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# On regularized plasticity models for strain-softening materials

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The paper analyzes and compares several regularization techniques for softening plasticity. It is shown that a basic nonlocal plasticity model with a nonlocal cumulative plastic strain in the softening law provides only a partial regularization. As an alternative, a refined nonlocal plasticity model with a combination of local and nonlocal cumulative plastic strain is investigated in detail. An efficient numerical algorithm solving the nonlocal consistency condition is outlined and a convergence proof is given. Furthermore, the behavior of the refined nonlocal model is compared to gradient plasticity with a softening law dependent on the gradient of the softening variable. The differences between plastic strain profiles localized inside the body and at the boundary are investigated and the correspondence between the boundary conditions in the gradient formulation and the rescaling of the weight function in the nonlocal formulation is discussed. A physical interpretation of the attractive or repulsive character of the boundary layer is suggested.

## 1 INTRODUCTION

Classical plasticity theories based on material models that are "simple" in the sense of Noll fail to provide an objective description of softening, since, after the onset of localization, the boundary value problem becomes ill-posed. The width of the localized zone is related to the heterogeneous material structure and can be correctly predicted only by models that have an intrinsic parameter with the dimension of length. Such a length scale is absent in standard theories of elasticity or plasticity, and it must be introduced by an appropriate enhancement.

Among the generalized continuum models that can serve as localization limiters and restore well-posedness of the boundary value problem, the most popular seem to be formulations that work with gradients or nonlocal averages of internal variables or their conjugate thermodynamic forces. Development of such formulations for damage models is relatively straightforward, because the concerned variable driving the dissipative process (e.g., the damage energy release rate, or the equivalent strain) is usually directly related to the total strain and thus can be easily evaluated in a displacement-driven finite element procedure. In plasticity, however, the problem is more delicate, since the concerned variable is typically the accumulated plastic strain, which must be computed from the consistency condition that has no longer a local character. The present study focuses on some fundamental aspects of nonlocal and gradient models for softening plasticity, the identification of common features and differences as well as their physical interpretation. Regarding a refined nonlocal model, the numerical implementation is also addressed. In view of the limited space and to keep the presentation simple, all considerations are done in one dimension, but most of the conclusions can be transplanted to the general case. Attention is restricted to the small-strain theory.

## 2 STANDARD AND ENHANCED PLASTICITY MODELS

### 2.1 *Local plasticity*

For a one-dimensional problem, standard local plasticity with linear isotropic hardening is described by the equations

$$\sigma = E(\varepsilon - \varepsilon_{\rm p}) \quad (1)$$

$$f(\sigma, \sigma_{\rm Y}) = |\sigma| - \sigma_{\rm Y} \quad (2)$$

$$\sigma_{\rm Y} = \sigma_0 + H\kappa \quad (3)$$

$$\dot{\varepsilon}_{\rm p} = \dot{\kappa}\frac{\partial f}{\partial \sigma} = \dot{\kappa}\,{\rm sgn}(\sigma) \quad (4)$$

$$\dot{\kappa} \geq 0, \quad f(\sigma, \sigma_{\rm Y}) \leq 0, \quad \dot{\kappa} f(\sigma, \sigma_{\rm Y}) = 0 \quad (5)$$

which represent the elastic law, definition of yield function, hardening law, flow rule, and loading-unloading conditions. In the above, $\sigma$ is the stress,

$\varepsilon$ is the strain, $\varepsilon_p$ is the plastic strain, $E$ is the elastic modulus, $H$ is the plastic modulus (positive for hardening and negative for softening), $\sigma_0$ is the initial yield stress, and $\kappa$ is the hardening variable. From the flow rule (4) and the first condition (5) it follows that $\dot{\kappa} = |\dot{\varepsilon}_p|$, which gives to the hardening variable $\kappa$ the physical meaning of the cumulative plastic strain.

Consider a bar of constant cross section fixed at one end and loaded by an applied displacement at the opposite end. For hardening, the response is unique, and the distribution of strain remains uniform. For softening, the governing equations admit infinitely many solutions with a nonuniform strain distribution. The stress must remain uniform and decrease, but plastic yielding does not need to occur at all sections of the bar. The plastic zone can become arbitrarily small and failure can occur at arbitrarily small dissipation. These physically inadmissible properties of the theoretical solutions lead to pathological sensitivity of the numerical results to the computational grid.

