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SIZE EFFECT IN SHEAR FAILURE OF REINFORCED CONCRETE BEAMS

Z.P. Bažant and E. Becq-Giraudon
Department of Civil Engineering, Northwestern University,
Evanston, Illinois 60208, U.S.A.

Abstract
The idea that the diagonal shear failure of reinforced concrete beams without or with stirrups exhibits a strong non-statistical size effect has been generally accepted in the fracture research community since the 1980's and during the last few years is gaining ground among design engineers as new evidence from properly scaled tests of large beams is emerging. Several competing theories to explain this size effect have been advanced. After mentioning the problems with some theories, the paper reviews the energetic theory of size effect and its recent application to the classical truss model (or strut-and-tie model). The formulae previously derived by energy analysis of the fracturing truss model are compared with numerous test data available in the literature, and a good agreement is demonstrated.
Key Words: RC beam, shear, failure, fracture, size effect, design.

1 Introduction and Overview of Current Status

The diagonal shear is a quintessential type of brittle failure of reinforced concrete beams, which has been extensively tested over the last fifty years in several hundred laboratories around the world (Bažant and Kim 1984, Bažant and Sun 1987), with a total expenditure that must have exceeded five hundred million in current dollars. Great advances of understanding have been achieved and incorporated in the design codes.

However, one salient feature—the size effect—received relatively little attention in this experimental research, perhaps because of overconfidence in the plastic limit state theory which implies no size effect, and because of the conviction (today known erroneous) that any size effect must be statistical in nature. Among about five hundred data sets available in the literature, only about twenty explored the size effect and, unfortunately, strict geometric scaling has not been followed until very recently, making separation of the size effect from other influences ambiguous. Thus it is not surprising that the size effect is still either ignored in the code specifications (as in ACI) or is dealt with in a questionable manner (as in CEB and JSCE).

There has nevertheless been a major positive change in attitude in regard to size effect during the last several years. Not only the research community in concrete fracture but also many engineers in the code-making committees world-wide came to agree that a significant (non-statistical) size effect does exist, that there is credible and extensive experimental evidence for it. Currently a wide-spread feeling that the size effect ought to be somehow reflected in code specifications has emerged. The question no longer is if, but how. But in that respect there is, unfortunately, little agreement.

The code specification in Japan, calibrated by the largest-scale tests ever made (Shioya et al. 1989), follow the most classical theory of size effect—the statistical theory of strength randomness originated by Mariotte (1686) and theoretically completed by Weibull (1939). However, whereas this theory has been very successful for fatigue-embrittled metals, its assumptions are not valid for reinforced concrete, for several reasons (the fact that the structure fails only after a large stable crack growth, the fact that concrete is a material possessing a 'material length', etc.; Bažant and Chen 1997, Bažant and Planas 1998).

The current CEB specification use a formula identical the so-called 'multifractal' scaling law proposed by Carpinteri on the basis of his idea that the size effects observed in experiments should be explained by the fractal nature of crack surface roughness and of microcrack distribution. However, this original and revolutionizing idea has been supported only by vague arguments which have been solely geometric in nature, while recent mechanical analysis of Bažant (1997b) reveals inconsistencies, leading to the conclusion that the hypothesis of fractality as a source of size effect cannot be valid.

Among several other proposals, one that is being advocated for ACI code specifications is that of Collins et al. (1996), which attempts to explain the size effect by a dependence of the width of diagonal cracks on the beam size. Yet this theory, too, runs into serious problems (Bažant 1997a).

The objectives of the present paper are (1) to briefly review the theory that explains the size effect in diagonal shear failure by energy release into a localized and propagating failure zone (Bažant and Kim 1984, Bažant and Sun 1987, Bažant 1997a), and (2) to present experimental evidence for this theory. This energetic theory of quasibrittle size effect, initiated by Bažant (1976, 1984) and based on a consistent theoretical framework, has had considerable success for many types of quasibrittle failures, not only for concrete but also for rocks, composites, ice and ceramics, and so far has not into any unanswerable fundamental theoretical problems (Bažant and Planas 1998).

Although generalization would be easy, attention will be restricted to rectangular cross sections because only for those there is extensive experimental evidence. The nominal shear strength of the beam is defined as $v_u = V_u/bd$ where V_u is the shear force at ultimate load, b and d are the beam width and depth (from top face to centroid of longitudinal steel bars). The study deals with a simply supported beam with two symmetric concentrated loads P, for which the shear span a is defined as the distance between the load and the support reaction.

