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Size Effects Quasibrittle Fracture: Apercu of Recent Results

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ABSTRACT: The paper consists of two parts. In view of length limitations, and because several comprehensive reviews behind the present lecture were already published in journals, the first part is limited to listing various recent advances that are surveyed in the conference lecture. The second, much longer, part then complements this survey by presenting in mathematical terms several new results that are only outlined in the lecture. These results concern: 1) a derivation of the first two terms of the small-size asymptotic expansion of size effect of the cohesive crack model; 2) a review of the derivation of the first two terms of the large-size asymptotic expansion of size effect ensuing from the smeared-tip method; and 3) a size effect formula for a very broad size range, unifying the fracture energies measured by the size effect method and the work-of-fracture method.

1. OVERVIEW OF SOME RECENT RESULTS

The size effect represents the most important practical consequence of fracture behavior as well as the clue to uncovering various fundamental characteristics of concrete fracture. Interest in the quasibrittle size effect, which started in the 1970s and surged throughout the 1990s, continues unabated and will probably persist for some time because significant open questions remain and applications in design, especially in terms of revisions of design code specifications, are still deplorably limited. The conference lecture addressing this broad subject reviews some selected recent results, dealing with:

- amalgamation of the deterministic (energetic) theory of quasibrittle size effect with the Weibull probabilistic theory of brittle size effect;
- derivation of small-size asymptotic properties of size effect ensuing from the cohesive crack model;
- size effect law for a very broad size range, explaining the difference between the fracture energies obtained by the work-of-fracture method (Hillerborg 1985a,b) and the size effect method (proposed by Bažant in 1987, see Bažant and Planas 1998) or Jenq and Shah's (1985) method;
- approximate statistical prediction of the fracture properties of concrete from its simple design characteristics;
- size effect in redundant beam structures failing by softening inelastic hinges; and
- size effect hidden in excessive dead load factor in the design codes (Bažant and Frangopol 2000).

In the closing of the lecture, it is pointed out that the size effect must have played a major, yet previously unrecognized, role in a host of famous structural catastrophes (e.g., Malpasset Dam, St. Francis Dam, Schoharie Creek Bridge, Sleipner Oil Platform, Hanshin Viaduct and Cypress Viaduct).

Since the aforementioned subjects have recently been reviewed in several journal articles of various foci and scopes (Bažant and Chen 1997, Bažant 1999a,b, 2000, 2001a), it would be superfluous, and inevitably duplicative, to devote this paper to still another review. Therefore, the rest of this paper will focus on several new, still unpublished, results that are outlined in the conference lecture.

2. NEW RESULTS ON ASYMPTOTIC SIZE EFFECT PROPERTIES

2.1 *Small-Size Asymptotics of Cohesive Crack Model*

The large-size asymptotic properties of the quasibrittle size effect have been determined on the basis of equivalent LEFM. That approach, however, is not possible for the small-size asymptotic properties. On the basis of the numerical solutions with the cohesive crack model, crack band model and nonlocal model, it has been well known for a long time that for a vanishing structure size D the nominal strength σ_N of a quasibrittle structure approaches a finite value, σ_N^0. This means that the size effect plot of $\log \sigma_N$ versus $\log D$ must approach on the left a horizontal asymptote. But how precisely this limit value or horizontal asymptote should be approached? It would be helpful to know. To this end, we will try to determine the second term of the small-size asymptotic expansion of size effect.

The static boundary value problem of linear elasticity is defined in Cartesian coordinates x_i ($i = 1, 2, 3$) as follows:

$$\sigma_{ij} = E_{ijkl}\tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

$$\sigma_{ij,j} + f_i = 0 \qquad (\text{in } \mathcal{V})$$

$$n_j\sigma_{ij} = p_i \quad (\text{on } \Gamma_s) \tag{2}$$

$$\bar{u}_i = 0 \quad (\text{on } \Gamma_d)$$

Here σ_{ij} = stress tensor components, $\frac{1}{2}(u_{i,j}+u_{j,i}) = \epsilon_{ij}$ = strain tensor components, E_{ijkl} = elastic moduli, f_i = body forces, p_i = surface tractions, prescribed on surface domain Γ_s, n_i = unit normal of the surface, and Γ_d is the surface domain where the displacements are fixed by supports.

Let us consider geometrically similar structures of various sizes D and introduce the dimensionless coordinates and variables, labeled by an overbar;

$$\bar{x}_i = x_i/D, \quad \bar{u}_i = u_i/D, \tag{3}$$

$$\bar{\sigma}_{ij} = \sigma_{ij}/\sigma_0$$

$$\bar{p}_i = p_i/\sigma_N, \quad \bar{f}_i = f_i D/\sigma_N, \tag{4}$$

$$\bar{E}_{ijkl} = E_{ijkl}/\sigma_0$$

The load magnitude is assumed to be characterized by σ_N as a single parameter, and so $\bar{p}_i$ is a size independent distribution of the dimensionless surface tractions on Γ_s, and $\bar{f}_i$ is a size-independent distribution of dimensionless body forces in volume $\bar{\mathcal{V}}$. The surface normals n_i at homologous points are independent of size D (and thus need no overbar).