### 2.2 *Gradient plasticity*

A gradient plasticity model inspired by the ideas of Aifantis (1984) has been described e.g. by de Borst and Mühlhaus (1992), its numerical implementation has been developed by Pamin (1994). It differs from the local model only by the dependence of the yield stress on the second derivative (in multiple dimensions, on the Laplacean) of the cumulative plastic strain:

$$\sigma_Y = \sigma_0 + H(\kappa + l^2\kappa'') \qquad (6)$$

This modified softening law introduces the length scale $l$, which controls the size of the plastic zone. The presence of the second derivative of $\kappa$ in the basic equations leads to the requirement of $C^1$-continuity for this variable (and, consequently, for the plastic strain).

The enrichment of the softening law by the second-order gradient term regularizes the problem and prevents localization of plastic strain into an arbitrarily small region. The regularizing effect of the gradient term can be explained as follows. In the absence of body forces, equilibrium conditions require the stress distribution to remain uniform at any stage of the loading process. In the plastic region $I_p$, the actual stress $\sigma$ must be equal to the current yield stress $\sigma_Y$, which means that, inside the plastic region, $\sigma_Y$ must be uniform. According to (6), this is possible only if

$$\kappa + l^2\kappa'' = c \quad \text{in } \mathrm{I_p} \qquad (7)$$

with $c =$ const. Since $\kappa$ must be continuously differentiable and must vanish outside the plastic region, it must start at the boundary of the plastic region with a zero slope and positive curvature. As we move into the interior of the plastic region, $\kappa$ is increasing, thus to satisfy (7), its second derivative must decrease. Equation (7) with initial conditions $\kappa(0) = 0$ and $\kappa'(0) = 0$ has a unique solution $\kappa(x) = c(1 - \cos(x/l))$, and the opposite boundary of the plastic region is reached at $x = 2\pi l$, (Fig. 1). This analytical solution shows that the size of the localized plastic zone is directly proportional to the internal length scale $l$ and independent of the softening modulus $H$.

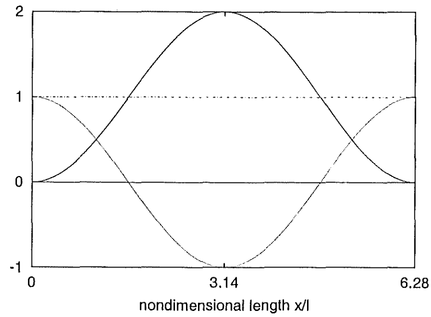


Figure 1: Gradient plasticity model; $\kappa$ (solid) and $l^2\kappa''$ (dot) are such that $\kappa + l^2\kappa''$ (dash-dot) is constant.

### 2.3 *Basic formulation of nonlocal plasticity*

Nonlocal material models admit that the local state of the material at a given point may not be sufficient to evaluate the stress at that point. This can be physically explained by the fact that no real material is an ideal continuous medium, and on a sufficiently small scale the effects of heterogeneity and discontinuous microstructure become nonnegligible. For metals, this scale is in the order of microns, but for concrete and other highly heterogeneous composite materials, it is substantially larger. If the strain distribution is sufficiently smooth, as is usually the case in the elastic regime, the standard local theory provides a good approximation and no important deviations from the actual behavior can be observed. After localization, the characteristic wave length of the deformation field becomes much shorter and this activates the nonlocal effects. For this reason, nonlocal theories that aim at regularizing the localization problem usually neglect the nonlocal elastic effects and apply nonlocal averaging only to an internal variable (or thermodynamic force) linked to the dissipative processes. In plasticity, this is naturally the softening variable (cumulative plastic strain), or the plastic strain itself.

Perhaps the simplest nonlocal plasticity theory can be constructed if the softening law (3) is reformulated as

$$\sigma_Y = \sigma_0 + H\bar{\kappa}. \qquad (8)$$

In this model, the yield stress depends on the nonlocal softening variable

$$\bar{\kappa}(x) = \int_L \alpha(x,\xi)\kappa(\xi)\mathrm{d}\xi, \qquad (9)$$

$\alpha$ is a certain weight function decaying with the distance between $x$ and $\xi$, and the integral is taken over the length $L$ of the bar (in general over the entire elasto-plastic body). In an infinite domain, the weight function $\alpha_\infty(r)$ would depend only on the distance $r = |x - \xi|$. In a finite domain, the weight function is often rescaled by

$$\alpha(x,\xi) = \frac{\alpha_\infty(|x-\xi|)}{\int_L \alpha_\infty(|x-\zeta|)\mathrm{d}\zeta} \tag{10}$$

