Fracture Mechanics of Concrete Structures, de Borst et al (eds) © 2001 Swets & Zeitlinger, Lisse, ISBN 90 2651 825 0

Material response fluctuations and correlation with internal lengths

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ABSTRACT: we use a model with random heterogeneity in order to study the effect of material microstructure on the fracture process of concrete-like materials. This aim is reached by analysing correlations of the (global force-loading) response, that reveals large fluctuations. This analysis gives information on the internal length involved in fracturing that could not be obtained with a continuum approach.

1 INTRODUCTION

Fluctuations on material response are often encountered during mechanical tests on quasi-brittle materials, like concrete. These fluctuations are known to be the effect of material heterogeneity. They are due to the accumulation of microscopic crack nucleation, propagation and arrest. However, most of models neglected these effects by averaging the material response. The aim of this work is to show that fluctuations analyses are able to reveal material intrinsic properties like internal length.

We are using a particular discrete model which produces a steady response. It allows to focus just on fluctuations due to heterogeneities whether they are due to the initial disorder or to progressive fracturing. The model is based on a semi-infinite set of perfectly brittle fibers that are connected between a rigid substrate and an elastic body. A displacement is imposed at one point of the elastic body, which may move along the interface. This model could be seen as a propagation of a one-dimensional crack. Three areas could be distinguished. The first one, near the imposed displacement, is called the broken area: all fibers are broken. The second one could be seen as a fracture process zone, where a finite fraction of fibers are surviving. The last one is called the safe area: no fibers are broken. The model response is the evolution of the global force, which are the sum of all surviving fibers contribution, versus the evolution of the imposed displacement. The fluctuations are analyzed through avalanches statistics, i.e. the series of microcrack events which are producted under a constant loading.

Two cases are considered: the first one uses a constant shape for the elastic body. This simple case allows an analytical resolution of the model. The most useful information is the avalanches distribution. Numerous studies have shown that this analysis is a pertinent tool to study fluctuations. For our model, the main observation is that the avalanche distribution exhibits two distinct behaviors according to the value of the avalanche size Δ with respect to a cross-over value Δ^*. The first regime, $\Delta < \Delta^*$, is a power law $p_1(\Delta) \propto \Delta^{-\tau_1}$ with an exponent $\tau_1 \sim 1.50$. The second regime, $\Delta < \Delta^*$, is also a power law but with a different exponent $\tau_2 \sim 2.00$. The cutoff Δ^* is proportional to the active area size ξ. The value of the two exponents could be easily understood. The first one corresponds to a regime where the force versus displacement response displays correlations similar to a random walk. It corresponds to avalanches with a size smaller than the active area one. The second regime corresponds to large size avalanches, i.e. $\Delta > \xi$. In this case, the forces are uncorrelated. We could make a parallel with the avalanche distribution for a random uncorrelated signal that gives a power law with an exponent of 2.

For this simple case, the avalanche distribution gives information of the fracture process zone size. For the second case, we consider a more general case where the elastic body is a flexible beam, which interacts with the fibers. In this case, the analytical solution could not be obtained and the fracture process zone is unknown. We make the same analysis through avalanches statistics. We find the same behavior with two power laws, with the same exponents τ_1 and τ_2. The cutoff between the two regimes provides the size of the fracture process zone. Hence, we show that the fluctuations analysis from the force-displacement response, a fully accessible experimental information, provides information on the existence of a well defined length scale, without having to know it forehand.

2 ZIP MODEL

Figure 1 illustrates our model, with the three parts described in the introduction. The position of fiber numbered i is x_i and its vertical displacement is y_i. For simplification, the spacing between fibers is set to one (i.e $x_{i+1} - x_i = 1$). All the fibers have the same stiffness κ, and an elastic perfectly brittle behaviour. The maximum extension of a fiber y_c before breaking is chosen randomly. For the analytical part of our study, a uniform distribution between 0 and 1 is chosen. Of course, the full study could be done with any other distributions.

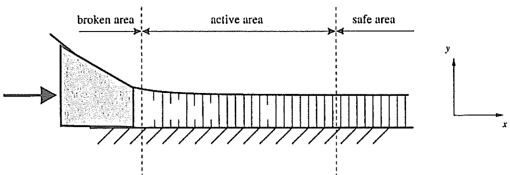

Figure 1: A schematic representation of the "Zip" model. The point where the displacement is imposed may move along the x-axis.