Denoting $\partial_i = \partial/\partial\bar{x}_i$ = partial derivatives with respect to the dimensionless coordinates, and noting that $\partial/\partial x_i = (1/D)\partial_i$, we can transform the foregoing equations to the following dimensionless form:

$$\bar{\sigma}_{ij} = \bar{E}_{ijkl}\tfrac{1}{2}(\partial_k\bar{u}_l + \partial_l\bar{u}_k), \tag{5}$$

$$\partial_j\bar{\sigma}_{ij} + \bar{f}_i\ \sigma_N/\sigma_0 = 0 \qquad (\text{in } \bar{\mathcal{V}})$$

$$n_j\bar{\sigma}_{ij} = \bar{p}_i\ \sigma_N/\sigma_0 \quad (\text{on } \bar{\Gamma}_s), \tag{6}$$

$$\bar{u}_i = 0 \quad (\text{on } \bar{\Gamma}_d)$$

where $\bar{\mathcal{V}}$ is the domain of structure volume in the dimensionless coordinates, and $\bar{\Gamma}_s$ and $\bar{\Gamma}_d$ are the surface domains in dimensionless coordinates corresponding to Γ_s and Γ_d.

Let coordinates x_i be positioned so that the crack would lie in the plane (x_1, x_3) and that the tip of the cohesive crack (and not the notch tip) would be at $x_1 = 0$. For a small enough D, the crack-bridging stress $\sigma > 0$ along the whole crack length L, and if D is small enough and if the compression strength is unlimited, the cohesive crack (with bridging stresses) will occupy at maximum load the entire area of the cross section or, in the case of a notch, the entire area of the ligament (note that if compressive stresses are needed in the ligament, they localize into a Dirac delta function); then the dimensionless crack length $\bar{L} = L/D$ = constant. If the compression strength is limited and the cross section is for instance subjected to bending, then a finite portion of cross section or ligament will be under compression, and then L/D will not be size independent; but we may assume it to be such, as an approximation for small D, since the strength in compression is much larger than in tension.

In the case of cohesive fracture, equations (5) and 6) must be supplemented by two conditions for the cohesive crack: 1) The dimensionless total stress intensity factor $\bar{K}_t = K_t\sqrt{D}/\sigma_N$ produced jointly by the applied load and the tractions $\bar{\sigma} = \bar{\sigma}_{22}$ acting on the crack faces must vanish in order to ensure the finiteness of the crack-tip stresses, i.e.

$$\bar{K}_t = 0 \tag{7}$$

2) The cohesive (crack-bridging) stresses σ must satisfy the softening law of the cohesive crack, i.e., the curve relating σ to the opening displacement $w = 2u_2$ on the crack plane. We will consider the law

$$\sigma = \sigma_0[1 - (w/w_f)^p] \tag{8}$$

(for $x_1 \in (-L, 0), x_2 = 0$); here p, w_f = positive constants, and σ_0 = tensile strength (also denoted as f_t'). In terms of the dimensionless variables corresponding to (3), the dimensionless form of the assumed softening law is

$$\bar{\sigma} = 1 - (\bar{D}\bar{w})^p \tag{9}$$

with

$$\bar{\sigma} = \sigma/\sigma_0, \quad \bar{w} = w/D,$$

$$\bar{D} = D/w_f \tag{10}$$

(for $\bar{x}_1 \in (-\bar{L}/D, 0), \bar{x}_2 = 0$).

We will now consider the dependence of the solution on structure size D. We will assume the dimensionless displacements, stresses and total stress intensity factor to approach their limit for $\bar{D} \to 0$ as power functions of $\bar{D}$ with exponent p, and will try to verify the correctness of this hypothesis. So, for small enough $\bar{D}$, we set:

$$\sigma_N = \sigma_N^0 + \sigma_N'\bar{D}^p, \tag{11}$$

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ij}^0 + \bar{\sigma}_{ij}'\bar{D}^p$$

$$\bar{\sigma} = \bar{\sigma}^0 + \bar{\sigma}'\bar{D}^p$$

$$\bar{u}_i = \bar{u}_i^0 + \bar{u}_i'\bar{D}^p,$$
$$\bar{w} = \bar{w}^0 + \bar{w}'\bar{D}^p,$$
$$\bar{K}_t = \bar{K}_t^0 + \bar{K}_t'\bar{D}^p \tag{12}$$

where $\sigma_N^0, \sigma_N', \sigma^0, \sigma', \sigma_{ij}^0, ..., K_t'$ are size independent. These expressions may now be substituted into (9), (7), (5) and (6), and the binomial expansion $\bar{w}^p = (\bar{w}^0)^p[1 + (\bar{w}'/\bar{w}^0)p\bar{D}^p + ...]$ should be noted. The resulting equations must be satisfied for various small sizes $\bar{D}$. For $\bar{D} \to 0$, the dominant terms in these equations are those of the lowest powers of D, which are those with $\bar{D}^0$ and $\bar{D}^p$. By collecting the terms without $\bar{D}$ and those with $\bar{D}^p$, we obtain two independent sets of equations. It so happens that each of these two sets defines a physically meaningful boundary value problem of elasticity for a body with given tractions applied on crack faces. This proves our hypothesis made in (11) and (12) to be justified.