with the argument that the nonlocal field corresponding to a constant local field should remain constant even in the vicinity of a boundary. Commonly used nonlocal weight functions are the Gauss-like function

$$\alpha_\infty^{\text{gauss}}(r) = \frac{2}{\sqrt{\pi}} \exp\left(-\frac{r^2}{l^2}\right) \tag{11}$$

which has unbounded support, and the bell-shaped polynomial function

$$\alpha_\infty^{\text{bell}}(r) = \begin{cases} \dfrac{15}{16R}\left(1-\dfrac{r^2}{R^2}\right)^2 & \text{if } 0 \le r \le R \\ 0 & \text{if } r \ge R \end{cases} \tag{12}$$

with support radius $R$.

Nonlocal plasticity based on a nonlocal softening law (8) has only a partial regularizing effect. After a proper calibration, it gives the correct dissipation and a mesh-insensitive load-displacement diagram, but the plastic strain is still localized into a single "point" (meaning here one cross section of the bar) and has the character of a Dirac distribution. Based on the fact that the nonlocal weight function decays with $r$ (which is a natural and generally accepted assumption), it is possible to show that the only solution with nonzero size of the plastic region is that with a constant plastic strain along the entire bar. The proof is especially easy for an infinite bar. Suppose that the plastic region $I_\text{p}$ is a finite interval of nonzero size. In the plastic region, the nonlocal softening variable $\bar\kappa$ must be constant

$$\bar\kappa = \frac{\sigma_\text{Y} - \sigma_0}{H} = \frac{\sigma - \sigma_0}{H} = \text{const.} \tag{13}$$

and thus its derivative

$$\begin{aligned} \bar\kappa'(x) &= \int_{-\infty}^{\infty} \frac{\partial\alpha_\infty(|x-\xi|)}{\partial x}\kappa(\xi)\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \alpha'_\infty(|x-\xi|)\operatorname{sgn}(x-\xi)\kappa(\xi)\mathrm{d}\xi \end{aligned} \tag{14}$$

must vanish there. The prime in $\alpha'_\infty$ denotes the derivative taken with respect to $r = |x - \xi|$. Since the local softening variable $\kappa$ vanishes outside the plastic region and is positive inside of it, and since $\alpha'_\infty(r)$ is negative for $r < R$ and zero for all $r \ge R$, the integral in (14) has a positive (and thus nonzero) value for $x$ located in the plastic region just next to its "left" boundary. Consequently, $\bar\kappa'(x) > 0$, which contradicts the assumption of constant $\bar\kappa$ in $I_\text{p}$.

Planas et al. (1993) showed that the plastic strain must localize into one single point, in which case the above arguments do not hold because the plastic interval collapses into a single point. The localized solution can be described by

$$\kappa(x) = \frac{\sigma - \sigma_0}{H\alpha(x_s, x_s)}\delta(x - x_s) \tag{15}$$

where $x_s$ is the (arbitrary) localization point, and $\delta$ denotes the Dirac distribution. The corresponding nonlocal field

$$\bar\kappa(x) = \int_L \alpha(x,\xi)\kappa(\xi)\mathrm{d}\xi = \frac{\sigma - \sigma_0}{H\alpha(x_s, x_s)}\alpha(x, x_s) \tag{16}$$

is a multiple of the weight function (taken as a function of $x$ with a fixed $\xi = x_s$). Despite the fully localized character of the local strain, the energy $G_F$ dissipated during the failure of the bar (taken per unit cross sectional area) is nonzero:

$$G_F = -\frac{\sigma_0^2}{H\alpha(x_s, x_s)} \tag{17}$$

The total bar elongation can be decomposed into the elastic part, which is proportional to the bar length, and the inelastic part, completely independent of the bar length. Therefore, the present basic formulation is essentially equivalent to a cohesive zone model, as pointed out by Planas et al. (1993).

### 2.4 *Refined formulation of nonlocal plasticity*

In the foregoing analyses we have repeatedly used the argument that the expression added to the initial yield stress in the hardening law must be constant along the plastic zone, in order to satisfy both the yield condition and the equilibrium condition. The gradient formulation achieves this by combining the local $\kappa$ with a multiple of its second derivative, which can really provide a constant function if $\kappa$ is selected as the shifted harmonic function $1 - \cos(x/l)$; see Fig. 1. On the other hand, a nonlocal average $\bar\kappa$ of any local distribution $\kappa$ with the expected characteristics (monotonically increasing from the boundary of the plastic region to its center) can never be constant across the plastic region; it will have a shape similar to the local distribution but more flat and spread to the sides (Fig. 2). But if the nonlocal distribution is amplified by a scalar factor larger than 1, it may coincide with the local distribution shifted by a constant.

