Statistical size effect in quasibrittle materials: Computation and extreme value theory

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ABSTRACT: This paper analyzes various modeling alternatives for the statistical size effect in quasibrittle structures. The role of reliability techniques, encompassing the classical reliability theory at random variables level, the theory of extreme values in Weibull form, and the stochastic finite element method with random strength field, is examined with a view toward capturing both the deterministic and statistical size effects. The theoretical development describing deterministic and statistical size effects is documented using the crack initiation problem. Theoretical predictions are compared with the existing test data.

Keywords: size effect, quasibrittle materials, fracture mechanics, stochastic finite element method, Latin hypercube sampling, extreme value statistics, Weibull theory, concrete.

1 INTRODUCTION

Quasibrittle materials such as concrete, fiber composites, rocks, tough ceramics, sea ice, dry snow slabs, wood and some biomaterials, fail at different nominal strengths with respect to their structure size. Smaller structures fail in a ductile manner which usually involves distributed cracking with strain-softening. The stress redistribution that is caused by fracture and distributed cracking engenders an energetic size effect, i.e., decrease of the nominal strength of structures with increasing structure size. A structure far larger than the fracture process zone (FPZ) fails in an almost perfectly brittle manner and, if the failure occurs right at the crack initiation, the failure load is governed by the statistically weakest point in the structure, which gives size to the statistical size effect.

A simple strategy for capturing the statistical size effect using the stochastic finite element method in the sense of extreme value statistics is presented. It combines a feasible type of Monte Carlo simulation based on nonlinear fracture mechanics. This is exemplified by the case of size effect of bending span in four-point bending tests of plain concrete specimens.

The interdisciplinary field of stochastic fracture mechanics is accessed by utilizing new advanced software developments which progress beyond the traditional approach and attempt to treat in a combined manner the reliability theory with fracture nonlinearity. This approach automatically yields not only the statistical part of size effect at crack initiation, but also the energetic part of size effect. Examples of statistical simulations of size effect with nonlinear fracture mechanics software ATENA, combined with probabilistic software FREET, are presented. Capturing the statistical size effect is made possible by (1) incorporating the analytical results of extreme value statistics into the stochastic finite element calculations, (2) implementing an efficient random field generation, and (3) exploiting small-sample Monte-Carlo type simulation called Latin hypercube sampling.

2 GENERAL SIZE EFFECT THEORY

2.1 *Energetic Size Effect*

There are two basic types of energetic size effect which are distinguishable (Bažant 1997, 2001a, 2002, Bažant and Chen 1997). Structures of positive geometry having no notches or preexisting cracks are classified as Type 1 size effect (Bažant and Li 1995, 1996, Bažant 1998, 2001a). For positive structure geometries, the maximum load occur as soon as the FPZ gets fully developed.

Positive geometry is one of the requirements for the applicability of Weibull-type weakest link model.

Type 2 size effect (Bažant 1984, 2002, Bažant and Kazemi 1990) occurs also for positive geometry structures but with notches, as in fracture specimens, or with large stress-free (fatigued) cracks that have grown in a stable manner prior to the maximum load. The mean nominal strength for this type of size effect is not significantly affected by material randomness (Bažant and Xi 1991, Bažant 2002), but the variance of course is. There exists also a Type 3 size effect (Bažant 2001a), occurring in structures with initially negative geometry. However, this type is so similar to Type 2 that it is barely distinguishable experimentally.

2.2 *Probabilistic Size Effect*

Traditionally, the probabilistic size effect has been explained by Weibull-type statistical weakest link model (Fisher and Tippett 1928; Weibull 1939, 1949, 1951, 1956; Epstein 1948; Freudenthal 1956, 1968, Freudenthal and Gumbel 1953; Gumbel 1958; Saibel 1969). Its basic hypothesis is that the structure fails as soon as the material strength is exhausted at one point of the structure. This is true for quasibrittle materials only if the size of the structure is much larger than the FPZ.