Elasticity Problem I: By isolating the terms that do not contain $\bar{D}$ (i.e., contain $\bar{D}^0$), we get:

$$\bar{K}_t^0 = 0, \quad \bar{\sigma}^0 = 1 \tag{13}$$
$$(\text{for } -\bar{L} \leq \bar{x}_1 < 0, \bar{x}_2 = 0)$$
$$\bar{\sigma}_{ij}^0 = \bar{E}_{ijkl}\,\tfrac{1}{2}(\partial_j\bar{u}_i^0 + \partial_i\bar{u}_j^0), \tag{14}$$
$$\partial_j\bar{\sigma}_{ij}^0 + \bar{f}_i\;\sigma_N^0/\sigma_0 = 0,$$
$$(\text{in } \bar{V})$$
$$n_j\bar{\sigma}_{ij}^0 = \bar{p}_i\;\sigma_N^0/\sigma_0 \quad (\text{on } \bar{\Gamma}_s), \tag{15}$$
$$\bar{u}_i^0 = 0 \quad (\text{on } \bar{\Gamma}_d)$$

Elasticity Problem II: By isolating the terms that contain $\bar{D}^p$, we get:

$$\bar{K}_t' = 0, \quad \bar{\sigma}' = -(\bar{w}^0)^p \tag{16}$$
$$(\text{for } -\bar{L} \leq \bar{x}_1 < 0, \bar{x}_2 = 0)$$
$$\bar{\sigma}_{ij}' = \bar{E}_{ijkl}\,\tfrac{1}{2}(\partial_j\bar{u}_i' + \partial_i\bar{u}_j'), \tag{17}$$
$$\partial_j\bar{\sigma}_{ij}' + \bar{f}_i\;\sigma_N'/\sigma_0 = 0,$$
$$(\text{in } \bar{V})$$
$$n_j\bar{\sigma}_{ij}' = \bar{p}_i\;\sigma_N'/\sigma_0 \quad (\text{on } \bar{\Gamma}_s), \tag{18}$$
$$\bar{u}_i' = 0 \quad (\text{on } \bar{\Gamma}_d)$$

Note that parameter $\bar{w}'$ does not appear in this problem.

The role of stresses and displacements is played by $\bar{\sigma}_{ij}^0$ and $\bar{u}_i^0$ in problem I, and by $\bar{\sigma}_{ij}'$ and $\bar{u}_i'$ in problem II. In problem I, the crack faces are subjected to fixed uniform tractions equal to 1. In problem II, in which σ' plays the role of the cohesive stress, the crack faces are subjected to tractions $-(\bar{w}^0)^p$ which vary along the crack faces but can be determined in advance from the $\bar{w}^0$-values obtained in solving problem I. The fact that isolation of the terms with the zero-th power and the p-th power of D happens to yield two separate boundary value problems of elasticity is crucial for our goal. The rest of the argument is easy and may be stated as follows.

The magnitude of the loads (surface tractions and body forces) is proportional to σ_N^0 in problem I, and to σ_N' in problem II. These elasticity problems are known to have a unique solution. If σ_N^0 were zero, i.e., if the applied load in problem I vanished, the crack face tractions equal to 1 would cause K_t^0 to be nonzero, in violation of (13). Likewise, if σ_N' were zero, i.e., if the applied load in problem II vanished, the nonuniform crack face tractions $-(\bar{w}^0)^p$ in problem II would cause K_t' to be nonzero, in violation of (16). If the loads for problems I and II were infinite, then K_t^0 or K_t' would be infinite as well, which would again violate (13) or (16). Therefore, the only possibility left is that both σ_N^0 and σ_N' are finite. Thus we have proven the following:

THEOREM I: If the softening law of the cohesive crack model has a finite strength and starts its descent as w^p, then the size effect law for nominal strength approaches for $D \to 0$ a finite value and does so as D^p.

2.2 *Some Implications for Size Effect Formulae*

As widely agreed, the softening cohesive law for quasibrittle materials such as concrete begins its descent with a tangent of a finite slope (e.g., Guinea et al. 1997); hence, $p = 1$. Consequently, according to (11), the size effect law must begin near zero size D as a linear function of D, and as an exponential in the logarithmic plot [the latter ensuing from the approximation $\ln\sigma_N - \ln\sigma_N^0 = \ln(1 + \sigma_N'\bar{D}/\sigma_N^0) \approx (\sigma_N'/\sigma_N^0)\, e^{\ln\bar{D}}$].