This motivates a nonlocal plasticity formulation with softening driven by a suitable linear combination of the local and nonlocal softening variable, proposed

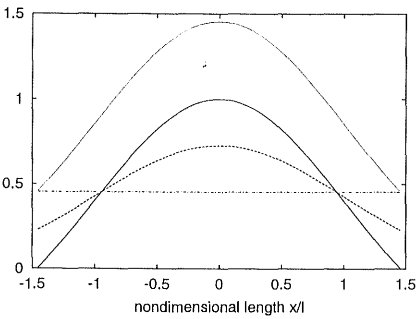


Figure 2: Refined nonlocal plasticity model for $m = 2$: $\kappa$ (solid) and $\bar{\kappa}$ (dot) are such that $2\bar{\kappa}$ (dash) is constant.

independently by Strömberg and Ristinmaa (1996) and Planas et al. (1996); see also Bažant and Planas (1998), p. 497. The modified hardening law,

$$\sigma_Y = \sigma_0 + H[m\bar{\kappa} + (1-m)\kappa] \tag{18}$$

can be interpreted as the original law (8) with the nonlocal average $\bar{\kappa}$ evaluated using a special singular weight distribution

$$\alpha_m(x,\xi) = m\alpha(x,\xi) + (1-m)\delta(x-\xi) \tag{19}$$

This generalized formulation of nonlocal plasticity includes the local and the basic nonlocal model as special cases with $m = 0$ and $m = 1$, respectively. Strömberg and Ristinmaa (1996) call it the "mixed local and nonlocal model", while Planas et al. (1996) speak of the "nonlocal model of the second kind", the case $m = 1$ being called the "nonlocal model of the first kind". It is important to note that $0 < m < 1$ does not lead to any improvement compared to the basic model with $m = 1$. This is intuitively clear from Fig. 2, and can be rigorously proven, cf. Planas et al. (1996). The plastic zone is finite if and only if $m > 1$.

## 3 ANALYSIS OF NONLOCAL MODEL

### 3.1 *Continuity of plastic strain distribution*

Inside the plastic region, the yield and equilibrium conditions imply

$$m\bar{\kappa} + (1-m)\kappa = \frac{\sigma - \sigma_0}{H} \quad \text{in } \mathrm{I_p}. \tag{20}$$

For a continuous nonlocal weight function, the nonlocal average $\bar{\kappa}$ is always continuous, thus (20) implies that the function $\kappa$ must also be continuous inside $I_p$. Repeating this argument recursively, one can show that all the derivatives of $\kappa$ must be continuous, i.e., that the distribution of the plastic strain is infinitely smooth. This is, however, true only inside the plastic region (and also inside the elastic region, where $\kappa$ vanishes). On the elasto-plastic boundary, the degree of regularity can be lower. Planas et al. (1996) nevertheless have shown that $\kappa$ must be continuous even on the elasto-plastic boundary. As we approach the elasto-plastic boundary from the interior of the plastic region, $\kappa$ must tend to zero, otherwise one of the loading-unloading conditions would be violated. A positive limit would violate the condition $f \leq 0$ in some subdomain of the elastic region close to the boundary, and a negative limit would violate the condition $\dot{\kappa} \geq 0$ in some subdomain of the plastic region close to the boundary.

Thus, in contrast to gradient plasticity, $\kappa$ is only at least $C^0$-continuus everywhere because of the elasto-plastic boundary. This is why the formal equivalence between gradient and nonlocal plasticity, "derived" from the expansion into a truncated Taylor series

$$\bar{\kappa}(x) = \int_{-\infty}^{\infty} \alpha_\infty(|x-\xi|)\kappa(\xi)\mathrm{d}\xi \approx \kappa(x) + c^2\kappa''(x) \tag{21}$$

where the constant $c$ is given by

$$c = \frac{1}{2}\int_{-\infty}^{\infty} \alpha_\infty(|x-\xi|)(x-\xi)^2\mathrm{d}\xi \tag{22}$$

does not hold, since $\kappa$ would have to be $C^1$-continuos everywhere.

