Crack width prediction in RC members in bending: a fracture mechanics approach

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ABSTRACT: Cracking is a very common occurrence in reinforced concrete(RC) structures. Cracks in RC structures are characterized by crack width and crack spacing. In the present study an expression is developed using a cohesive crack model having a bilinear strain softening relationship to predict crack widths in RC beams. One of the assumptions made in the development of the model is that there is complete loss of bond between the bar and the concrete. However, this crude assumption leads to too conservative values for crack width, since the frictional forces at the bar-concrete interface limit bar slip and consequently crack width. Therefore, it is necessary to introduce a restraining force to model bond and to find crack-closing displacement. The crack width values so obtained from the proposed model are compared with code predictions and with experimental results available in literature. The results show that the proposed approach is sound, consistent and realistic.

1 INTRODUCTION

The occurrence of cracks in reinforced concrete structures is inevitable because of the low tensile strength of concrete. Cracks form when the tensile stress in concrete exceeds its tensile strength. Cracking in reinforced concrete structures has a major influence on structural performance, including tensile and bending stiffness, energy absorption capacity, ductility, and corrosion resistance of reinforcement. Cracking at the service load should not extend to such a limit that it spoils the appearance of the structure or leads to excessive deformation of the members. This may be achieved by specifying an allowable limit on crack width values. In order to assure a satisfactory performance of the structure even under service loads, an important limit state i.e., the limit state of serviceability (cracking) is introduced into the limit state design procedure. This limit state is assumed to be satisfied if crack widths in a concrete member are within a maximum allowable limit. While the need for a crack limit state has been universally agreed on, the formulae for predicting the crack width extensively vary in the various codes of practice. Inspection of crack width prediction procedures proposed by various investigators indicates that each formula contains a different set of variables. A literature review also suggests that there is no general agreement among various investigators on the relative significance of different variables affecting the crack width, despite the large number of experimental work carried out during the past few decades. Taking all the parameters into account in a single experimental program is not normally feasible due to the large number of variables involved, and the interdependency of some of the variables.

In this paper, an attempt is made to predict an expression for crack width by incorporating a bilinear strain softening function and all the variables which influence crack widths. The proposed formulas are also compared comprehensively with the test results available in the literature (Hognestad, 1962; Kaar and Mattock, 1963; Clark, 1956). To access the relative performance of the proposed crack width equation, it is compared with the international codes of practice.

2 CRACK WIDTH EXPRESSION

Gerstle et al (1992) developed simplified assumptions that allow analytical solutions for flexural cracks in singly reinforced beams in bending while retaining the significant features of the fictitious crack model (FCM) which was introduced by

Hillllerborg et al. (1976). The FCM has the potential of being very useful in understanding the fracture and failure of concrete structures. It assumes that the fracture process zone at the crack tip is long and infinitesimally narrow. The fracture process zone is characterized by a normal stress versus a crack opening displacement curve which is considered as a material property. The shape of this stress-crack opening displacement (softening) curve can be either linear/bilinear/tri-linear or a power law. Gerstle et al.(1992) assumed a linear softening relation in his formulation and predicted the crack width as a product of a constant *C* (a function of brittleness of concrete or a function of reinforcement) and critical crack opening displacement COD_{cr} (a function of the softening curve or the fracture energy). This expression for crack width does not explicitly include such parameters as the diameter/perimeter of reinforcement which influences the values of crack widths. In the literature, bilinear softening seems to describe the behavior of concrete in tension more appropriately than the linear softening. An attempt is made here to work out an expression for crack width based on bond mechanics (bar slip included), and formulated as the product of the crack spacing times the mean strain in the reinforcement by incorporating the bilinear softening function.