The case $p > 1$ means that the softening law begins its descent from a horizontal initial tangent, which is reasonable to assume for ductile fracture of plastic yielding materials. The case $p < 1$ means that the cohesive law begins its descent with a vertical tangent, which would be an unrealistic superbrittle behavior.

The condition that $p = 1$ for quasibrittle materials such as concrete happens to be satisfied by the classical size effect law for bodies with large and similar cracks proposed by Bažant in 1984. Indeed, $\sigma_N \propto (1 + D/D_0)^{-1/2} \approx 1 - D/2D_0$ for small D (D_0 = constant). But this condition is satisfied for none of the formulae

$$\sigma_N = \frac{Bf_t'}{1 + \sqrt{D/D_0}},$$
$$\sigma_N = \frac{Bf_t'}{[1 + (D/D_0)^r]^{1/2r}},$$
$$\sigma_N = \sigma_0\sqrt{1 - e^{-D_0/D}},$$
$$\sigma_N = \sigma_0\left(1 - e^{-(D_0/D)^s}\right)^{1/2s} \tag{19}$$

(with D_0, r, s = positive constant, $r \neq 1$) even though each of these four formulae (the first being a special case of the second, and the third of the fourth) has correct small-size and large-size asymptotes. As for the case $r > 1$ ($p = r$), the softening law begins its descent from a horizontal asymptote, which means that this case might be suitable for ductile fracture of plastically yielding materials.

The foregoing analysis also applies to structures failing at crack initiation from a smooth surface. It may now be noted that the formula

$$\sigma_N = \sigma_\infty \left(1 + \frac{rD_b}{D}\right)^{1/r} \qquad (r > 0) \tag{20}$$

derived from equivalent LEFM by Bažant (1998) (which includes, as a special case for $r = \frac{1}{2}$, the 'MFSL' law of Carpinteri et al. 1994a,b) does not satisfy the small size asymptotic properties of the cohesive crack model (for $p = 1$). However, a simple adjustment of this formula, proposed in 1998 by Bažant, does (η = positive constant):

$$\sigma_N = \sigma_\infty \left(1 + \frac{rD_b}{\eta D_b + D}\right)^{1/r} \tag{21}$$

It must be admitted that our imposition of the small-size asymptotic properties of the cohesive crack model on the size effect law is debatable since, for cross section thicknesses less than several aggregate sizes, the material is not a continuum. For this reason, it may well be considered admissible to have an infinite σ_N for $D \to 0$ (which is the property of the widely used Hall-Petch formula for the yield strength dependence on the crystal size in metals; Petch 1954). Some researchers might even regard the preceding asymptotic analysis invalid because of heterogeneity of the material on the small scale.

Yet such counter-arguments have a somewhat nihilistic flavor. They could in fact be used to shoot down all asymptotic methods, since the infinitely large and the infinitely small are never attainable in reality. Imposition of the small-size asymptotic requirements is advantageous from the viewpoint of *asymptotic matching*, i.e., approximations that have two-sided exact asymptotic support (popularly, 'interpolation between opposite infinities; Bender and Orszag, 1978). Although the cohesive crack model is not amenable to a simple analytical solution in the middle size range, its validity in that range is not in question. In the spirit of asymptotic matching, an approximation for the middle range will be better if it satisfies the (easily solvable) small-size and large-size asymptotic properties of the theory that applies in that range.

2.3 *Large-Size Asymptotics via K-Profile of Cohesive Crack*

For very large sizes, the asymptotic size effect must again be determined from the theory valid for the middle size range—i.e., from the cohesive crack model (even though geometric scaling becomes in practice impossible, since the own weight dominates). In their mathematically rigorous and sophisticated analysis, Planas and Elices (1992, 1993) used the smeared-tip method to establish the first two terms of the large-scale asymptotic expansion of the size effect of the cohesive crack model for the case of notched structures of totally positive geometry. In this method (Bažant and Planas 1998), a cohesive crack is modeled as a weighted superposition of infinitely many LEFM solutions with different crack lengths. In the original version of this methods, used by Planas and Elices (1992, 1993), the weights were characterized in terms of the profile of nominal strength density, the p-profile. This profile depends, even for the large size limit, on the structure geometry, which is a disadvantage.