### 3.2 *Plastic region far from the boundary*

For a given stress rate $\dot{\sigma} < 0$, the rate of the softening variable (which is for tensile yielding identical with the rate of plastic strain) can be found by solving the rate form of equation (20), written as

$$m\int_{I_p} \alpha(x,\xi)\dot{\kappa}(\xi)\mathrm{d}\xi + (1-m)\dot{\kappa}(x) = \frac{\dot{\sigma}}{H}. \tag{23}$$

Here we have taken into account that $\dot{\kappa}(\xi) = 0$ for $\xi \notin I_p$, and so it is sufficient to integrate over the plastic region only. Equation (23) is a Fredholm integral equation of the second kind for the unknown function $\dot{\kappa}(x)$, and it can be approximately solved using, e.g., the collocation method. A nonstandard feature of the problem is that the interval $I_p$ is not known in advance. The numerical procedure starts from an assumed interval $I_p$ and computes the values of $\kappa$ at the collocation points by solving a set of linear algebraic equations that approximate the integral equation. The formal solution must then be tested for admissibility. First of all, the condition $\dot{\kappa} \geq 0$ implies that the values of $\dot{\kappa}$ at the collocation points must be nonnegative. Second, if the yield condition $f = 0$ is satisfied at some points outside $I_p$ at the beginning of the step (as is the case at the onset of localization from a perfectly uniform state), the rate of the yield function at those points must be nonpositive. This leads to the condition

$$m\int_{I_p} \alpha(x,\xi)\dot{\kappa}(\xi)\mathrm{d}\xi \leq \frac{\dot{\sigma}}{H} \tag{24}$$

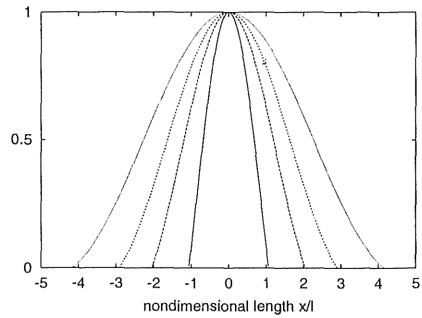


Figure 3: Plastic strain profiles for nonlocal model far from the boundary. Values of $m$: 1.5 (solid), 3 (dot), 5 (dash), 9 (dash-dot).

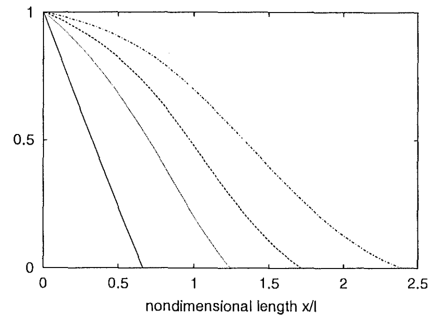


Figure 4: Plastic strain profiles for nonlocal model at the boundary. Values of $m$: 1.5 (solid), 3 (dot), 5 (dash), 9 (dash-dot).

to be verified for those $x$ outside $I_{\mathrm{p}}$ at which $|\sigma(x)| = \sigma_{\mathrm{Y}}(x)$.

If any of the admissibility conditions is violated, the assumed plastic region is adapted accordingly, i.e., reduced if some resolved values of $\dot{\kappa}$ are negative, and extended if (24) is violated. This is iteratively repeated until an admissible solution is found.

Suppose that the plastic zone is situated at a certain distance $a$ from the boundary. The boundary influences the solution only through the rescaling factor in the nonlocal weight function $\alpha$. Furthermore, for a weight function with bounded support, the rescaling is only activated if the distance $a$ is smaller than the support radius $R$. Thus, if no rescaling is performed or if $a \geq R$, the solution is the same as for an infinite domain, and it is symmetric with respect to the center of the plastic region.

For a function with unbounded support, like the Gaussian weight function, there is always an influence of the boundary in the analytical problem. In a numerical solution, however, the support is bounded; the "numerical" support radius $R$ depends on the computer tolerance. This is why, for sufficiently large values of $a$, the numerical solution is the same as for an infinite domain.

The shape of the plastic strain profile is strongly influenced by the parameter $m$, see Fig. 3 for the case of the bell-shaped weight function. The length of the plastic region vanishes for $m = 1$ and continuously increases with $m$.

### 3.3 *Plastic region close to the boundary*

One peculiar property of the refined nonlocal plasticity model is that no admissible solution exists for $0 < a < R$. At points in the boundary layer of thickness $R$, the nonlocal weight function is rescaled according to (10). If the assumed plastic region has a nonempty intersection with the boundary layer but does not touch the boundary, the formal solution of equation (23) becomes nonsymmetric and it is impossible to adjust the size of the plastic region such that the plastic strain tends to zero on both elasto-plastic boundaries at the same time. Only if the plastic region is assumed to start directly at the physical boundary, with no elastic layer interposed, the condition $\kappa = 0$ can be relaxed on the physical boundary and remains valid only on the internal elasto-plastic boundary. For the admissible solution, plastic strain attains its maximum value at the physical boundary and monotonically decreases to zero at the elasto-plastic boundary, (Fig. 4). For values of $m$ close to 1, the plastic strain distribution is almost linear.