2.1 *Significance of the strain softening curve*

The stress-crack opening law for concrete in tension is found to have a descending branch in the post-peak region. The simplest idealization for this behavior is a linear softening relation, but it is more realistic to consider a bilinear softening relationship. A typical bilinear shape of the softening curve is shown in Figure 1. Linear softening seems to be an obvious choice when the data describing the actual material behavior is limited. However, linear softening proved to overestimate structural capacity. Therefore, bilinear curves have been accepted as reasonable approximation of the softening curve for concrete, although there seems to be no agreement about the precise location of the kink point. In the literature, several researchers have given the kink positions (break points) on the basis of experiments, and there are quite a few simple methods to identify any bilinear softening to fit particular experimental data (Guinea et al. 1994). Brincker and Dahl (1989) reformulated the substructure method introduced by Petersson (1981) for the three point bending specimen in order to obtain complete load displacement relations. From the sensitivity analysis of their method using linear, bilinear and tri-linear models, it is evident that the shape of the stress crack opening displacement relation has significant influence on the results. However, tri-linear approximation does not seem to deviate significantly from the bilinear approximation indicating the sufficiency of the bi-linear approximation. In this study, the crack width is calculated considering specific kink positions(break points) as suggested by Brincker and Dahl (1989), having the value of $k_1 = 0.308$ and $k_2 = 0.161$.

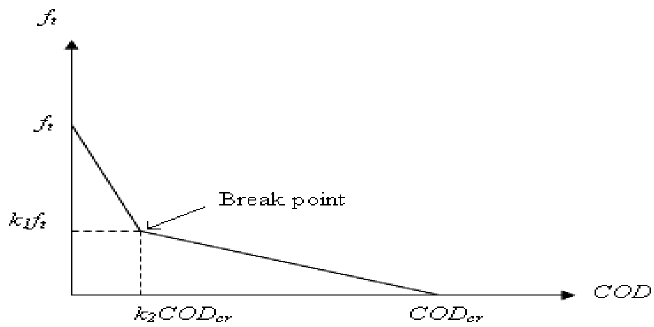

Figure 1. Typical bilinear stress versus crack opening displacement curve

2.2 *Proposed methodology using a flexural cracking model which incorporates the bilinear softening function in tension with non linearity of concrete in compression*

The main assumptions of the model are as follows:

1. Plane sections remain plane before and after deformation within the central elastic band.
2. The beam is considered rigid outside the central elastic band.
3. Fictitious crack surfaces remain plane after deformation.
4. The stress versus crack opening displacement curve is assumed as bilinear softening in tension.
5. Concrete is homogeneous, isotropic and non-linear elastic.
6. The steel has a perfectly plastic material model.
7. The reinforcement can slip with respect to concrete within the central elastic band (2ka).
8. The centroid of the steel is located at the bottom of the beam and the concrete cover below the steel level is deliberately neglected for simplicity in the derivation of the expressions.

As shown in Figure 2, this model considers a varying central elastic band whose width varies *k* times the length of crack from the crack surface. The width of the central elastic band considered is *2ka*, i.e., at a distance of *ka* on either side from the crack surface, where *k* is a constant and *a* is the crack length. The beam is considered rigid outside the central elastic band. The model is capable of predicting the flexural behavior of concrete beams

Figure 3, shows an idealization of the deformed shape (greatly magnified) of a crack in a reinforced concrete beam, together with normal stress distribution considering a bilinear stress crack opening displacement relationship.

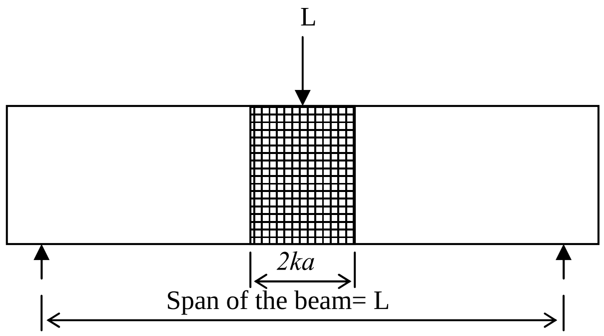

Figure 2. Schematic diagram of the beam showing central elastic band width

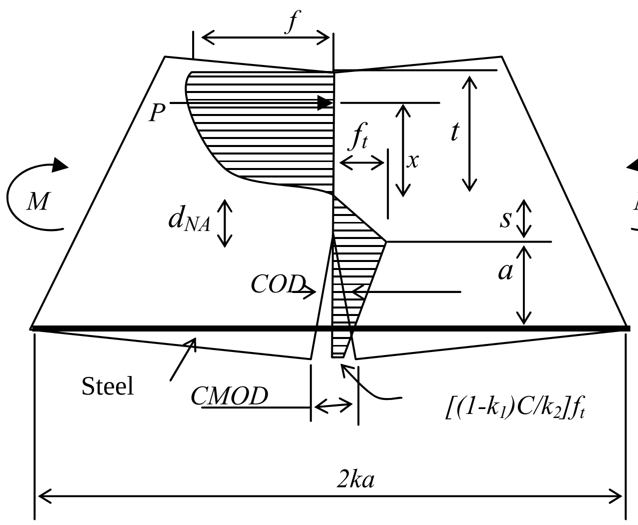

Figure 3. Schematic of stress variation (Case I - Stage I)