For quasibrittle failures of smaller sizes, there are other avenues of research which could explain the stress redistribution before failure. Daniel's (1945) fiber bundle model is one of the earliest generalizations of the extreme value statistics of the weakest link model, in which a hypothesis of load-sharing among fibers is invoked. This avenue of approach has been thoroughly investigated by S. Leigh Phoenix and co-workers (Harlow and Phoenix 1978a,b; Smith and Phoenix 1981; Smith 1982; Phoenix and Smith 1983; McCartney and Smith 1983; Phoenix 1983; Phoenix et al. 1997, 200; Mahesh et al. 2002).

The other, more recent, avenue of approach attempts to amalgamate the statistical and deterministic theories by means of a nonlocal generalization of Weibull theory (Bažant and Xi 1991, Bažant and Novák 200a,b, Bažant 2001b). This allows stochastic numerical simulations of the mean as well as variance of the deterministic-statistical size effect in structures of arbitrary geometry. In particular, this approach automatically captures the dependence of stress redistribution and energy release rate on the structure size D.

3 ASYMPTOTICS OF SIZE EFFECT

3.1 *Small-size asymptotes*

The small-size mean asymptotic properties should agree with the theoretical small-size asymptotic properties of the underlying continuum model, which can be the cohesive crack model, the crack band model, or the nonlocal damage model. Each of these models implies that the value of the nominal strength σ_N for $D \to 0$ should be finite and should be approached linearly in D (Bažant 2001a,b, Bažant 2002), as shown in Fig. 1. Agreement with these small-size asymptotic properties can be achieved by modeling the failure mechanism for the small-size limit with a fiber bundle. For a vanishing size, the failure tends to follow the theory of plasticity, and it is well known (e.g., Jirásek and Bažant, 2002) that in plasticity the failure proceeds according to a single-degree-of-freedom mechanism, i.e., is simultaneous, non-propagating. It follows that the failure probability distribution for $D \to 0$ ought to obey Daniel's (1945) \`fiber bundle', model rather than the Weibull-type weakest link model for a chain.

For $D \to 0$, a body with a cohesive crack (or crack band) approaches the case of an elastic body containing a perfectly plastic cohesive crack. Fig. 2

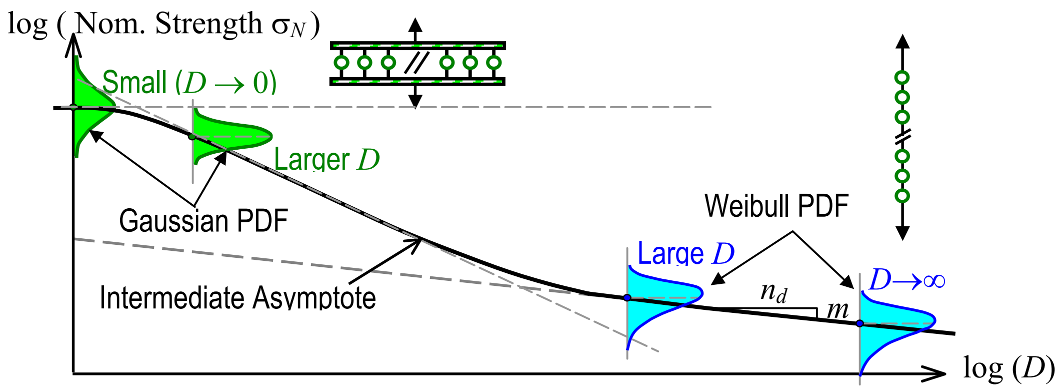

Figure 1: The curve of mean size effect for structures failing at macroscopic fracture initiation, and its probability distributions for various sizes

shows the fits of the energetic size effect to extensive experimental data on the modulus of rupture (or flexural strength) of unreinforced concrete beams (Bažant and Novák 2000a,b) and of fiber-polymer composite laminates (Bažant et al., 2003), which reveal that the small-size asymptote is closely approached only for extrapolation to specimen sizes much smaller than a representative volume of the concrete, considered here to be about three aggregates in size. This volume fractures simultaneously, which is why its statistics should be adequately described by the fiber bundle model, in which the breakages of fibers correspond to the breaks of microscopic bonds along the failure surface.