### 3.4 *Finite element solution*

In view of the preceeding analysis, only the bell-shaped function with bounded support of well-defined radius $R$ is considered in order to avoid dependence on computer arithmetics.

Incremental finite element analysis requires the implementation of a procedure for the evaluation of the stress and plastic strain increments that correspond to a given increment of strain. The solution must satisfy the loading-unloading conditions (5) at the end of the step. In the plastic region active during the step, characterized by nonzero increments of plastic strains, the yield condition $f = 0$ must be fulfilled at the end of the step. For the local plasticity model, this condition can be enforced at each material point independently, using the conventional stress return algorithms. In contrast to that, in nonlocal plasticity, the material points interact and the yield function at one point depends on the plastic strain increments at all points in the neighborhood of radius $R$. The equations for the evaluation of plastic strain increments become coupled. For one-dimensional nonlocal plasticity with linear softening under tension, we obtain a set of linear equations written in the compact form as

$$[E\boldsymbol{I} + (1-m)H\boldsymbol{I} + mH\boldsymbol{A}]\,\Delta\boldsymbol{\kappa} = \boldsymbol{f}_T \qquad (25)$$

where $\boldsymbol{I}$ is the unit matrix, $\boldsymbol{A}$ is a square matrix representing the discretized nonlocal averaging operator, $\Delta\boldsymbol{\kappa}$ is a column matrix with the unknown increments of the softening variable at the individual Gauss integration points in the plastic region, and $\boldsymbol{f}_T$ is a column matrix with the trial values of the yield function at the individual Gauss points, evaluated as $f_T = \sigma + E\Delta\varepsilon - \sigma_Y$. Here, $\sigma$ and $\sigma_Y$ are the stress and the yield stress at the beginning of the step, and $\Delta\varepsilon$ is the strain increment.

With a proper numbering of the Gauss points, the matrix $\boldsymbol{A}$ is banded but typically has a large bandwidth, so a direct solution technique would be quite expensive. In any case, an iterative procedure must be used because the number of plastic Gauss points in (25) may change after the solution of the system, since neighboring Gauss points may start yielding. An iterative solution can be based on the additive split of the system matrix into the local part, $[E + (1-m)H]\boldsymbol{I}$, and the nonlocal part, $mH\boldsymbol{A}$. If the nonlocal part were not present, the solution of (25) would be

$$\Delta\boldsymbol{\kappa}^{(1)} = \frac{1}{E + (1-m)H}\boldsymbol{f}_T \qquad (26)$$

This step is easy to perform; it corresponds to the standard local stress return algorithm with a modified value of the softening modulus.

Due to the presence of the nonlocal part, $\Delta\boldsymbol{\kappa}^{(1)}$ is not the exact solution of (25). In the spirit of the Jacobi iterative method, we can define a sequence of successive approximations $\Delta\boldsymbol{\kappa}^{(n)}$ by the formula

$$[E + (1-m)H]\Delta\boldsymbol{\kappa}^{(n)} + mH\boldsymbol{A}\,\Delta\boldsymbol{\kappa}^{(n-1)} = \boldsymbol{f}_T \qquad (27)$$

from which

$$\Delta\boldsymbol{\kappa}^{(n)} = \Delta\boldsymbol{\kappa}^{(1)} - \frac{mH}{E + (1-m)H}\boldsymbol{A}\,\Delta\boldsymbol{\kappa}^{(n-1)}. \qquad (28)$$

The difference between the exact solution $\Delta\boldsymbol{\kappa}$ and its approximation $\Delta\boldsymbol{\kappa}^{(n)}$ is in each iteration multiplied by the matrix $(mH/[E + (1-m)H])\boldsymbol{A}$, and convergence is guaranteed if the norm of this matrix is smaller than 1. Matrix $\boldsymbol{A}$ has only nonnegative elements, because the nonlocal weight function is nonnegative. The sum of elements in each row is equal to 1, provided that the nonlocal weight function is rescaled according to (10), and smaller than 1 without rescaling. If the vector norm is chosen as the max-norm

$$\|\boldsymbol{x}\|_{\max} = \max_i |x_i| \qquad (29)$$

then the corresponding matrix norm of $\boldsymbol{A}$ is equal to 1. Thus, convergence is guaranteed for

$$\left|\frac{mH}{E + (1-m)H}\right| < 1 \quad \Leftrightarrow \quad E + H > 0 . \qquad (30)$$

This condition must be satisfied anyway, since for $E + H \leq 0$ snapback occurs even if the plastic strain remains uniform, which is inadmissible. This means that the iterative nonlocal stress return algorithm always converges.