Two cases are considered and they are as follows:

i) Case I, the fictitious crack is not sufficiently open to relieve the normal stress at its mouth i.e., ($CMOD < COD_{cr}$). Case I is further subdivided into two stages, viz. Stage I (just before the kink) , and Stage II (just after the kink), and

ii) Case II, in which the fictitious crack is sufficient open to relieve the normal stress at its mouth ($CMOD > COD_{cr}$).

2.3 *Normalization of Parameters*

– Crack mouth opening displacement

$$C = \frac{CMOD}{COD_{cr}}$$

– Crack length = $A = \frac{a}{h}$

– Distance from crack tip to neutral axis

$$S = \frac{s}{h}$$

– Distance from neutral axis to top fiber of beam

$$T = \frac{t}{h}$$

– Material-scale parameter for concrete=

$$\beta = \frac{f_t\, h}{E_c\, COD_{cr}}.$$

– Material parameter(for reinforcement), $\alpha = \rho n$

Where, m is the applied moment, $n = E_s/E_c$ is the modular ratio, ρ is the geometric reinforcement ratio and α is the mechanical reinforcement ratio.

Deriving on similar lines to that suggested by Gerstle et al., (1992) the expression for the crack mouth opening displacement (*CMOD*) is obtained at various stages of loading as equal to constant *C* multiplied by critical crack opening displacement (COD_{cr}). The *CMOD* is nothing but the crack width at any point of loading in the reinforced concrete flexural member at the level of steel.

$$CMOD = C^*COD_{cr}$$

From the derivation of the flexural cracking model with bilinear softening in tension and nonlinearity in compression, we have the strain at the bottom of the beam which is also equal to the strain at the level of steel.

$$\varepsilon_s = \varepsilon_b = \frac{k_1 f_t (1-C)}{E_c (1-k_2)} \quad (2)$$

Rearranging the above equation,

$$\left[C = 1 - \frac{E_c (1-k_2)}{k_1 f_t} \varepsilon_s \right] \quad (3)$$

Substituting (3) in (1),

$$CMOD = \left[1 - \frac{E_c (1-k_2)}{k_1 f_t} \varepsilon_s \right] COD_{cr} \quad (4)$$

$$=\left[\frac{1}{\varepsilon_s}-\frac{E_c(1-k_2)}{k_1 f_t}\right]COD_{cr}\varepsilon_s$$

$$=\left[\frac{E_s}{f_s}-\frac{E_c(1-k_2)}{k_1 f_t}\right]COD_{cr}\varepsilon_s$$

$$=\left[\frac{nE_c}{f_s}-\frac{E_c(1-k_2)}{k_1 f_t}\right]COD_{cr}\varepsilon_s$$

$$=\left[\frac{n}{f_s}-\frac{(1-k_2)}{k_1 f_t}\right]E_c COD_{cr}\varepsilon_s \qquad (5)$$

As we know,

$$\beta=\frac{f_t.h}{E_c.CODcr}, \alpha=\rho n, \rho=\frac{A_s}{bh}=\frac{\varepsilon.\pi.\phi.\phi}{4.b.h},$$

ε = number.of.bars

and ϕ = Diameter of the reinforcing bar.