Mean deflection of the beam The first part of our study deals with an imposed deformed shape of the beam. In order to have a good approximation of this shape, we compute its average deflection. Introducing E and I, respectively the young modulus and transverse geometrical inertia of the beam, we can write an equation for the mean deflection of the beam $y(x)$ as

$$EI\frac{d^4y(x)}{dx^4} = -\kappa y(x)(1 - Y(x)) \tag{1}$$

where

$$Y(x) = \max_{x' \geq x}(y(x')) \tag{2}$$

This equation is true under the hypothesis of a uniform distribution of critical fiber extension between 0 and 1, and for $y < 1$. For larger y,

$$\frac{d^4y}{dx^4} = 0. \tag{3}$$

The boundary conditions are

$$y(\infty) = \frac{dy(\infty)}{dx} = 0 \tag{4}$$

(no deformation far from the edge),

$$y(0) = 1 \tag{5}$$

(imposed displacement) and

$$\frac{d^2y(0)}{dx^2} = 0 \tag{6}$$

(no torque being applied at the loading point). The reason why we have to distinguish between y and Y is that the deflection is not a monotonous function of x.

Since damage is irreversible, we have to compute the maximum damage having been met by the corresponding section of fibers. The analytical solution to this problem is not known, but we can see that the quadratic non-linear term becomes unimportant at large distance from the origin. Thus the asymptotic shape will have the following expression

$$y(x) = Ae^{-x/\xi}\cos(x/\xi + \phi) \tag{7}$$

where

$$\xi = \sqrt{2}\left(\frac{EI}{\kappa}\right)^{1/4} \tag{8}$$

The oscillatory component is the one which makes the deflection non monotonous, and thus requires the distinction between y and Y.

3 IMPOSED DISPLACEMENT PROFILE

The imposed displacement profile allows to obtain analytical results. We impose an exponential shape which captures some of the features of a beam deflection that are mentioned in the previous part. For any abscissa x, the profile y is given by

$$y(x) = \exp\left(\frac{(U(t) - x)}{\xi}\right) \tag{9}$$

where the length scale ξ is considered as a fixed parameter, U is the time-dependant horizontal displacement of the edge. This kind of expression is closed to the exact beam deflection and allows analytical expression of the loading.

3.1 *Mean behavior and fluctuations*

The total force exerted on the wedge is defined as the sum of all fiber contributions,

$$F = \sum_i f_i(U) \tag{10}$$

where each individual force f_i is

$$\begin{array}{lll} f_i(U) & = \kappa y(x_i, U) & \text{if} \quad y(x_i, U) < y_c(x) \\ & = 0 & \text{if} \quad y(x_i, U) > y_c(x) \end{array} \tag{11}$$

where κ is the elastic modulus of the fibers before breaking. After few developments, one can compute the mean value of $\langle F \rangle$ as

$$\langle F \rangle \approx \kappa\frac{\xi}{2}\left(1 - \frac{1}{6\xi^2}\right) \tag{12}$$

and the total force variance $\sigma^2(F)$ as

$$\sigma^2(F) \approx \frac{\xi\kappa^2}{12}\left(1 - \frac{1}{\xi^2}\right) \tag{13}$$

Then, the entire distribution of the total force can be obtained analytically in the context of this simplified model. In the limit of a large length ξ, we can simply observe that F is given by a sum of statistically independent random variables and the law of large numbers applies. Consequently, F will have a Gaussian distribution. Thus the above computed average and variance are sufficient to specify entirely the distribution $p(F)$

$$p(F) = \frac{\sqrt{6}}{\sqrt{\pi\xi}\,\kappa} \exp\left(-\frac{3(2F-\kappa\xi)^2}{2\kappa^2\xi}\right) \tag{14}$$

3.2 *Correlations*

When the steady state regime is obtained, the global force response gives a fluctuating signal which could be seen as the effect of the model heterogeneity, i.e. the succession as micro-events that are fiber failures. In the previous section, we have computed just the mean properties of the global force, but we do not characterize the correlation of its response with time, i.e. the relation between $F(U)$ and $F(U+\Delta U)$. This can be computed from the relation $dF(\Delta U) = F(U+\Delta U) - F(U)$. Because F is the sum of independent statistical variables, one can write

$$dF = \sum_j df_j \tag{15}$$

where j extends over the unbroken fibers after the crack tip U. The random variables df_j assumes the following values:

$$df_j = \kappa(y_j(U+\Delta U) - y_j(U)) \tag{16}$$

with probability $(1 - y_j(U+\Delta U))$,

$$df_j = -\kappa y_j(U) \tag{17}$$

with probability $(y_j(U+\Delta U) - y_j(U))$,

$$df_j = 0 \tag{18}$$

with probability $y_j(U)$ where the last condition has been written so that the sum can extend over all fibers for $j > U + \Delta U$. One has to consider also the fibers in the range $U < j < U + \Delta U$, which are broken with probability 1, but were surviving under the displacement U with probability $1 - y_j(U)$. At the steady state value of the force, $\langle dF \rangle = 0$, by definition. The expectation value of $\langle dF^2 \rangle$ can thus be obtained by summing up the variances of the df_j.