For the numerical simulations of strain localization in a one-dimensional bar, an initial imperfection in the form of a reduced yield stress value is placed into one finite element. The numerical results match the solutions obtained in section 3.2 for $a \geq R$. The solution is centered over the initial imperfection. On the other hand, if the initial imperfection is located within the distance $R + L_s/2$ to the boundary ($L_s$ being the size of the plastic region on an infinite domain), the strain localizes at the boundary, which corresponds to the solution obtained in section 3.3.

### 3.5 *Thermodynamically based nonlocal plasticity model*

Borino et al. (1999) have analyzed the thermodynamic aspects of nonlocal plasticity models and proposed an extension of the postulate of maximum plastic dissipation to nonlocal models. They denoted the usual nonlocal averaging operator as $R$ and constructed the adjoint operator $R^*$, defined by the identity

$$\int_L f\, R^*(g)\,\mathrm{d}x = \int_L R(f)\, g\,\mathrm{d}x \qquad (31)$$

that must hold for any functions $f$ and $g$ for which the right-hand side of (31) makes sense. It is easy to show that if $R$ is the integral operator (9), the adjoint operator $R^*$ is a similar integral operator with swapped arguments of the kernel (weight function):

$$[R^*(g)](x) \equiv \int_L \alpha(\xi, x) g(\xi)\mathrm{d}\xi \qquad (32)$$

If the weight function is symmetric with respect to its arguments (which is the case on an infinite domain or if no rescaling around the boundaries is applied), the operator $R$ is self-adjoint, i.e., $R^* = R$.

For one-dimensional plasticity with linear softening, the model proposed by Borino et al. (1999) differs from the local model only in the softening law (3), which now takes the form

$$\sigma_Y = \sigma_0 + R^*\left(H\, R(\kappa)\right). \qquad (33)$$

Writing the averaging operators explicitly and changing the order of integration, we obtain

$$\begin{aligned} \sigma_Y(x) &= \sigma_0 + \int_L \alpha(\xi,x) H \int_L \alpha(\xi,\eta)\kappa(\eta)\mathrm{d}\eta\mathrm{d}\xi = \\ &= \sigma_0 + H \int_L \int_L \alpha(\xi,x)\alpha(\xi,\eta)\mathrm{d}\xi\kappa(\eta)\mathrm{d}\eta = \\ &= \sigma_0 + H \int_L \beta(x,\eta)\kappa(\eta)\mathrm{d}\eta \end{aligned} \qquad (34)$$

where

$$\beta(x,\eta) = \int_L \alpha(\xi,x)\alpha(\xi,\eta)\mathrm{d}\xi. \qquad (35)$$

The final expression in (34) has the same structure as the softening law (8) used in basic nonlocal plasticity. The nonlocal weight function $\beta$ is now defined indirectly by (35), and it is always symmetric with respect to its arguments. On a finite domain, nonlocal averaging with $\beta$ as the weight function transforms a uniformly distributed local variable into a nonlocal variable that is not uniform in the vicinity of the boundary.

The solutions of the one-dimensional localization problem obtained with the model of Borino et al. (1999) remain essentially the same as those obtained with the basic nonlocal plasticity model. If the function $\alpha$ is regular, $\beta$ is regular as well and the plastic region localizes into a single point. In multiple dimensions, mesh-induced directional bias are expected to persist.

Even though Borino et al. (1999) considered the usual, regular nonlocal weight function, their approach is quite general and admits defining $R$ as an operator combining the nonlocal average with the local value in the spirit of the modified nonlocal model presented in section 2.4. It remains to be investigated whether such a model leads to localized plastic regions of a finite size.

Finally, let us mention that the symmetric form of the nonlocal weight function is not dictated by the requirements of thermodynamic admissibility but by the adoption of the postulate of maximum plastic dissipation, which provides a fully associated model. The basic formulation of nonlocal plasticity does not seem to violate the second law of thermodynamics.

