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Mixed mode fracture of an higher-order beam model in gradient plasticity

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ABSTRACT: A higher-order shear-deformable beam model with gradient-dependent plasticity regularisation is presented in this work. The model takes into account a more realistic non-linear variation of stresses and strains through the beam thickness. Numerical results are presented for a multilayered beam. The discrete problem size is significantly reduced through this semi-local approach.

1 INTRODUCTION

Shear banding and strain localization of ductile and quasi-brittle materials can be understood as an instability in the macroscopic constitutive description of inelastic deformation. From a mathematical point of view, the appearance of localization in classical continuum mechanics is associated with a change of type of the governing equations. The governing equations loose ellipticity in statics, and this can be considered as an indicator of the initiation of a material surface discontinuity. Since the localization zone can be infinitely thin, a displacement discontinuity may develop, leading to numerical difficulties when conventional FE-methods are used. The calculated energy dissipation tends to zero upon refinement of the FE-mesh. Thus, the global postlocalized response becomes infinitely brittle and shows a critical mesh dependence.

Higher-order continuum theories can be used to remedy the mesh dependence. Aifantis (1992) and Vardoulakis et al (1990) demonstrated the regularizing role of higher order strain gradients in localization phenomena. The governing equations possess a mathematical structure, which is amenable to non-linear stability analysis. The inclusion of the equivalent plastic strain gradient into the yield condition leads to the prediction of the shear-band widths.

A shear-deformable beam model with cross section warping incorporating gradient-dependent plasticity regularization is presented in this work. The model is able to describe localized failure under shear, tension and mixed-mode conditions. An appropriate form of the warping function was chosen in order to provide more accurate solutions, thus eliminates the use of shear correction coefficients, as it is the case in Timoshenko's and Reissner's theory for beams.

2 FINITE ELEMENT FORMULATION

Although gradient dependent models bear significant advantages of mesh independence and preservation of ellipticity, the increment of the plastic strain can not be obtained at a local level. The consistency condition which governs the plastic flow, becomes a second order partial differential equation thereof. One may use a finite difference method as proposed by Belytschko and Lasry (1989); the algorithm is then a sequence of separate approximate solutions of the equilibrium problem. de Borst and Mühlhaus (1992) showed a more general approach which uses only finite elements and solves the problems of the functional dependence of the yield function on the plastic strain and its Laplacian. The plastic strain Laplacian acting as an internal length scale, the localization zones have finite widths and mesh dependency is prevented.

The yield function and the consistency condition in the second gradient theory read:

$$f(\sigma, \kappa, \nabla^2\kappa) = 0. \tag{1}$$

$$\dot{f}(\sigma, \kappa, \nabla^2\kappa) = 0. \tag{2}$$

where κ is the equivalent plastic strain.
The discretization of the plastic multiplier was required to satisfy weakly the yield condition. The nodal displacements and the degrees of freedom Λ related to the plastic multiplier λ give:

$$\boldsymbol{u} = \mathbf{N}\mathbf{a} \quad \lambda = \mathbf{H}^{T}\Lambda \quad \nabla^2\lambda = \mathbf{P}^{T}\Lambda. \tag{3}$$

where $\mathbf{N}$ and $\mathbf{H}$ are shape functions.

The weak satisfaction of the equilibrium and the Laplacian-dependent yield condition lead to the functions that govern the equilibrium process (de Borst et al. 1992):

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{\lambda a}^{T} \\ \mathbf{K}_{\lambda a} & \mathbf{K}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{da} \\ \mathrm{d}\Lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e - \mathbf{f}_i \\ \mathbf{f}_\lambda \end{bmatrix}. \tag{4}$$

with the elastic stiffness matrix K_{aa}, the off diagonal matrix $K_{a\lambda}$, the gradient dependent matrix $K_{\lambda\lambda}$, f_e is the external force vector, f_i the internal force vector and f_λ is the vector of the residual forces resulting from the inexact fulfilment of the yield condition.

3 THE MULTILAYERED FINITE ELEMENT MODEL

The multilayered finite element has originally been developed by Meftah (1997). It is a method allowing finite element analyses with reduced degrees of freedom: $\mathbf{a} = [u,v,\beta]^T$, where u and v are respectively the axial and the transverse displacements of the beam middle plane, β is the rotation thereof. The plastic multiplier field is divided into superposed layers trough the beam depth, giving its variation by the mean of nodal parameter$\Lambda^k=(\Lambda_i^k,\Lambda_j^k)$ at each layer k. C^1 continuous interpolation polynomials are considered for the plastic multiplier field. Therefore, the equation of the equilibrium process for n layers becomes:

$$\begin{bmatrix} [K_{aa}] & [K_{\lambda a}^1]^T & \cdots & [K_{\lambda a}^k]^T & \cdots & [K_{\lambda a}^n]^T \\ [K_{\lambda a}^1] & [K_{\lambda\lambda}^1] & \cdots & [0] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [K_{\lambda a}^k] & [0] & \cdots & [K_{\lambda\lambda}^k] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [K_{\lambda a}^n] & [0] & \cdots & [0] & \cdots & [K_{\lambda\lambda}^n] \end{bmatrix} \begin{bmatrix} \mathbf{da} \\ \mathrm{d}\Lambda^1 \\ \vdots \\ \mathrm{d}\Lambda^k \\ \vdots \\ \mathrm{d}\Lambda^n \end{bmatrix} = \begin{bmatrix} f_e - f_i \\ f_\lambda^1 \\ \vdots \\ f_\lambda^k \\ \vdots \\ f_\lambda^n \end{bmatrix}. \tag{5}$$

The multilayered beam model was previously approximated by the Euler-Bernoulli theory of bending which leads to serious discrepancies in the case of beams with small aspect ratio. The Euler-Bernoulli theory is not appropriate for shear failure and mixed-mode fracture.

4 HIGHER-ORDER SHEAR THEORY

The hypothesis that the cross section normal to the undeformed middle surface remains so after bending may adequately approximate the behaviour of structures like thin beams. This theory satisfies the shear free boundary condition on the lateral surface of the beam, but it implies that the transverse shear strain becomes zero.

In the Timoshenko's theory for beams, the transverse shear strain remains constant through the thickness. It does not satisfy the shear-free boundary condition mentioned above. To avoid discrepancies in the shear constitutive equations, a shear correction coefficient is then introduced.

The higher-order deformation theory incorporates a more realistic non-linear variation of the longitudinal displacements through the beam thickness thus, eliminates the use of shear correction coefficients. This theory allows the cross-section to rotate and to warp into a non-planar surface. The following kinematical assumption is made:

$$u(x,y) = u(x) + y^*\beta(x) - \frac{4}{3h^2} * \left(\beta(x) + \frac{dv}{dx}\right)*y^3, \tag{6}$$

$$v(x\,,\,y) \ = v(x). \tag{7}$$

where u and v are respectively the axial and the transverse displacements of the middle plane (x-axis), β is the rotation thereof and h is the beam depth (Levinson 1981).

This higher-order theory allows the non-uniform shearing of the cross section as well as the possibility of satisfying the shear free boundary conditions on the lateral surface of the beam. No shear coefficient is required. A similar effort has also been made by Kant et al. (1989), but their model requires additional degrees of freedom.

This kinematical assumption was introduced in the displacement field of the multilayered beam previously developed with the Bernoulli theory (Salomon 2000). The gradient regularization concerns only the tensile principal stress. It is only obtained from the shear and normal stresses which are coupled in a Drucker-Prager yield criterion adapted for tension, since the model does not take into account the transverse normal stress.

5 NUMERICAL EXAMPLES

5.1 *Four-point bending*

The geometry and the material data for the notched concrete beam are based on the experimental values of Hordjik (1991): Young's modulus $E = 40000\ N/mm^2$, tensile strength $ft = 3.3\ N/mm^2$, fracture energy $G_f = 0.12$ N/mm and the internal length $l = 3$ mm. The half of the beam is discretized using 18 elements along the beam length, 41 layers through the depth and 37 layers in the notched area. Deformation control is used.

The comparison models are the plane stress case analysis of Pamin (1994) and the multilayered approach with Euler-Bernoulli kinematics of Meftah (1997). Although Mode I fracture occurs, where shear effects are insignificant, the shear-deformable beam leads to better results than the one with Euler theory which overestimates the peak load by 28% and the deflection by 78%.

A comparison of the present analysis with the experimental data is in good agreement. The present beam model overestimates the peak load by 11% and the displacement by 7%. Due to gradient regularisation, fracture is distributed to elements neighbouring the notch.

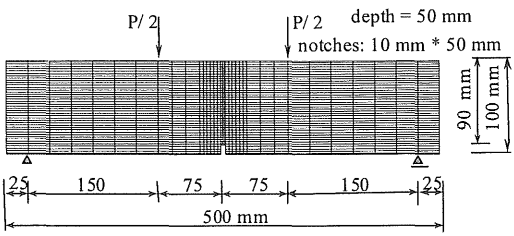

Figure 1. Geometry of the notched beam

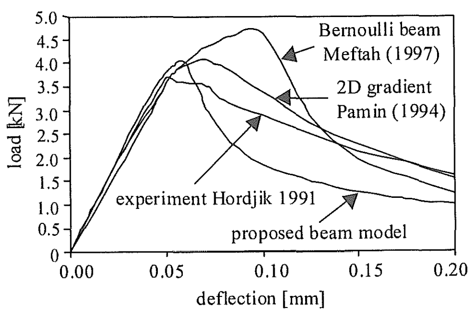

Figure 2. Load deflection diagram of the four-point-bending test

5.2 *Four-point shear test*

The four-point-shear test was initially proposed to study the yield strength of welded joints. The geometry was adapted to concrete materials in order to improve the behaviour of the shearing zone (Bažant et al. 1986). Those beams do not follow Saint-Venant's hypothesis, since the displacement control point is next to the loading platen. This simulation should test the ability of the present model to describe mixed mode fracture.