2 Energetic Size Effect in Fracturing Truss Model

The latest version of the energetic theory of size effect in diagonal shear represents an extension of the classical truss model (also called the strut-and-tie model). The diagonal shear failure begins by formation of inclined tensile cracks in concrete. These cracks, however, develop before the maximum load and do not control the value of maximum load (i.e. do not control failure, or stability loss, under dead load). It is assumed that the diagonal shear crack at maximum load have the principal stress direction, and that the principal tensile stresses (cohesive stresses) bridging the diagonal shear cracks at maximum load are negligible compared to the compressive principal stresses parallel to cracks carried by the so-called 'compression struts'.

Assuming the reinforcement to be designed strong enough, the truss can fail only in the compression strut. So the failure necessarily starts as a compression failure (although, during post-peak deflections, it may evolve into what looks at the end as a shear failure). The classical plastic limit analysis could be valid only if the compression strut failed simultaneously everywhere (and if the load-deflection

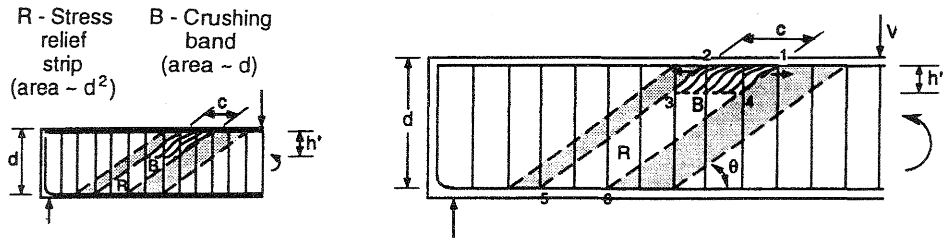

Fig. 1 Energy release zone in similar beams of various sizes.

diagram exhibited a horizontal yield plateau). But this is not the case.

Owing to the strain-softening character of concrete, the material failure must concentrate into a small zone within the strut (e.g., zone 12341 in Fig. 1) and must propagate across the cross section of the strut. At maximum load the failure zone crosses only a part (of length c) of the compression strut width. Such a kind of propagating (progressive) failure necessitates energy analysis, which calls for the use of fracture mechanics.

The necessity of an energetic size effect can be made clear by Fig. 1 even without any calculations. Compression of the strut produces axial splitting cracks (with material buckling) with zone 12341 whose depth h' is essentially a material property governed by the maximum aggregate size (that the zone is pictured at the top of the strut and that it propagates horizontally is unimportant; if it were located elsewhere and propagated vertically or in an inclined direction, the conclusions would be similar). Formation of this failure zone reduces the stress in the blank strip 12561 extending (because of preexisting diagonal cracks) over the whole length of the strut. When the beam is scaled, the width of this strip is scaled in the same proportion, and since the energy release is proportional to area 12561 of the strip, it is proportional to the beam size squared, or to $v_u^2 d^2$. On the other hand, the energy consumed by the formation of the axial splitting cracks in the compression failure zone is proportional to the area $h'd$ of this zone (zone 12341), which is proportional to d rather than d^2. The mismatch—the energy release increasing with the size as d^2 and the energy consumed increasing as d—must obviously cause size effect.

Even if the other competing theories of size effect have some merit (which they might), they cannot ignore this energetic source of size effect, which operates inevitably. They would have to be regarded as a secondary source of size effect, additional to the energetic one.

It is important to realize that *there is no limit state* (Bažant and Kim 1984), i.e., a state at which the material strength would be mobilized simultaneously along the entire failure surface corresponding to a strictly kinematic failure mechanism, which is the central hypothesis of plastic limit analysis underlying the current code specifications. A kinematic failure mechanism, seen at the end of laboratory tests, develops only in the post-peak softening regime, after the load has been reduced to a small value.

Since the failure band width is roughly size independent, it becomes more localized relatively to the beam size if the beam is larger, and less localized if it is smaller. For very small sizes, the failure zone in the compression strut occupies nearly the beam depth, and in that case the material strength is mobilized almost simultaneously over the entire failure surface. That is why the test results for small beams seem to follow plastic limit analysis relatively well and do not reveal appreciable size effect.