$$\langle dF^2 \rangle = (1 - \exp(-\Delta U/\xi))\sigma^2(F) \tag{19}$$

Thus, the squared force difference increases first linearly with ΔU, and saturates to a constant value equal to twice the variance of the force. The interpretation of this property is easy to deduct: The total force is the sum of the order of ξ uncorrelated random variables. Thus over this length scale, $F(u)$ behaves as a random walk. However, for larger distances, the fluctuations of F becomes uncorrelated.

As a consequence of this observation, we note that the typical variation of the force over a short displacement ΔU scales as $\sqrt{\Delta U}$. Consequently, the fluctuating part of the signal become non-differentiable when the microstructural size goes to 0, keeping ξ fixed (note that this construction implies a redefinition of the physical scale since we chose here to measure distances in terms of the microscopic distance, the fiber separation distance).

If the limit of a bundle with an infinite number of fibers is expected to represent a continuum response, the fact that the constitutive response is not differentiable is a striking departure from traditional assumptions. In fact, it means that when this continuum limit is considered, a smoothing of the constitutive response is performed at the same time so that the response becomes differentiable. It follows that upon taking this limit, the information contained in the fluctuation of the response, which still exist for large size systems, is lost. As we will see in the next section, this information enlights the crack propagation regime and yields a parallel between the fiber bundle model and cohesive crack models in the context of the present study.

4 NUMERICAL STUDY OF FLUCTUATIONS

An efficient tool that allows to study fluctuations is obtained through the definition of avalanches [5], [2], [4]. An usual way to define avalanche consists in selecting a level of force, and computing the distance ΔU over which the crack can propagate. The avalanches are characterized by their statistical distribution, $p_1(\Delta U, F)$. Fig. 2 shows the computed for-

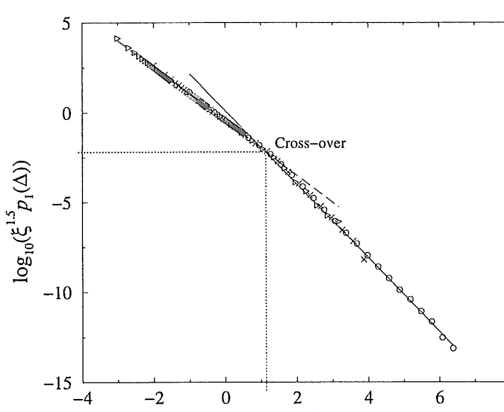

Figure 2: The avalanche distribution for zip model with a constant beam shape. The active area size ξ are 100 (○), 1000 (×) and 10000 (▷). The dashed line is a guide line with a slope of -1.5. The continuous lines is a guide line with a slope of -2.05.

ward avalanche distribution for the present model where a fixed displacement profile is set. Three values of ξ have been used. The first one is $\xi = 100$ ($\circ$) with 10^8 broken fibers, the second one is $\xi = 1000$ ($\times$) with 2.10^7 broken fibers and the third one is $\xi = 10000$ ($\triangleright$) with 5.10^6 broken fibers.

The main observation is that the distribution exhibits two distinct behaviors according to the value of the avalanche size Δ with respect to a cross-over value Δ^*. The first regime, $\Delta < \Delta^*$, is a power-law $p_1(\Delta) \propto \Delta^{-\tau_1}$ with an exponent

$$\tau_1 = 1.50 \pm 0.05 \tag{20}$$

The second regime, $\Delta > \Delta^*$, is also a power-law but with a different exponent τ_2.

$$\tau_2 = 2.05 \pm 0.10 \tag{21}$$

Fits to both of these power-laws are plotted on Figure 2. Finally, the cross-over scale Δ^* scales as ξ: we have used in the graph the scaled variables Δ/ξ, and scaled distribution $\xi^{\tau_1} p_1(\Delta)$, to show that the three curves collapse onto a single master curve. This data collapse shows that indeed, the cross-over scale Δ^* is proportional to ξ.

One can understand the value of the two exponents as follow. The first one corresponds to a regime where the force versus crack length U displays correlations similar to a random walk. The forward avalanche, in this case can be interpreted as the time required for a random walk to return to the origin. This well-known statistical problem is indeed a power-law of exponent $3/2$ in agreement with the first regime. Note that this behavior is exactly the one which has been established for the global load sharing fiber bundle with rigid boundaries [2], [1], using essentially also a mapping onto a random walk problem.