The geometry and the material data for the notched concrete beam are based on the experimental values of Schlangen (1993): Young's modulus $E = 35000\ N/mm^2$, tensile strength $ft = 3.0\ N/mm^2$, compressive strength $f_c = 46.6\ N/mm^2$, fracture energy $G_f = 0.10$ N/mm and the internal length $l = 3$ mm. Arc length method is used.

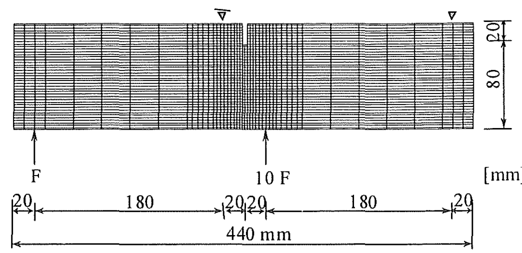

Figure 3. Geometry of the four-point-shear test beam

In the neutral axis of the beam, high shear stresses rise in the notched area, while the bending moment vanishes. According to the beam kinematics, this means no axial stress in the notched area (Salomon

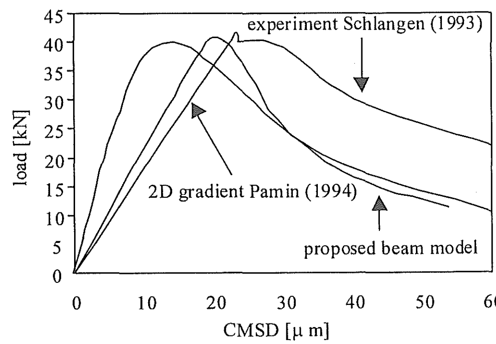

Figure 4. Load versus Crack Mouth Sliding Displacement [CMSD] of the four-point-shear test

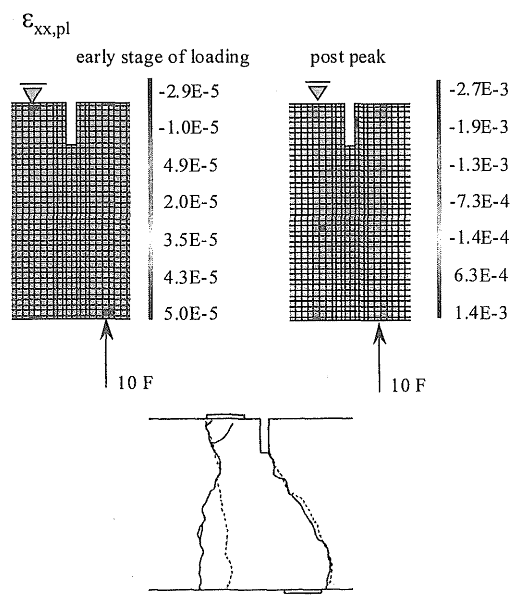

Figure 5. Crack pattern of the shear test a) beam model, b) experiment (Schlangen 1993)

2000). This should give a better insight in the global response of the beam model (cf. figure 4).

The present model underestimates the peak load by 2.5%, the sliding displacement by 32 %. At early load stage, the model could not simulate the normal stress concentration at the notched. As a consequence the response in the pre-peak regime is too stiff. However, the failure mode and the peak load have been properly simulated. Mixed mode fracture occurs, since the fracture zone has a zigzag shape (cf. figure 5). The cracking of the notched zone is due to the shear stress that compensates for the zero normal stresses.

6 CONCLUSION

In this paper, a beam element based on a higher-order shear deformation theory with gradient plasticity regularization is developed and studied. For thin beams, shear stress is often justifiably neglected in the analysis using the Bernoulli theory. When high shear gradients are present in the beam, excluding warping due to transverse shear may not be justified. The chosen warping function provides accurate solutions thus, eliminates the use of shear correction coefficients.

The higher order beam element accurately predicts the peak load for the four-point bending analysis. Mixed mode fracture was properly simulate, and the response of the model are almost similar to those given by a full bidimensional analysis.

The numerical results indicate that the higher order beam theory coupled with gradient plasticity suffices for the examination of many beam problems and provides a beam element, which accounts for shear deformation effects. This beam model consists in a quasi two-dimensional method, allowing finite element analyses with reduced degrees of freedom. This means that memory workload and calculation cost decreases.

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