By formulating and solving a recursive relation for the failure probability distribution, Daniels (1945) showed that the failure probability $G_n(x)$ follows the standard normal distribution

$$G_n(x) \approx \Phi\left(\frac{x-\mu^*}{\gamma^*/\sqrt{n}}\right) \tag{1}$$

where the mean μ^* and variance γ^{*2}/n can be expressed implicitly as follows:

$$\mu^* = \max_{x \geq 0}\{x[1-F(x)]\} = x^*[1-F(x^*)] \tag{2}$$

$$\gamma^{*2}/n = (x^*)^2\, F(x^*)[1-F(x^*)] \tag{3}$$

$F(x)$ is the probability distribution of failure of the fibers, are assumed to be identical and statistically independent.

3.2 *Large-size asymptotes*

On the other extreme, the failure for a very large structure of positive geometry occurs as soon as the FPZ becomes fully developed. Structures of positive geometry are those in which the stress intensity factor, or the energy release rate, increases if the crack extends at constant load. The failure of such a structure could be modeled with a single chain of elements, each representing a FPZ and the failure probability of such a structure follows the weakest link model.

$$P_N(\sigma) = 1-[1-P_1(\sigma)]^N \tag{4}$$

where $P_I(\sigma)$ is the cumulative probability distribution of the element, which represents the FPZ in this case and $P_N(\sigma)$ is the cumulative distribution function of the chain.

Although there are no substantial amount of experimental data that test on very large structures to verify the correctness of the weakest link model, numerical simulation on such large scale such as dams (to be presented in another paper) and the theoretical argument that very large positive definite structures fail at crack initiation, provides strong argument for the weakest link model.

Fisher and Tippet (1928) has proved that there exist three and only three asymptotic forms of the extreme value distribution:

1. Weibull distribution
2. Fisher-Tippett-Gumbel distribution
3. Fréchet distribution

In this paper, we focus on generic Weibull distribution with zero threshold for each FPZ; the