3 Size Effect in Reinforced Beams Without Stirrups

The energy analysis of the truss model (Fig. 2) has led to the formula (Eqs. 9–11 in Bažant 1997):

$$v_u = v_p \left(1 + \frac{d}{d_0}\right)^{-1/2} \tag{1}$$

where

$$d_0 = w_0 \frac{d}{c}; \qquad v_p = c_p K_c \left(\frac{a}{d} + \frac{d}{a}\right)^{-1} \tag{2}$$

$$K_c = \sqrt{E_c G_f}; \qquad c_p = k \sqrt{\frac{2h_0}{w_0 s_c} \frac{c/d}{a/d}} \tag{3}$$

Here c = length of the compression failure band at maximum load, which may be considered to be roughly proportional to the beam depth; G_f, K_c, E = fracture energy, fracture toughness and Young's modulus of concrete; h_0 = maximum failure band width (or length of compression splitting cracks), s_c = typical spacing of the splitting cracks; and w_0 = positive constant. Eq. (1) coincides with that proposed without recourse to the truss model by Bažant and Kim (1984) and has the usual form of the size effect law.

Eq. (1) has been fitted to the test data of Leonhardt and Walther (1962), Rüsch et al. (1962), Kani (1967), Bahl (1968), Taylor (1972), Chana (1981), Bažant and Kazemi (1991), Shioya et al. (1989), Walraven (1978), and Walraven and Lehwalter (1994), and Kim and Park's data (1994). The values of parameters d_0 and v_p were optimized for each of these data by optimum fitting in the plot of $\log v_u$ versus $\log d$. This has been done by means of he standard library subroutine for Levenberg-Marquardt nonlinear optimization algorithm.

An alternative way to fit the data and optimize the values of parameters d_0 and v_p is to convert Eq. (1) to a linear regression equation $Y = AX + C$ in which

$$X = 1/d, \quad Y = 1/v_u^2, \quad A = C/d_0, \quad C = 1/v_p^2 \tag{4}$$

The linear regression with uniform data weights, however, implies a different weighting of the data than the nonlinear optimization in the aforementioned doubly logarithmic plot. The weighting of the latter is more realistic, for good reasons (see Sec. 6.3.6 in Bažant and Planas 1998).

On the other hand, linear regression plots are most suitable for visual evaluation, and are therefore used for presenting the optimum fits of the data in Figs. 2 and 3. However, the optimum values of d_0 and v_p shown in each figure have been obtained by nonlinear regression in the doubly logarithmic plot. The agreement with the energetic theory of size effect seen in the plots is quite satisfactory, especially for the data of Leonhardt and Walther, Bažant and Kazemi, Shioya et al., Walraven, Walraven and Lehwalter.

Fig. 4 shows all the data sets put together in a doubly logarithmic size effect plot in relative coordinates, taken as the relative shear stress at ultimate load (v_u/v_p) versus the relative depth of the beam d/d_0, where v_p and d_0 have the values obtained by nonlinear optimization of the each data set separately. If there were no scatter and the energetic theory were perfect, all the data points would have to lie in this plot on one curve, and so the deviation from the size effect curve shows the errors. The plot in Fig. 4 shows the overall scatter under the assumption that the correct parameter values are know, i.e., it shows how good is the *form* of the formula but not how good are its parameter values. But it cannot be regarded as a validation of the size effect theory because the different data sets were put together by using the theory. The validation depends on the aggregate of all individual data fits (Figs. 2 and 3).

The problem of prediction of parameters d_0 and v_p from the strength, composition, maximum aggregate size, fracture energy (or toughness), effective fracture process zone length, characteristic length of concrete, etc., is theoretically formidable. Empirical rules will have to be developed.

Eq. (1) was derived by Bažant (1997a) under the assumption that the stress transmitted across the compression failure band is vanish-