Using the above relationships, we get

$$CMOD=\left[\frac{\alpha}{\rho f_s}-\frac{(1-k_2)}{k_1 f_t}\right]\frac{f_t\, h}{\beta}\varepsilon_s \qquad (6)$$

$$=\left[\frac{\alpha f_t h}{\rho f_s \beta}-\frac{(1-k_2)h}{\beta\, k_1}\right]\varepsilon_s$$

$$CMOD=\left[\frac{(4bh^2\alpha f_t)}{(\varepsilon\pi\phi\phi\beta f_s)}-\frac{(1-k_2)h}{k_1\beta}\right]\varepsilon_s \qquad (7)$$

Where, *CMOD* = Maximum crack width at the level of steel & k_1 k_2 = Kink Positions(Break points). Therefore, Maximum crack width in the above expression is a function of nine variables viz: reinforcement ratio (α), brittleness of concrete (β), tensile strength of concrete (f_t), stress in steel (f_s), diameter of bar (ϕ), perimeter of bar $(\varepsilon\pi\phi)$, cross section dimensions of the beam (b, h) and the strain in steel (ε_s), which directly influence the crack width prediction and hence exhibits a physically realistic expression. The crack width in question is at the level of the reinforcement. The crack width at the bottom of the beam with the reinforcement cover shall be equal to [*(h-x) / (d-x)*] times the crack width at the level of steel, where, *h* and *d* are overall depth and effective depth respectively and *x* is the neutral axis position from the top fiber. In the above expression for crack width Eq. 7 is derived from a flexural cracking model considering bilinear strain softening in tension and non-linearity of concrete in tension and one of the assumptions is that there is a complete slip between the crack face and the steel. However, this need not be the case and the predictions made using Eq. 7 are likely to be conservative estimates of the crack width. Further, it can be understood physically that there cannot be a complete loss of bond between the steel and concrete in ordinary bond conditions. Therefore, in order to account for this anomaly, a restraining bond force is introduced against complete loss of bond between the steel and concrete, to predict crack width values that are realistic. The effect of this crack closing force on the predicted values of the crack width is discussed in the following section.

3 CRACK WIDTH CORRECTION

Assuming total loss of bond at the bar-concrete interface within the distance $\pm$ *kA* from the cracked plane is too conservative, since there is the restraining action due to bond. Therefore, in order to realistically model cracking and to avoid any crack-width overestimation, it is necessary to introduce a restraining force at the level of the reinforcement and to find the crack-closing displacement at that level. This is done using the expression as given in the hand book stress intensity factors (Gdoutos 2003),

$$U_\theta=\frac{K_1^{total}}{4\mu}\left(\frac{r}{2\pi}\right)^{1/2}\left[-(2k+1)\sin\theta/2+\sin 3\theta/2\right] \qquad (8)$$

Where, k = $\left[\frac{3-\nu}{1+\nu}\right]$ for plane stresses and

k = $(3-4\nu)$ for plane strains

$\mu=E/2(1+\nu)$ = Rigidity Modulus, $\nu_{concrete}=0.15$,

r = distance to the crack tip.

K_1^{total} = Stress intensity due to applied load 'P' and tensile stress 'f_t' of concrete.

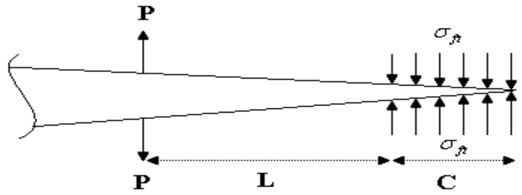

Figure 4. Schematic of the restraining force and the crack closing stresses

The stress intensity factor K_I is calculated from Dugdale's model (Gdoutos 2003). According to this model there is a fictitious crack equal to the real crack (L) plus the length of fracture process zone (C). The crack is loaded by a restraining force (P) at the level of steel and an additional crack closing stress which is equal to the tensile strength of concrete. Therefore, the stress intensity factor $K_I^{(total)}$ acting at tip of the fictitious crack is expressed as:

$$K_1^{(P)}+K_1^{(f_t)}=K_1^{(total)} \qquad (9)$$

$K_I^{(P)}$ = Stress intensity factor due to applied loads P is

$$K_1^{(P)} = \frac{2*P}{[2*\pi*(C+L)]^{1/2}} \qquad (10)$$

Where,

C = length of fictitious crack obtained from the model.

L = Real crack obtained from the model.

P = Restricting force

$P = B*\left(D/2\right)*f_t*0.09$ when D/B = 1.

$= B*\left(D/2\right)*f_t*0.2$ when D/B = 2.

$= B*\left(D/2\right)*f_t*0.26$ when 2 < D/B < 3.

$P = B*\left(D/2\right)*f_t*0.13$ when 3 < D/B<3.5.