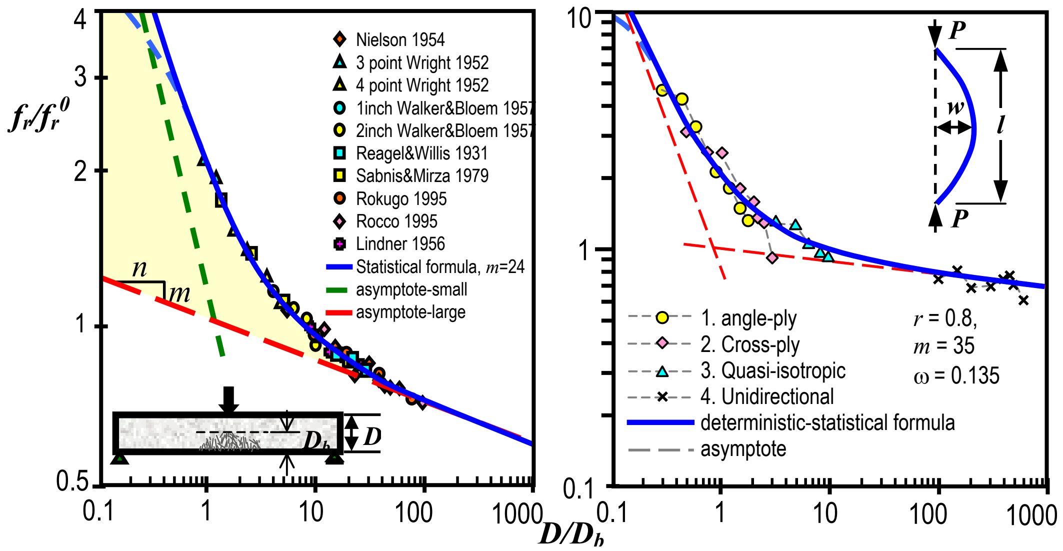

Figure 2: Dimensionless size effect curve for modulus of rupture f_r (flexural strength) obtained after the values D_b and $f_r^0=s_0f_0$, different for each data set, have been identified by separate fitting. Left: Concretes. Right: Fiber-composite laminates.

cumulative probability distribution can be expressed as follows:

$$P_1(\sigma) = 1 - e^{-(\sigma/\sigma_0)^m} \tag{5}$$

where m and s_0 are the Weibull shape and scale parameters respectively (m = Weibull modulus).

The asymptotic probability distribution for the weakest link model will remain Weibull at varying D but the mean and standard deviation will shift as follows:

$$\mu_N = \mu_1 (N)^{-1/m} = \sigma_0 (N)^{-1/m}\, \Gamma(1+1/m) \tag{6}$$

$$\delta_N^2 = \delta_1^2 \left(\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1 \right) \tag{7}$$

According to the expressions in Eq. (6) & (7), it is clear that the coefficient of variation of σ_N depends only on the shape parameter and can be expressed as follows:

$$\omega_N = \sqrt{\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1} \tag{8}$$

Note that the coefficient of variation of σ_N is independent of the structure size D. This implies that, if the size effect is purely statistical, the Weibull modulus, m, which is completely determined by the experimentally observed scatter of the results of tests of identical specimens of one size, must be the same as the m identified from the size effect tests. This is a check on the validity of the statistical theory which has been omitted in many studies. For small and intermediate size structures, the Weibull statistical theory does not apply and this is most easily recognized by the fact that moduli m obtained according to (8) from tests at very different sizes do not match each other.

4 TRANSITION BETWEEN SMALL AND LARGE SIZE ASYMPTOTES

4.1 *Chain of Bundle Model*

For intermediate size structures, the size of FPZ is large as compared to the size of the structure. Stress redistribution and energy release are significant for these structures, which suggests that the deterministic effect should not be neglected. The size effect curve could be determined by numerous simulations of intermediate size structures using a nonlinear stochastic finite element program. This is reviewed later in the paper. Now an alternative approach with a transition based on the chain of bundles model (Fig. 3) proposed in Bažant (2003a,b) will be studied.

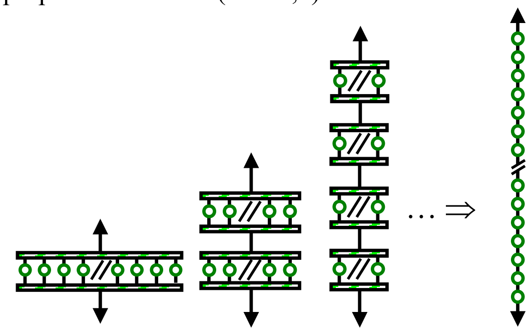

Figure 3: Chain of bundles as a model for probability distribution describing a continuous transition between Weibull and Gaussian distributions

Visible macro-cracks are assumed to appear at a minimum crack spacing equal to characteristic length which is approximately three maximum aggregate sizes. For hypothetical specimens smaller than this characteristic length, the failure should follow the fiber bundle model in which each fiber in the bundle of the lowest hierarchy represents a micro-bond. The failure probability distribution of the fiber bundle (Fig. 3 left) consisting of a large number of fibers can be described well by Daniels's approximation in Eq. (1).