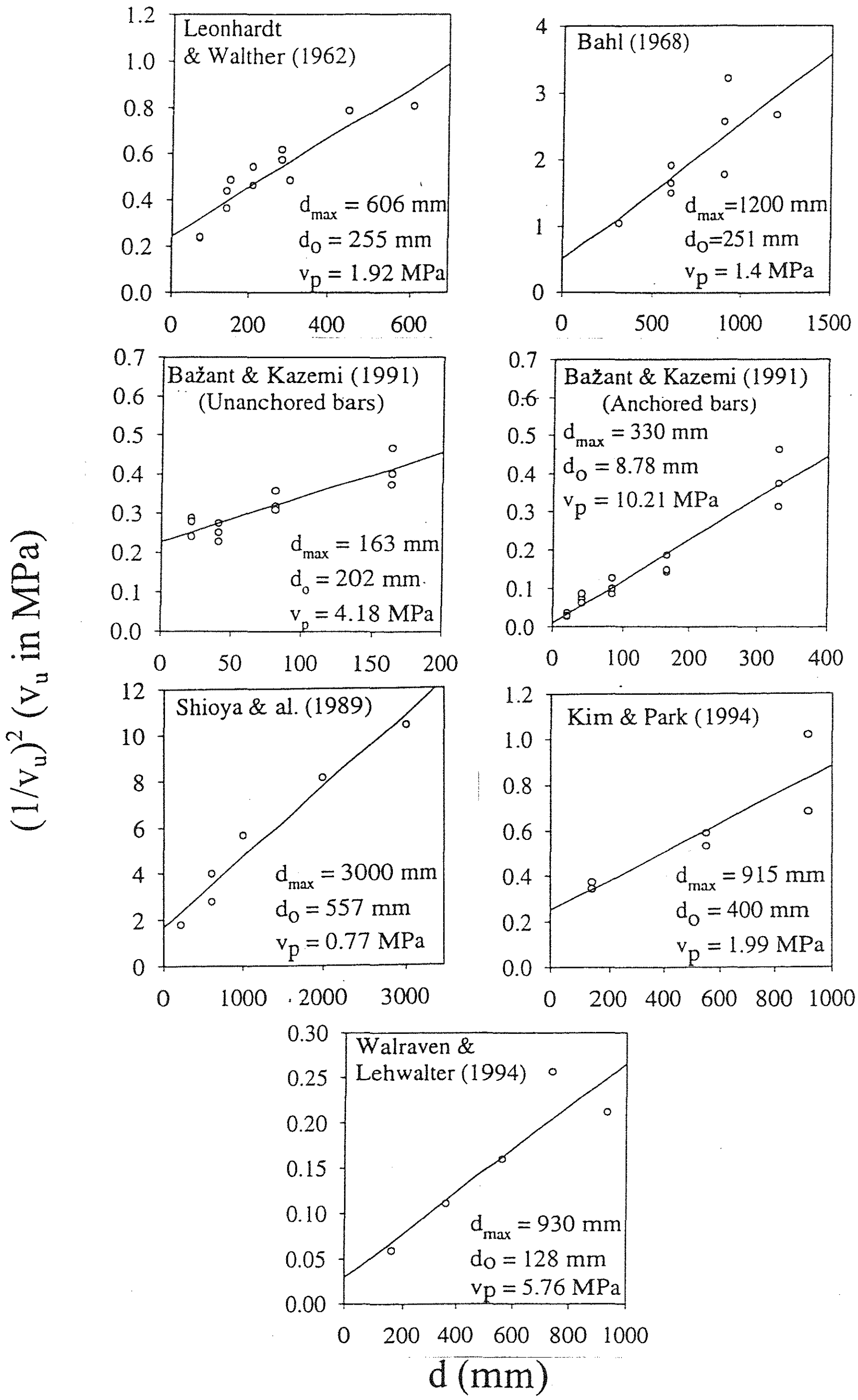

Fig. 2 Beams without stirrups: linear regression fits of test data by various investigators.