$= B*\left(D/2\right)*f_t*0.15$ when 3.5 < D/B< 4.

$= B*\left(D/2\right)*f_t*0.41$ when D/B = 4.

B = Width of the beam in mm.

D = Overall depth of the beam, mm.

f_t = Tensile strength of concrete in N/mm^2.

$K_1^{(f_t)}$ = The stress intensity factor due to tensile stress (f_t) acting along the length of the fracture process zone.

$$K_1^{(f_t)} = -\frac{4*f_t*C^{1/2}}{(2*\pi)^{1/2}} \qquad (11)$$

Here, f_t = Cohesive force acting over the fictitious region.

Substituting Eq.10 and Eq.11, in Eq. 9

$$K_1^{(total)} = \frac{\left[(2P)-\left(4*f_t*\sqrt{C}*\sqrt{(C+L)}\right)\right]}{\sqrt{2\pi}*\sqrt{(C+L)}} \qquad (12)$$

Substituting $\theta = \pi$, in Eqn. 8, we obtain the expression for the crack closing displacement at the level of steel as

$$U_\theta = \frac{-K_1}{2\mu}(k+1)\left(\frac{r}{2\pi}\right)^{1/2} \qquad (13)$$

where, r = (C+L) = Total crack, k = $\left(\frac{3-\nu}{1+\nu}\right)$ for plane stresses & (3-4ν) for plane strains

$\mu = \frac{E}{2(1+\nu)}$ = Rigidity Modulus, $\nu(concrete)$= 0.15

Substituting $K_1^{(total)}$ from Eqn. 12 into Eqn. 13

Therefore, U_θ =

$$= -\left\{\left[\frac{(2*P)-\left(4*f_t*\sqrt{C}*\sqrt{(C+L)}\right)}{2*\mu*\sqrt{2\pi}*\sqrt{(C+L)}*\varepsilon_S}\right]*\left(\sqrt{C+L}\right)*(k+1)\right\}*\varepsilon_s \qquad (14)$$

The restricting force P used in the expression is calculated as the product of the effective area of concrete A_{ct} around the reinforcement contributing to this effect and the maximum tensile stress f_t of concrete. The restricting force P for various combinations of depth to width ratio (D/b) is calculated and it is substituted in Eq (3.10) to obtain the stress intensity factor due to an applied load. Therefore the crack is loaded by a restricting force (P) at the level of steel and an additional crack closing stress which is equal to the tensile strength of concrete and the stress intensity factor $K_I^{(total)}$ acting at the tip of the fictitious crack will be the combination of the stress intensity factor due to the applied load and stress intensity factor due to tensile stress (f_t) acting along the length of the fracture process zone. Therefore, the maximum crack width at the level of steel is computed as the difference in the crack width values with a complete loss of bond and the crack closing displacement due to the restricting force.

i.e., $(CMOD)_{w\,c\,f} = CMOD_{w\,o\,c\,f} - U_\theta$ (15)

Where, $(CMOD)_{w\,c\,f}$ = Maximum Crack width with closing force

$CMOD_{w\,o\,c\,f}$ = Maximum Crack width without closing force

U_θ = Crack closing displacement due to closing force

4 RESULTS AND DISSCUSSION

In order to assess the soundness of the proposed expression (Equation 7 and 15), they are compared with the test results available in literature (Kaar and Mattock, 1963; Hognestad, 1962; Clark, 1956) and also with the expressions adopted in the international codes of practice.

4.1 *Test results of Kaar and Mattock*

Kaar and Mattock (1963) of the Portland Cement Association (PCA) modified the CEB equation (1959) to express the maximum crack width at the level of reinforcement on the concrete surface. Two full scale T-beams and a half and quarter scale model of one of these beams were tested. The T-beam specimens were loaded by hydraulic rams under the center diaphragm and were restrained by tie rods near the beam ends. This loading arrangement was used to simulate a negative moment region in a continuous T-beam. A 40-power microscope graduated in thousandths of an inch hydraulic actuators placed at mid-span was used to measure crack width.