For large avalanches, $\Delta > \Delta^*$, the forces are uncorrelated. We thus may resort to this simple case to work out the avalanche statistics: let $\eta(t)$ be a random uncorrelated noise, with a distribution $p(\eta)$, and cumulative distribution $P(\eta) = \int^{\eta} p(x)dx$. Starting at a given value of $\eta = \eta_0$, the probability that an avalanche is larger than Δ is $Q(\Delta, \eta_0) = P(\eta_0)^{\Delta}$ since the different η values are uncorrelated. The cumulative distribution P_1 of forward avalanches is obtained from the integration of the above Q distribution over all starting points of distribution $p(\eta)$, hence

$$P_1(\Delta) = \int P(\eta_0)^{\Delta} p(\eta_0) d\eta_0 = \int_0^1 u^{\Delta} du = \frac{1}{\Delta} \tag{22}$$

where we have used $\Delta \gg 1$. The avalanche distribution p_1 is obtained from the derivative of the cumulative distribution and leads to the power-law $p_1(\Delta) = \Delta^{-2}$, for *all* distributions $p(\eta)$. In our problem, for large avalanches we are precisely in this case, and indeed we do observe an exponent $\tau_2 = 2$.

One sees on this particular example that a simple statistical analysis performed on the force signal allows to extract the correlation length $\xi \propto \Delta^*$ without knowing it beforehand. This correlation length defines the size of the fracture process zone [3].

5 ZIP MODEL WITH DEFORMABLE BEAM

In this part, the full problem is solved, i.e. the mechanical interaction between beam and fibers is taking into account. The problem requires a much longer computation time compared to the simplified model, due to the larger number of degrees of freedom. For numerical convenience, we only deal with the process zone, and thus we neglect the presence of a few broken fibers ahead of this region. However, we take into account this zone in the computation, by introducing a boundary condition at the end of the process zone which represent an infinite beam connected to the substratum through intact fibers. This involves two relations between the derivatives of order 0 to 3 of the deflection function $y(x)$. The length of the domain considered numerically is set equal to ξ.

Figure 3 shows forward avalanches distributions for two stiffnesses ratio, and hence two values of ξ. We recover in both cases the previous results, i.e. two power-laws with exponents $\tau_1 = 1.50 \pm 0.05$ and $\tau_2 = 2.1 \pm 0.15$. We check the reliability of the measure by plotting and evaluating the mean shape of the beam during crack propagation. The results are very close to the simplified model. Hence, we show that the fluctuation analysis from the force-displacement response, an accessible experimental information, does provide informations on the existence of a well defined length scale related to the fracture process zone in the interface.

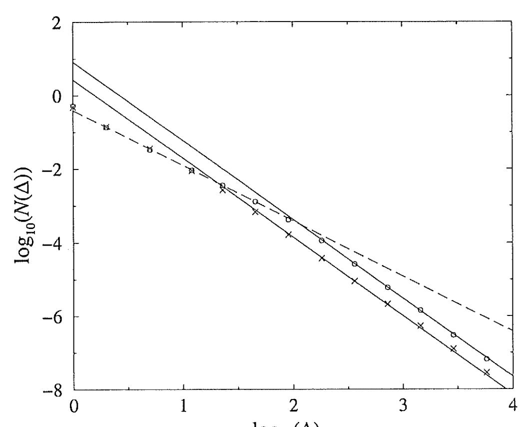

Figure 3: The avalanche distribution for Zip model with beam deformation. Two beam stiffnesses are considered. The dashed line is a guide line with a slope of -1.5. The continuous lines are guide lines with a slope of -2.1.

6 CONCLUSIONS

We have introduced a discrete model that represents the propagation of a crack through a set of parallel brittle fibers. The fibers are clamped between an elastic body and a rigid substrate, that could be seen as a symmetric axe. The response of the model, that is the global force versus the loading displacement, is analysed. First, we make the analytical study when the elastic body has a constant shape. Then, the numerical analysis of the response, with avalanche study, reveals two power law with distinguished exponents. The first exponent is close to 1.5 as the second one is close to 2. We show that the cutoff between this two regimes is nothing else that the size of the so-called fracture process zone, where both broken fibers and surviving fibers are present. Finally, we make the numerical study of the real problem with mechanical interaction between the elastic body end the fibers. We find again two power laws with similar exponents, that gives the length of the fracture process zone without knowing it forehand. It shows the importance of fluctuations that are directly the response of material heterogeneity.

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