## 4 DISCUSSION OF BOUNDARY EFFECTS

### 4.1 *Plastic region far from the boundary*

For a localization zone sufficiently far from the boundary, the plastic strain profiles for gradient plasticity and nonlocal plasticity are plotted in Fig. 1 and Fig. 2. The shapes of the curves exhibit small differences, especially close to the elasto-plastic boundary, where the profile obtained with the gradient plasticity model is smoother due to the imposed $C^1$-continuity.

### 4.2 *Plastic region close to the boundary*

In the vicinity of a boundary, the plastic strain profile strongly depends on the adopted boundary condition (for the gradient model) or rescaling rule (for the nonlocal model).

For the gradient model, the boundary condition must be formulated in terms of either the softening variable $\kappa$ or its first derivative $\kappa'$. Obviously, it does not make sense to prescribe a nonzero value of $\kappa$ or $\kappa'$, because the condition must be satisfied at all stages of the loading process, including the elastic stage, during which the softening variable vanishes. If a zero value

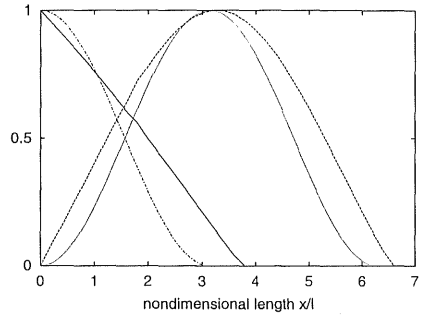


Figure 5: Comparison of refined nonlocal and gradient model in the vicinity of a boundary: nonlocal model with rescaling (solid), without rescaling (dash); gradient plasticity with $\kappa'(0) = 0$ (dot), and with $\kappa(0) = 0$ and $\kappa'(0) = 0$ (dash-dot).

of $\kappa$ is prescribed on the boundary, the distribution of plastic strain in a plastic zone starting right at the boundary is exactly the same as far from the boundary, Fig. 5. On the other hand, if a zero value of $\kappa'$ is prescribed, the foregoing solution remains admissible, but another solution emerges, with maximum plastic strain right at the boundary. This solution gives a steeper post-peak slope of the load-displacement diagram than the solution with a plastic zone far from the boundary. According to the stability criterion discussed in Bažant and Cedolin (1991), this is the solution that would actually occur.

For the nonlocal model, the effect of boundary treatment is even more pronounced. The model of course does not require any boundary condition, but it is necessary to specify how the nonlocal averaging operator treats the case when a part of the neighborhood that contributes to the nonlocal average protrudes out of the body. If the original weight function is kept without any changes and the integral is computed only over the part of the contributing neighborhood located inside the body, there is no solution with a localized plastic region touching the boundary. In fact, not even a solution with a localized plastic region separated from the boundary by an elastic layer of a thickness smaller than $R + L_s/2$, where $L_s$ is the size of the plastic region in an infinite bar, exists. On the other hand, if the nonlocal weight function is rescaled according to (10), a solution with a maximum plastic strain right at the boundary emerges, and this solution again leads to a steeper load-displacement diagram than that obtained far from the boundary.

In conclusion, the boundary can either repel or attract localization, depending on the details of the formulation. From the physical point of view, this dichotomy could be related to the microstructure of the boundary. For example, for concrete one may think of the two cases schematically presented in Fig. 6. For a boundary layer of soft matrix without any hard ag-

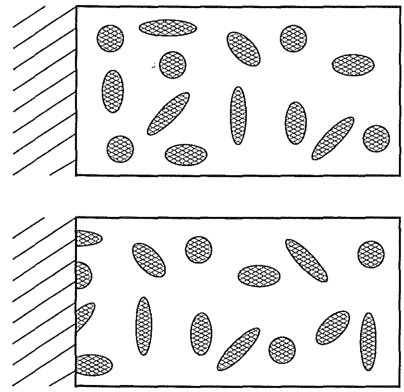


Figure 6: Micromechanical structure at the boundary.

gregates, localization at the boundary would be easier than inside the body, Fig. 6 a. If, on the other hand, the hard particles are present at the boundary and are strongly glued to the rigid support, localization at the boundary would be more difficult than inside the body, Fig. 6 b. The two types of boundary conditions or of boundary averaging rules discussed above should be seen only as the extreme cases. It would be possible to construct intermediate boundary conditions (e.g., a linear combination $a\kappa + b\kappa' = 0$) or rescaling rules. Their design should be guided by micromechanical analyses of the boundary region.

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