4.2 *Test results of Hognestad*

Hognestad (1962) tested reinforced concrete members with high-strength deformed bars and concluded that (i)the mechanism of crack formation is such that a wide experimental scatter must inherently occur. (ii) both maximum and average crack width are essentially proportional to the stress in steel and (iii) the crack width that developed in the case of beams reinforced with state-of-the-art deformed bars was less than one half of that for plain bars. He reported crack widths at the centroid of reinforcement for steel stresses ranging from 20000 lb/in^2 (137.9 N/mm^2) to 50000 lb/in^2 (344.7 N/mm^2) for every 10000 lb/in^2 (68.9 N/mm^2) increments.

4.3 *Test results of Clark*

Clark (1956) tested 54 specimens and reported maximum crack width and spacing for steel stresses ranging from 15000 lb/ in^2 (103.4 N/mm^2) to 45000 lb/ in^2 (310.2 N/mm^2) at every 5000 lb/ in^2 (34.5 N/mm^2) increment. Crack widths on the tensile face were determined by the use of Tuckerman optical strain gages, strains in the tensile reinforcement were measured with electrical resistance strain gages. The location and extent of cracks were observed and recorded. A number of R/C Slabs and beams with different geometries and bar arrangements were tested in 4-point bending.

4.4 *Consolidated test results*

The proposed method for predicting the maximum crack width is compared using the test results reported in the literature (Hognestad 1962), Kaar and Mattock (1963), (Clark 1956). For each beam, the theoretical crack width obtained by means of the proposed expression is divided with the corresponding experimental crack width i.e., (W_{cal}/W_{exp}) and the average ratio is obtained at steel stress of 275.8 N/mm^2 (40000 Psi). The respective standard deviation and coefficient of variation are also obtained as shown in the Table 1.

4.5 *Comparison of the crack widths from the proposed expression along with the Codes of Practice with reference to the test results of Hognestad*

In order to assess the relative performance of the proposed expression (Equation 7 and 15), the average crack width ratios, standard deviation and coefficient of variation are obtained for the test results of Hognestad (1962) and compared with the corresponding values obtained from the expression adopted for crack width prediction in the international codes of practice.

4.6 *Discussion of the test results*

From the results obtained (Tables 1 & 2) the following points are noted:

From Table 1, it is observed that an average crack width ratio of 1.081 and the coefficient of variation of 22.71% is obtained at kink position of ($K_1=0.308$ $K_2=0.161$) for the test results of Hognestad (1962), indicating that theoretical values of crack width obtained from the proposed expression is closer to the experimental results.

For the test results of Kaar and Mattock, the proposed expression produces a crack width ratio of 1.093 with a standard deviation 0.226 and a coefficient of variation of 20.714% at kink position of (K_1=0.308 K_2=0.161) These values indicate that the crack width predicted by the proposed expression is consistent and reliable, and that the coefficient of variation is lower.

For the test results of Clark, the proposed expression provides a crack width ratio of 1.154 at kink position of (K_1=0.308 K_2=0.161). The deviation of theoretical crack width from experimental crack width was 0.284 with a coefficient of variance of 24.614%.

From Table 2, it can be observed that the BS 8110 equation underestimates the crack width by 27.4% for the test results of Hognestad at a steel stress of 275.8 N/mm^2 (40000 Psi) with an average crack width ratio of 0.726 and a coefficient of variation of 29.46%.

The Model code equation 1990 also underestimates the values of crack width by 38% for the test results of Hognestad at a value of steel stress equal to 275.8 N/mm^2 with an average crack width ratio (W_{cal}/W_{exp}) and coefficient of variation as 0.620 and 43.55% respectively.

The Gergely and Lutz equation which is based on a statistical analysis provides an average crack width ratio of 0.892 with a coefficient of variation of 23.57% at a steel stress of 275.8 N/mm^2 (40000 Psi) for experimental values of Hognestad. It can be observed that even though the coefficient of variation is lower, the average crack width is still underestimating by 10.8%.

For the test results of Hognestad, the Chinese code underestimates the values of crack width by 16.7% with a coefficient of variation of 24.02% at a steel stress of 275.8 N/mm^2 (40000 Psi). The average crack width ratio is 0.833, which clearly shows that even though the coefficient of variation is lower, the crack width ratio still underestimates.

From Table 2, it is also observe that the proposed expression provides better crack width ratio (1.081) and coefficient of variation (22.708%). These statistics indicate that this proposed expression is able to predict consistent crack width values with a significantly lower coefficient of variation as compared to the crack width values provided by the codes.