Recently (Bažant and Zi 2001), the smeared-tip method was reformulated with the weights characterized by the profile of a continuously distributed (smeared) stress intensity factor, called the K-profile, which has the advantage that asymptotically for large sizes it is independent of the structure geometry. Thus the K-profile can be used to characterize the softening stress-displacement curve of the cohesive crack model; it can be derived from that curve by solving a certain integral equation, and the stress-displacement can be derived from the K-profile. While this equivalence of the stress-displacement curve and the K-profile is exact only asymptotically for very large sizes, it is approximately valid even for normal structure sizes, and since the cohesive crack model itself is only an approximation, the K-profile may be used as an alternative general characterization of the cohesive fracture properties, except perhaps for very small structure sizes. One advantage of this alternative approach is that the asymptotic properties of size effect can be derived more easily. This advantage was exploited (Bažant and Zi 2001) to re-derive the Planas and Elices' (1992, 1993) results on the asymptotic size effect for notched specimens of totally positive geometry in a shorter way, and to extend the analysis to other size effect types (Types 1 and 3 defined later).

For an elastic body with a sharp crack, the applied load P and the mode I LEFM stress intensity factor are related as $P = b\sqrt{D}\ K_I(\alpha)/k(\alpha)$ where $\alpha = a/D$, a = crack length, b = body thickness; $k(\alpha)$ = dimensionless stress intensity factor = K_I for $D = b = P = 1$. For an elastic body with a cohesive crack, the applied load P, the crack-bridging (cohesive) stresses σ and the crack opening w are expressed in the smeared-tip method as a superposition of the LEFM solutions for infinitely many cracks with continuously distributed (smeared) tips;

$$P = \int \mathrm{d}P = b\sqrt{D} \int_0^{L/D} \frac{\mathrm{d}K_I(\alpha)}{k(\alpha)} \tag{22}$$

$$\mathrm{d}K_I(\alpha) = \frac{K_c D}{2c_f} q[\rho(\alpha)]\,\mathrm{d}\alpha = K_c\, q(\rho)\,\mathrm{d}\rho \quad (23)$$

where $K_c = \sqrt{E'G_f}$ = fracture toughness (critical K_I) (G_f = fracture energy); E' = effective Young's modulus); c_f = half-length of the fracture process zone (FPZ); L = final length of the crack at total break; $\mathrm{d}K_I(\alpha)$ is the stress-intensity factor of the smeared tips lying between α and $\alpha + \mathrm{d}\alpha$; $\rho = (\alpha - \alpha_1)/2\theta$ where $\theta = c_f/D = \frac{1}{2}(\alpha_2 - \alpha_1)$; α_1 is the end of the stress-free crack portion and α_2 is the tip of the cohesive crack (end of FPZ); $q(\rho)$ is the dimensionless K-profile, such that the function $q(\rho)/\sqrt{|\omega - \rho|}$ be integrable for $0 < \omega < 1$. For $D \to \infty$, the FPZ in the relative coordinate α becomes a point, and so LEFM must apply, which means that $\int dK_I = K_c$ or $I_1 = \int_0^1 q(\rho)\mathrm{d}\rho = 1$. The use of $q(\rho)$ contrasts with the original version of the smeared-tip method, in which (22) and (23) are replaced by $\mathrm{d}P = p(\alpha)D\mathrm{d}\alpha$ where $p(\alpha)$ is the load-sharing distribution.

Assuming the structure size D to be large enough compared to the fracture process zone length $2c_f$,

$$\sigma(\rho) = f'_t \frac{S(\rho)}{S(1)}, \qquad S(\rho) = \int_0^\rho \frac{q(\omega)\, d\omega}{\sqrt{\rho - \omega}} \quad (24)$$

$$w(\rho) = w_f \frac{W(\rho)}{W(0)}, \qquad W(\rho) = \int_\rho^1 q(\omega)\sqrt{\omega - \rho}\, d\omega \quad (25)$$

If the softening stress-displacement law of the cohesive crack model is written as $\sigma/f'_t = \phi(w/w_f)$ then the following equation must be satisfied:

$$\frac{S(\rho)}{S(1)} = \phi\left(\frac{W(\rho)}{W(0)}\right) \quad (26)$$

This represents an integral equation from which the K-profile $q(\rho)$ may be solved if function ϕ is given.

It can be shown (Bažant and Zi 2001) that for large sizes the nominal strength $\sigma_N = P/bD$ for the cohesive crack model may be expressed as follows:

$$\sigma_N = \frac{K_c}{\sqrt{D}} \int_0^1 \frac{q(\rho)\,\mathrm{d}\rho}{k[\alpha(\rho)]} \quad (27)$$

The asymptotic analysis of this equation yields the size effect curve of the cohesive crack model, provided that function $k(\alpha)$ is known. The analysis shows that the size effects can be classified into three basic types (rather than two, as previously thought).