For typical test specimen sizes (larger than the aforementioned characteristic length), the failure mechanism is modeled with a hybrid of series and parallel coupling as shown in Fig. 3 (middle). The statistical effect of the stress redistribution causing energy release can be modeled by the parallel coupling of elements, each of a characteristic volume. The number of characteristic volumes in a normal structure would be small and the probability distribution could not be approximated accurately by Eq. (1). The failure probability could be computed exactly by a recursive formula (Smith and Phoenix 1981, Smith 1982) expressed as follows:

$$G_n(x) \approx \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} F(x)^k\, G_{n-k}\!\left(\frac{nx}{n-k}\right) \tag{9}$$

where n is the number of elements of characteristic volumes in a bundle and $F(x)$ is the probability distribution of each element.

On the other hand, the possibility of cracks appearing along the span of a flexed beam or along the length a tensioned bar can be accounted for by coupling the bundles in a chain-like manner. In this way, the deterministic and statistical size effect can be fused in a single approximate model which also provides an asymptotically correct transition from small to large size asymptotes. In one extreme, in which the specimens are very small, the chain-of-bundles model would collapse into a bundle of micro-bonds. In the other extreme for large structures, the size of the bundles is fixed since the FPZ has been fully developed while the number of bundles in the chain scales according to the size of the structure and the model behaves as a weakest link.

The size effect curves for the coefficient of variation (COV) have also been computed and are found to match the small and large size asymptotic properties (Fig. 4). This demonstrates that the chain-of-bundles model is able to capture the first and second moments.

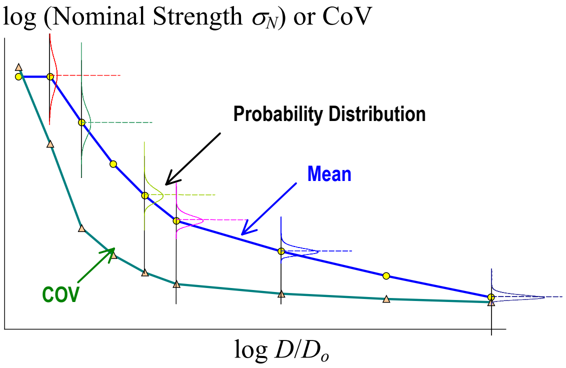

Figure 4: Mean and CoV size effect curve using Weibull elements and its probability distributions for various sizes

The mean size effect curve can be fitted to different types of loading which is shown in Fig. 5. The mean size effect curve (MSEC) resembles the size effect curve that is derived theoretically (Bažant 2003a,b) by asymptotic matching in Fig. 1. A different MSEC is obtained for each type of loading because the failure mechanism is different and the critical regions also differ.

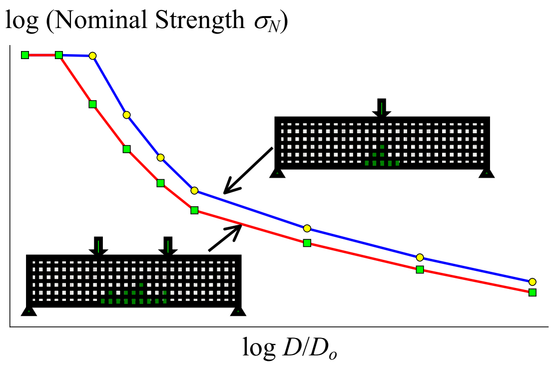

Figure 5: Effect of different types of loading on MSEC

The chain of bundles model offers flexibility in the choice of the generic probability distribution for the micro-bonds, resulting in different size effect curves. Existing literature on limited tensile test data suggested different probability distributions, namely Weibull, Normal and Log-normal distribution and the chain of bundles model can be used to gain insight into the probability distribution for the generic probability distribution by using the computed size effect curve with different probability distribution (Fig. 6) to match to experimental data.

