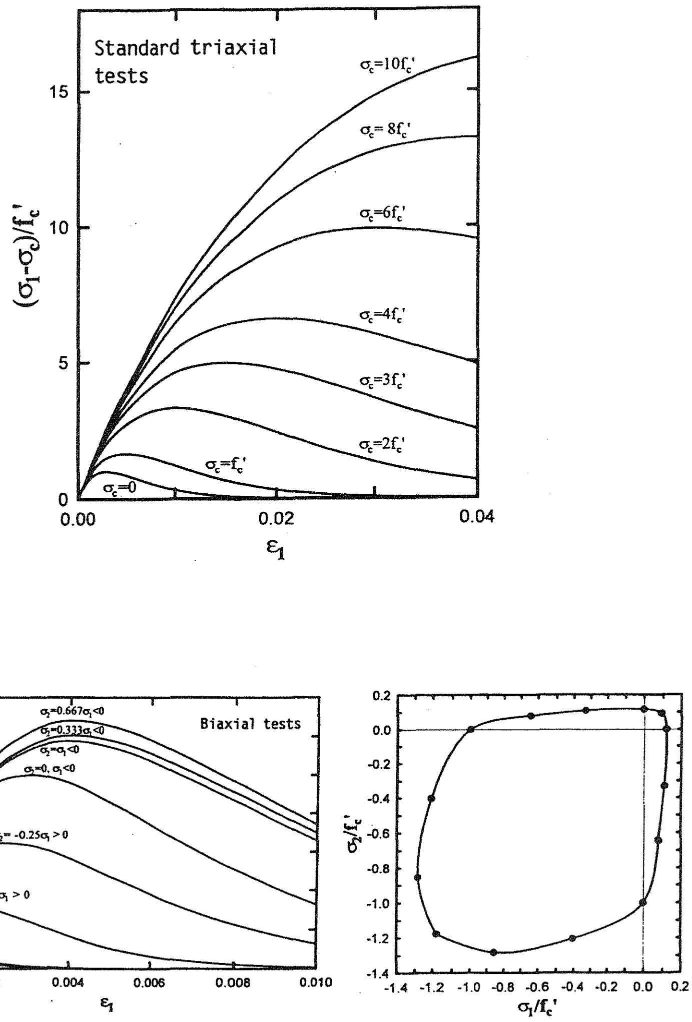

Fig. 2 Response curves calculated with a previous simplified version of the model for the standard triaxial tests and biaxial tests, and corresponding biaxial failure envelope (after Prat and Bažant 1997).

5 Concluding Comment

Modeling of concrete on the basis of the composite material theories for cracked solids appears to give realistic results. It offers the promise of becoming a general constitutive model for concrete with a better physical foundation than the classical plasticity or damage models based on tensorial invariants as well as the microplane model. However, plasticity on the microscale would probably need to be incorporate into the model.

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