Type 1, $k(0) = 0, k'(0) > 0$. Unnotched structure ($\alpha_0 = 0$) of positive geometry, reaching maximum load at the initiation of fracture growth. The size effect is obtained in the form of (21), i.e.,

$$\sigma_N = \sigma_\infty \left(1 + \frac{rD_b}{\eta D_b + D}\right)^{1/r} \quad (28)$$

(Bažant and Zi 2001) where $r > 0$, $\eta > 0$, $\sigma_\infty = K_c I_3/\sqrt{2g'_0 c_f}$, $D_b = I_4 c_f(-g''_0)/2I_3 g'_0$ $I_3 = \int_0^1 q(\rho)\mathrm{d}\rho/\sqrt{\rho} \geq 1$, $I_4 = \int_0^1 q(\rho)\sqrt{\rho}\mathrm{d}\rho \leq 1$.

Type 2, $k(\alpha_0) > 0, k'(\alpha_0) > 0$. Structure of a positive geometry containing a notch or a pre-existing traction-free crack (i.e., a fatigued crack). Asymptotic analysis of (27) yields the classical size effect proposed by Bažant in 1984, i.e.,

$$\sigma_N = \sigma_0 \left(1 + \frac{D}{D_0}\right)^{-1/2} \quad (29)$$

(Bažant and Zi 2001) where $\sigma_0 = K_c/k_0\sqrt{D_0}$, $D_0 = 4I_2 c_f k'_0/k_0$, $I_2 = \int_0^1 q(\rho)\, \rho\,\mathrm{d}\rho$. Note that D_0 depends on the softening curve.

Type 3, $k(\alpha_0) > 0, k'(\alpha_0) = 0$. Negative-positive geometry—the geometry is initially negative, i.e., $k'(\alpha) < 0$, which means that the crack initially grows in a stable manner at increasing load, but later becomes positive, i.e., $k'(\alpha) > 0$. The maximum load is reached at crack length at which $k'(\alpha) = 0$. For this type it is found that

$$\sigma_N = \sigma_0 \left(\frac{D_1}{D + D_1} + \frac{D}{D_0}\right)^{-1/2} \quad (30)$$

(Bažant and Zi 2001) where $D_1 \geq D_0$ is required for the curvature of $\log\sigma_N$ versus $\log D$ to be everywhere negative; $D_1 = 4k_0k''_0 I_5\,(c_f\sigma_0/K_c)^2$, $D_0 = (K_c/k_0\sigma_0)^2$, $I_5 = \int_0^1(\rho - \frac{1}{2})^2 q(\rho)\mathrm{d}\rho$. Since in Type 3 the crack at maximum load is large, it was thought that the size effect should be the same as in Type 2. Yet it is not. But, at the same time, the difference is not very pronounced. It consists merely of a more abrupt transition between the same asymptotes.

2.4 *Summary of Required Asymptotic Properties of Size Effect*

THEOREM II. For quasibrittle materials, the (deterministic) size effect curve of σ_N versus D must have the following small-size and large-size asymptotic properties:

For $D \to 0$: (31)

$$\sigma_N \propto 1 - \frac{D}{D_s} - ... \quad (32)$$

For $D \to \infty$:

$$\text{Type 1:} \quad \sigma_N \propto 1 + \frac{D_b}{D} + ... \quad (33)$$

$$\text{Type 2} \quad \sigma_N \propto \frac{1}{\sqrt{D}}\left(1 - \frac{D_0}{2D} + ...\right) \quad (34)$$

$$\text{Type 3:} \quad \sigma_N \propto \frac{1}{\sqrt{D}}\left(1 - \frac{D_a{}^2}{D^2} + \ldots\right) \quad (35)$$

(Fig. 1). Here D_0, D_a, D_b, D_s = constants, $D_a = D_0 D_1/2$, $D_s = rD_b$, and $\propto$ is the proportionality sign. Note that types 2 and 3 are verified by the following expansions:

$$\left(1+\frac{D}{D_0}\right)^{-1/2} = \sqrt{\frac{D_0}{D}}\left(1+\frac{D_0}{D}\right)^{-1/2} = \sqrt{\frac{D_0}{D}}\left(1-\frac{D_0}{2D}+\ldots\right) \quad (36)$$

$$\left(\frac{D_1}{D+D_1}+\frac{D}{D_0}\right)^{-1/2} = \sqrt{\frac{D_0}{D}}\left(1+\frac{D_0 D_1}{D^2(1+D_1/D)}\right)^{-1/2} = \sqrt{\frac{D_0}{D}}\left[1+\frac{D_0 D_1}{D^2}\left(1-\frac{D_1}{D}+\ldots\right)\right]^{-1/2} = \sqrt{\frac{D_0}{D}}\left(1-\frac{D_0 D_1}{2D^2}+\ldots\right) \quad (37)$$

2.5 *Broad-Range Size Effect and Unification of G_f and G_F*

In principle, the fracture energy of a material with a large FPZ must be equal to the energy dissipated by fracture propagation in an infinitely large specimen. The basis of the size effect method of fracture energy testing is the extrapolation of nominal strength to an infinite size. So why the fracture energy G_f obtained by size effect testing is systematically less than the fracture energy G_F obtained by the work-of-fracture method?