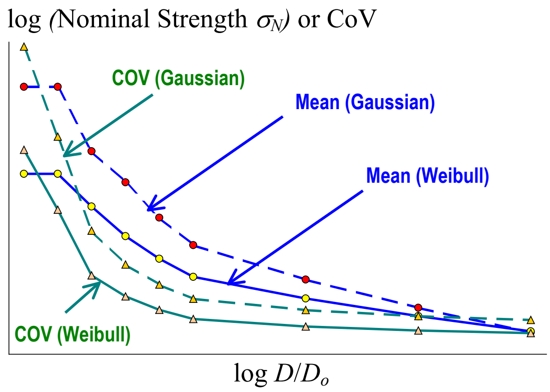

Figure 6: Comparison of size effect curves with Weibull and Gaussian elements

4.2 *Stochastic Finite Element*

A simple but primitive approach to stochastic finite element analysis is to subdivide a structure into elements of the size of the characteristic volume. Such an approach is feasible for small structures but would be hardly possible to implement in very large structures. As proposed by Bažant et al. (2003), this difficulty could be overcome by using stochastic macro-elements where each macro-element has stochastic properties that are scaled according to Fisher & Tippett's (1928) fundamental stability postulate of extreme value distributions. The advantage is that the number of macro-elements can be kept fixed but while their size is increased in proportion to structure size D. This allows efficient stochastic computations for very large structures.

The treatment of the macro-element and the selection of the extreme value for each macro-element is described in detail in Bažant et al.(2003). The scaling of the mean strength and variance of the macro-element are given by Eqs. (6) and (7).

The scaling formula is applicable only if the probability distribution of each micro-element of characteristic length could be described by Weibull distribution of Weibull modulus m and scale parameter σ_0. An additional condition of validity is that the structure must reach the peak load at crack initiation.

The macro-element approach is checked against Koide et al. (1998, 2000) tests of plain concrete beams under four-point bending with different bending spans (200, 400 and 600mm) but identical cross sections (100 by 100mm) (which eliminates the energetic part of size effect). Nonlinear fracture mechanics software ATENA (Červenka and Pukl 2002) is integrated with probabilistic software FREET (Novák et al. 2002) to perform statistical simulation of Koide's beam. With the meshing of Koide's beam into 6 macro-elements for each of 3 sizes (Fig. 7), 16 simulations of Latin

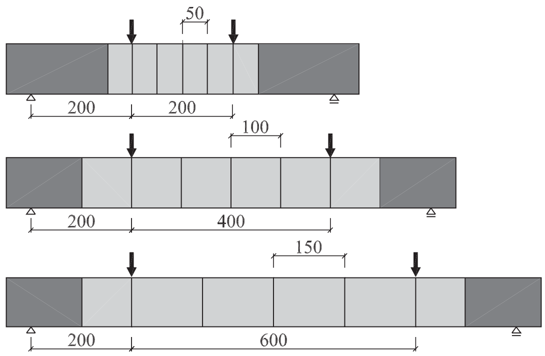

Figure 7: Koide's beams of bending spans 200, 400 and 600mm

Hypercube Sampling (LHS) are performed for 6 random strengths (and random fracture energies G_f). These independent (or correlated) variables are sampled according to the optimization techniques of Vořechovský and Novák (2002). Three alternatives are tested (Fig. 8): Alt. I with random tensile strength, Alt. II with statistical correlation between tensile strength and fracture energy, and Alt. III with a change in material parameters from Alt. II needed to shift the size effect curve in order to fit the experimental data.

The comparison of these three alternatives reveals the influence of the basic parameters: A decrease of modulus m or the characteristic length causes a stronger size effect, reflected in a larger slope in the MSEC. An increase of tensile strength shifts the entire MSEC upwards. A decrease of the fracture energy, or the correlation factor between strength and fracture energy, leads to a stronger size effect (slope of MSEC) and at the same time shifts the MSEC downwards, which represents better the behavior of a chain.

Alt. III is able to fit the size effect curve by changing the mean tensile strength and fracture energy of the finite elements, which is admissible because Koide's tests did not include measurement of these parameters. Despite the good fit of Koide's test data, the Weibull modulus, found to be m = 8, is surprisingly low compared to Weibull modulus m = 24 obtained by fitting many data with a nonlocal generalization of Weibull theory (Bažant 2000b).