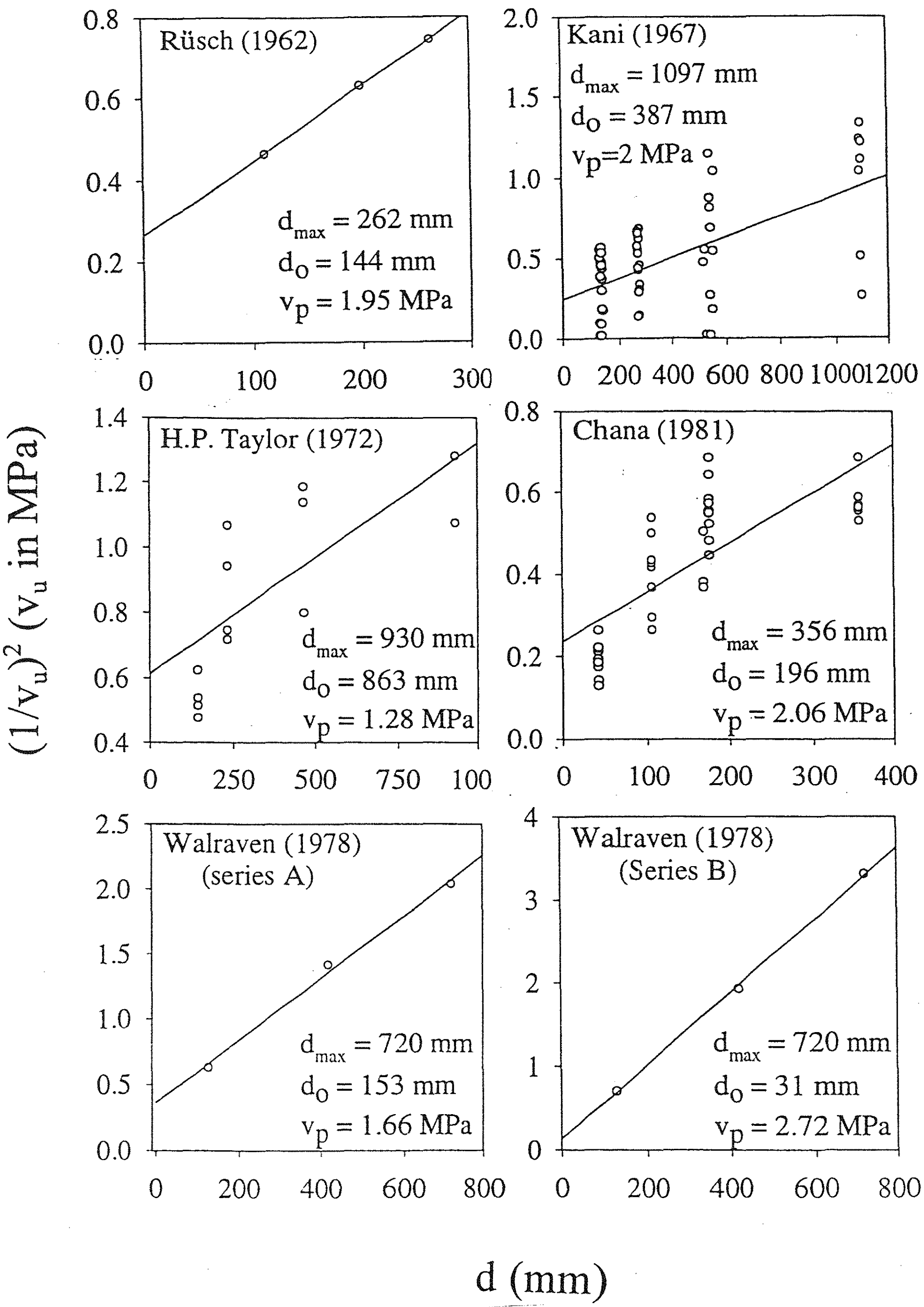

Fig. 3 Beams without stirrups: linear regression fits of further test data.