There are several explanations (Bažant 2001), and one of them is that the simple classical size effect law used for notched specimens (Type 2) is not valid for a sufficiently broad size range. A broad range size effect law whose large size asymptote can be made to agree with the G_F-value without sacrificing the fit of the σ_N-values for ordinary specimen sizes can be written as follows:

$$\sigma_N^2 = \frac{\sigma_0^2}{1+D/D_0}\left(1+\frac{\gamma_1}{1+\lambda D_0/D}+\ldots+\frac{\gamma_n}{1+\lambda^n D_0/D}\right) \quad (38)$$

(Bažant 1999b, 2001) (Fig. 2). This formula satisfies all the asymptotic requirements stated in (33) (Type 2); λ, γ_k are non-negative constants, $\lambda > 1$. With $n = 1$ and $\lambda = 30$, formula (38) can fit broad-range finite element results on size effect, such as those in Bažant (1985).

The law (38) provides a unification of the size effect fracture energy G_f, which corresponds to the final asymptote of the first term ($n = 0$) representing the classical size effect law, with the work-of-fracture energy G_F (Hillerborg 1985a,b, Nakayama 1965, Tattersall and Tappin 1986), which corresponds to the final asymptote of (38). Choosing $n = 2$, $\gamma_1 = \sqrt{2.5} - 1 = 1.58$ and $\gamma_2 = 2.5 - 1 - \gamma_1 = 0.92$, one has a formula that gives the size effect on nominal strength agreeing within a size range of about 1:20 with the fracture energy G_f, yet for extrapolation to infinite structure size gives fracture energy $G_F = 2.5G_f$. Here 2.5 is the widely accepted ratio of G_F and G_f, as found by Planas et al. (1992) and Guinea et al. (1994a,b) (see also Bažant and Planas 1998), and recently confirmed as optimal for a database involving 238 test series from different laboratories (Bažant and Becq-Giraudon 2000; see also a paper in these proceedings).

The values $\Gamma_k = \gamma_k G_f$ may be regarded as partial fracture energies associated with sizes $H_k = \lambda^k D_0$, and the plot of Γ_k versus $\log H_k$ may be regarded as the fracture energy spectrum of the cohesive crack model. The first value of the spectrum is the fracture energy G_f corresponding to the area under the initial tangent of the softening curve of the cohesive crack model, and the sum $G_f + \Gamma_1 + \ldots + \Gamma_n = G_F =$

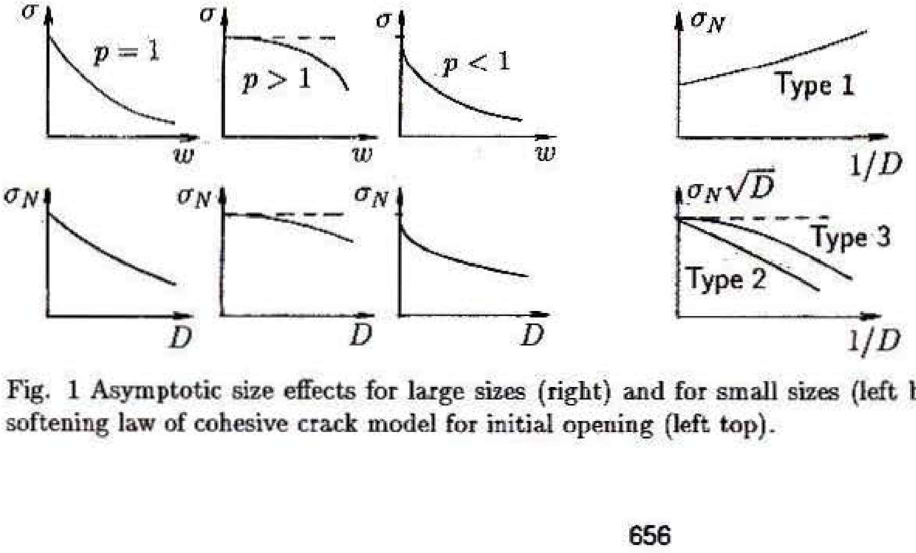

Fig. 1 Asymptotic size effects for large sizes (right) and for small sizes (left bottom), with the corresponding softening law of cohesive crack model for initial opening (left top).