The low Weibull modulus obtained for Koide's beam could be explained by the mechanism of failure and the applicability of stability postulate for small or intermediate size structures. The stress redistribution that occurs before failure magnifies the deterministic size effect which gets coupled with the statistical size effect, resulting in a stronger size effect.

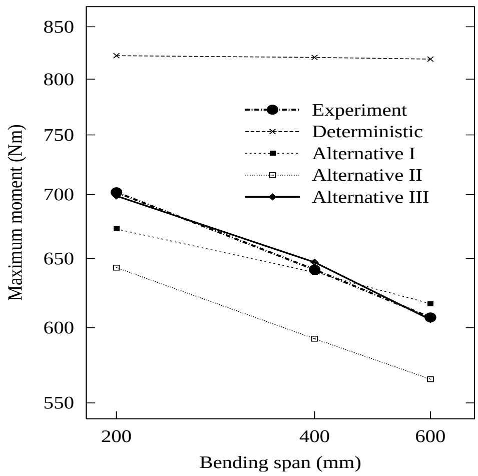

Figure 8: Comparison of means of Koide's data with deterministic and statistical simulations by ATENA

The Weibull modulus found for Koide's beams could reflect the slope near the intermediate asymptote, which however decreases when the deterministic size effect wears out as the size $D \rightarrow \infty$. Although Koide's results could not be extended or directly applied to very large structures, they reveal strong Weibull-type stochastic behavior when a slender structure is scaled longitudinally.

Koide's beam has been simulated in ATENA with a macro-element, the parameters of which are scaled in one dimension (1D). This represents the weakest-link chain model which can be imagined to describe the bottom layer of finite elements in the beam. Despite the good match with 1D treatment, the material parameters used in these computations could not be reproduced on a different set of experiment. The deterministic size effect is not properly treated with the 1D model as this model does not fail at crack initiation (Fig. 9). The scaling of Koide's beams for the strength should better be done in two dimensions (2D) because it involves considerable stress redistribution across the beam depth.

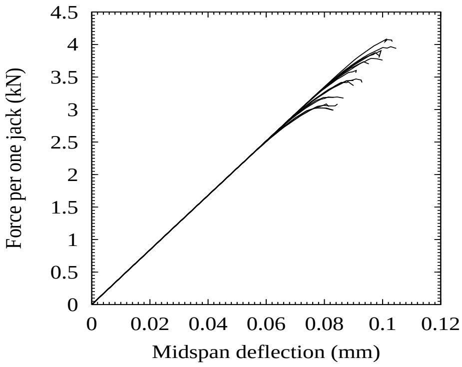

Figure 9: Random load-deflection diagrams for one size of Koide's beams (note the curvatures indicating appreciable stress redistributions before peak)

The difficulty with the 2D treatment is that the stability postulate could not be directly applied to the macro-elements. The coupling of the macro-elements in a load-sharing manner violates the assumption of the weakest-link chain model. The probability distribution of the macro-element could be derived by other methods, e.g. a bundle model capturing the coupling effect of the macro-element, and the shift in mean and variance could be computed accordingly using the fiber bundle model. Such an approach is currently being pursued at Northwestern by S.D. Pang and M. Vořechovský.

5 CONCLUDING REMARKS

The paper shows how the statistical size effect at fracture initiation can be captured by a stochastic finite element code based on extreme value statistics, simulation of the random field of material properties, and chain of bundles transition. The computer simulations of the statistical size effect in 1D based on stability postulate of extreme value distributions match the test data. However, the correct behavior cannot be achieved for other tests using a 1D treatment. A proper way of treating the stress redistribution is by the recently proposed macro-elements in 2D (or 3D), the scaling of which is based on the fiber bundle model capturing partial load-sharing and ductility in the finite element system.

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