Table 1. Statistical comparison of the proposed expression at kink positions (break points) of bilinear curve (K_1=0.308 K_2=0.161) with the reported test results.

Source	No. of observation	(Crack width ratio)W_{cal}/W_{exp} for Bilinear (K_1=0.308 K_2=0.161)		
		Avg	Std. Dev	C.O.V
Clark	15	1.154	0.284	24.614
Kaar & Matock	6	1.093	0.226	20.714
Hognestad	27	1.081	0.246	22.708

Table 2. Statistical comparison of various codes with the proposed method.

Source	No of observation	(Crack width ratio)W_{cal}/W_{exp} for Bilinear (K_1=0.308 K_2=0.161)		
		Avg	Std. Dev	C.O.V
BS 8110 equation	32	0.726	0.214	29.46
Model code equation	32	0.620	0.270	43.55
Gergely and Lutz equation	32	0.892	0.210	23.57
Chinese code equation	32	0.833	0.200	24.02
Bilinear	27	1.081	0.246	22.708

The graphical illustration of the statistical comparison of the proposed expression at kink position (break point) of the bi-linear curve (*k_1 = 0.308 & k_2 = 0.161)* as given in Table 1 is presented in Figure 5 and the graphical illustration of the statistical comparison of the various codes with the proposed method as given in Table 2 is presented in Figure 6.

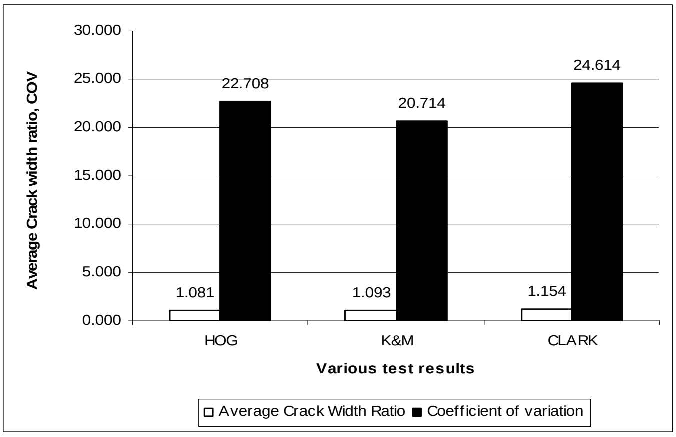

Figure 5. Graphical representation of the average crack width ratio and Coefficient of variation with reference to test results of various investigators.

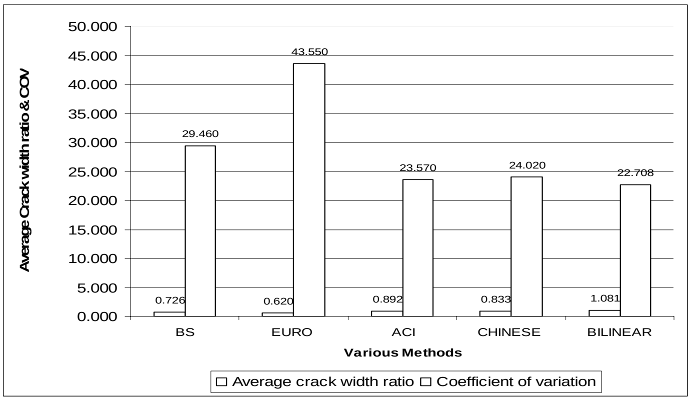

Figure 6 Comparison of Average crack width ratio and Coefficient of variation of proposed expression along with various codes of practice using the test results of Hognestad (1962).

5 CONCLUSION

In the present study an expression is developed to predict crack width in R/C beams, taking advantage of the cohesive-crack model. This expression is a function of the brittleness of concrete (a function of the tensile strength of concrete, beam depth, elastic modulus of concrete and the fracture energy), reinforcement ratio, crack length, bar diameter, stress in steel and Young's modulus of steel. To assess the validity of the expression, it was compared with other test data on crack width and crack spacing and also with various international codes of practice. The results show that the proposed approach obtained from the model using the bi-linear softening makes it possible to evaluate more accurately the crack width, compared to other formulations. Furthermore, the proposed approach has a rational and mechanically-sound basis, since it is rooted in concrete fracture mechanics.

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