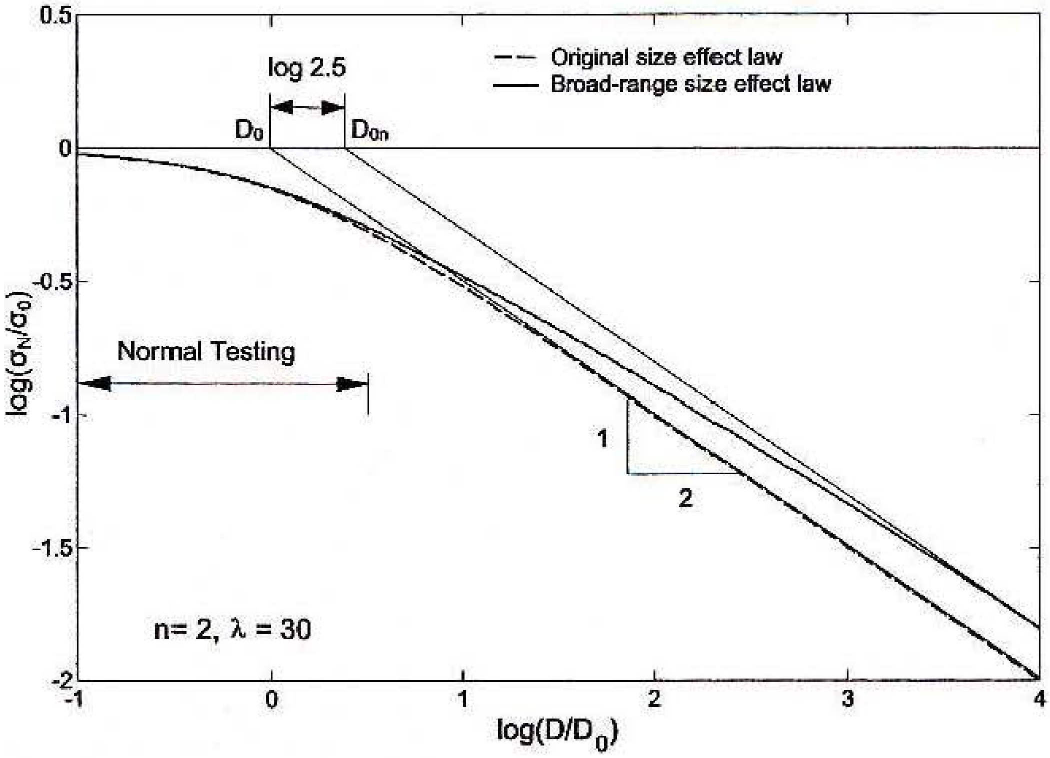

Fig. 2 Broad-range size effect law.

area under the whole softening curve. For predicting the maximum loads of structures in the normal size range, only G_f is needed, although G_F, which governs the far post-peak response, would in theory govern the size effect on nominal strength for an infinite size. In this manner, the long debated discrepancy between G_f and G_F can be reconciled.

With regard to Sec. 2.1, note that the broad-range size effect law (38) has correct small-size asymptotics—it gives a finite σ_N-value for $D \to 0$ and approaches this value linearly.

Other formulae satisfying Theorem II are possible, e.g., $\sigma_N^2 = \sigma_0^2 \sum_k \gamma_k/(1 + D/D_k)$ where γ_k, D_k = constants. However, formula (38) is more convenient for generalization from narrow-range data.

2.6 *Effect of Strength Randomness*

Weibull type extreme value statistics of material strength randomness has no influence on the mean size effect at sufficiently small sizes (Bažant and Xi 1991, Bažant and Novák 2000a). The same for large structure sizes if the structure is notched or if it fails only after a large stable crack growth. But if the structure is not notched and fails at the initiation of fracture growth, then the asymptotic size effect for $D \to \infty$ approaches Weibull-type size effect, $\sigma_N \propto D^{-n/m}$ where n = the number of dimensions in geometric similarity (1, 2 or 3) and m = Weibull modulus ($n \neq 12$, as previously thought, but about 24 for concrete; Bažant and Novák 2000b). Eq. (33) giving the first two terms of the large-size asymptotic expansion of Type 1 size effect must now be replaced by

For $D \to \infty$:

$$\text{Type 1:} \quad \sigma_N \propto \left(\frac{D_w}{D}\right)^{n/m} + \frac{D_b}{D} + \dots \tag{39}$$

where D_w = constant. The overall size effect law which replaces (28) and achieves asymptotic matching to the deterministic size effect of cohesive crack model for small sizes is (with $rn/m < 1$)

$$\sigma_N = \sigma_\infty \left[\left(\frac{D_b}{\eta D_b + D}\right)^{rn/m} + \frac{r D_b}{\eta D_b + D}\right]^{1/r} \tag{40}$$

The randomness of strength is not important for flexure of concrete beams less than about 1m thick. But is has a major effect for cross section thickness of the order of 10 m, which is important for arch dams. Material randomness of course influences the size effect on the variance of σ_N.

3. CLOSING REMARKS

Fracture mechanics of concrete is now in a golden period of research in which great rewards in engineering are within grasp. However, the existing applications in structural design still lie far below the potential of the theory. The case for introducing the available theory into practice needs to be made more convincingly, especially with regard to the size effects. Although many details relevant to structural design practice still remain to be researched and some fundamental questions settled, the theory now appears ripe for applications.

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