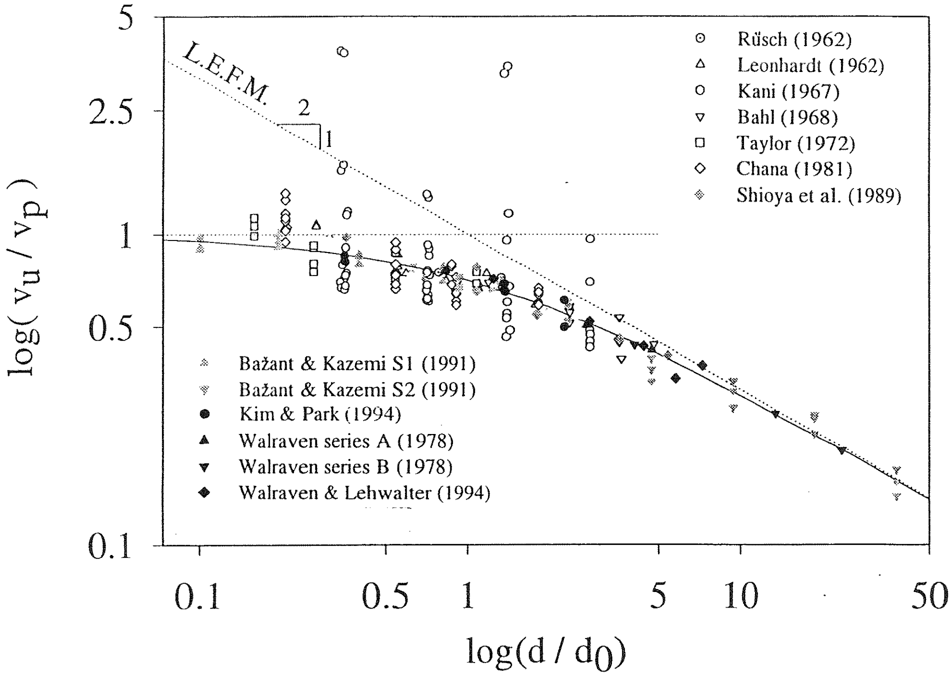

Fig. 4 Beams without stirrups: comparison of all test data sets with Bažant's size effect law.

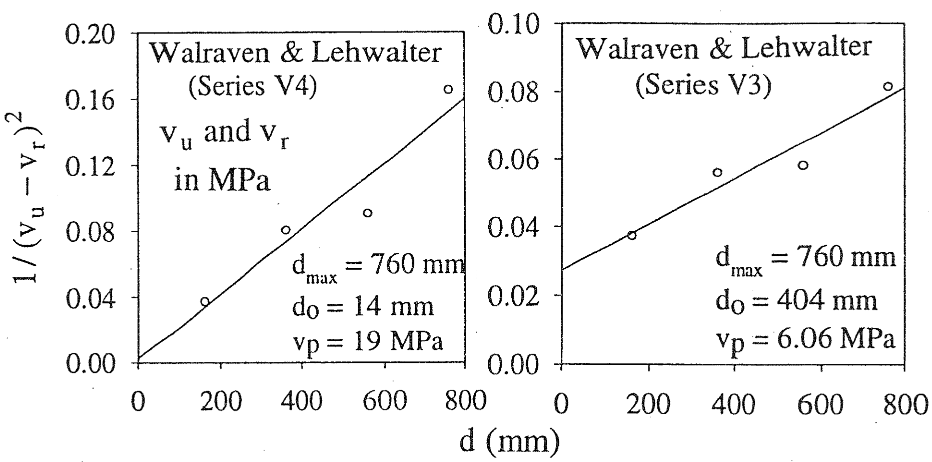

Fig. 5 Beams with stirrups: comparison of Kim and Park's data with Bažant's size effect law.

ingly small. This assumption is on the safe side, however, a study of compression failure in general (Bažant and Xiang 1997) indicates that, at maximum load, this stress might have a significant value σ_r, even in absence of stirrups. In that case, an analysis of the type presented in Bažant (1997a) leads to an extended formula

$$v_u = v_p \left(1 + \frac{d}{d_0}\right)^{-1/2} + v_r \tag{5}$$

in which v_r is a non-zero residual strength. The question whether σ_r, and thus also v_r, can have non-negligible values in a beam without stirrups needs to be settled in future research.

4 Size Effect in Reinforced Beams With Stirrups

We study only the usual case when the stirrups are uniformly and densely distributed. Because the stirrups help to confine concrete, it is now logical and safe to assume that the stress σ_r transmitted across the compression failure band has a non-negligible value. The energy analysis of the failure of the compression strut in the truss model (Eq. 21–23 in Bažant 1997a) leads to the same formula as Eq. (5) but with different expressions for the coefficients:

$$d_0 = w_0 \frac{d}{c} \tag{6}$$

$$v_r = \frac{\sin 2\theta}{2} \sigma_r; \qquad v_p = K_c \sqrt{\frac{h_0}{2 s_c w_0}} \sqrt{\frac{c}{d}} \sin 2\theta \tag{7}$$

Here θ is angle of inclination of the compression strut from the horizontal, which may be determined from the strain compatibility condition of the classical truss model (Collins 1978).

Unfortunately, only very few data showing size effect are available in the literature, for the case of beams with stirrup reinforcement. Only Walraven and Lehwalter's data (1994) are sufficiently relevant to the size effect. These data have been fitted using the Levenberg-Marquardt nonlinear optimization algorithm. Fig. 5 shows the linear regression plot of these data, with d as the coordinate and $1/(v_u - v_r)^2$ as the ordinate. The optimum values of the parameters are given in the figure.

Conclusion

The size effect formula obtained from the energetic fracturing truss model agrees quite well with the available test data for beams both without and with stirrups. For beams with stirrups, however, more extensive experimental validation is needed.

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