Path and dynamics of cracks propagating in a disordered material under mode I loading

L. Ponson
COPPE/UFRJ Programa de Eng. Civil, Rio de Janeiro, Brazil

D. Bonamy & E. Bouchaud
Fracture Group, Service de Physique et Chimie des Surfaces et Interfaces, CEA Saclay, France

G. Cordeiro, R. D. Toledo & E. M. R. Fairbairn
COPPE/UFRJ Programa de Eng. Civil, Rio de Janeiro, Brazil

ABSTRACT: The propagation of a single crack in a disordered material is theoretically investigated. Restricting first our analysis to perfectly brittle materials, we show within the framework of the linear elastic fracture mechanics that a crack is described by two independent equations: A *path equation*, that gives the trajectory followed by the crack within the disordered material, and a *motion equation* providing the local velocities along the crack front during its propagation. Theoretical predictions resulting from the path equation are found in good agreement with the statistical properties of experimental rough fracture surfaces resulting from perfectly brittle fracture. The motion equation is then studied and predictions for the crack growth velocity dependence with the applied stress intensity factor are given. The path and the dynamics of a crack propagating in a quasi-brittle material is also discussed using a natural extension of the previous description that takes into account the effects of damage and microcracks.

1 INTRODUCTION

The fracture of disordered media represents an important applied problem, with intriguing theoretical aspects. Models issued from statistical physics have been shown to be very promising to describe various aspects of the failure of heterogeneous media (Roux & Herrmann 1990; Bouchaud et al. 1993). In particular, the morphology of rough fracture surfaces as well as the spatio-temporal evolution of an interfacial crack front are found to exhibit scaling laws characterized by exponents the value of which depends very weakly on the materials (Bouchaud et al. 1990; Måløy et al. 1992; Ponson et al. 2006a; Ponson et al. 2006b; Måløy et al. 2006). For example, the height-height correlation function $\Delta h = \langle (h(z+\Delta z) - h(z))^2 \rangle_z^{1/2}$ computed on fracture surfaces perpendicularly to the direction of crack propagation is observed to scale as $\Delta h \sim \Delta z^{\zeta}$ where the roughness exponent is found to be $\zeta \simeq 0.75$ for a wide range of materials. These properties suggest that one theory could describe the statistical properties of a crack propagating in any disordered material. This theoretical description should be able to depict the competition between two antagonist aspects that govern such a phenomenon. The structure and the microstructural material properties that are inhomogeneous and the stress field that follows the law of elasticity.

Recent experimental results suggest in fact the existence of second class in failure problems: Fracture surfaces resulting from a perfectly brittle failure are characterized by a lower roughness exponent $\zeta \simeq 0.4$ (Boffa et al. 1998; Ponson et al. 2006c), while failure involving non-linear processes such as damage and microcracks in quasi-brittle materials was suggested to result in rougher fracture surfaces with $\zeta \simeq 0.75$ as for mortar (Mourot et al. 2005) and wood (Morel et al. 1998).

In this article, we extend a model (Bonamy et al. 2006) based on Linear Elastic Fracture Mechanics (LEFM) originally proposed to reproduce the statistical properties of fracture surfaces resulting from brittle fracture of heterogeneous materials (Section 2). An interpretation of the morphology of broken quasi-brittle material surfaces is then proposed. Then we investigate theoretically the dynamics of the crack front (Section 3). In particular, a relation between the mean crack growth velocity to the applied stress intensity factor is proposed for a perfectly elastic material. The case of quasi-brittle materials is then discussed. The experimental investigation of such a relation is currently under progress and the results will be published elsewhere.

2 PATH EQUATION

In this section, the path followed by a single crack propagating in a disordered material is theoretically investigated. At first, we will focus on perfectly brittle material so that the theoretical framework of LEFM can be used. We restrict the following analysis to the case where the crack speed is small enough compared to the sound speeds – speed of longitudinal, transverse and Rayleigh waves – in the material so that the quasi-static approximation is relevant. A pure mode I loading is considered. The crack front (oriented along the z-axis) is thus confined roughly to a plane (x, z) perpendicular to the tensile forces (along the y-axis) and propagates along the x-axis. In a homogeneous material, the crack would propagate at spatially uniform velocity within a plane, the plane (z, x). But the heterogeneities of the material induce both *in-plane* $f(z, t)$ and *out-of-plane* $h(x = x_0 + f(z, t), z)$ perturbations in the crack front. Schematic views of the in-plane $f(z, t)$ and out-of-plane $h(x = x_0 + f(z, t), z)$ displacements are shown in Figure 1. For simplicity, the out-of-plane perturbations have been represented for a crack front without in-plane perturbations ($f(z, t) = 0$). The morphology of fracture surfaces is then a direct measurement of the out-of-plane perturbations $h(x, z)$. For small enough perturbations, the out-of-plane displacements are independent of the in-plane displacements so that the shape of the fracture surface can be predicted independently of $f(z, t)$. On the other hand, this implies that the dynamical properties of the crack – the local velocities of the crack front $\frac{\partial f}{\partial t}(z, t)$ – investigated in Section 3 are decoupled from the crack path $h(x, z)$.

Let us consider a point M of the crack front characterized by its position $(x = x_0 + f(z, t), y = h(x, z), z)$. The local stress field around M determines its trajectory. The stress at a distance r ahead of the point M in the direction θ can be written as the sum of the contributions of each of the three fracture modes, each mode being developed as a $r^{k/2}$ expansion with $k \geq -1$ (Irwin 1958)

$$\sigma_{ij} = \sum_{p=I}^{III} \frac{K_p}{\sqrt{2\pi r}} g_p^{ij}(\theta) + T_p k_p^{ij}(\theta) + A_p l_p^{ij}(\theta)\sqrt{r} + ... \quad (1)$$

where $K_p(M)$ (stress intensity factors), $T_p(M)$ (T-stress) and $A_p(M)$ are depending on the remote loading, the geometry of the sample, the shape of the crack front, and the coordinate z of the point M. The functions $g_p^{ij}(\theta)$, $k_p^{ij}(\theta)$ and $l_p^{ij}(\theta)$ are universal. Even though we focus here on a dominantly mode I loading situation, K_{II} and K_{III} are not equal to zero. The perturbations h and f of the crack shape induce a small shearing loading around the crack front.

The path chosen by the crack in M is the one for which the local stress field is of mode I type ("criterion of local symmetry") (Gol'dstein & Salganik 1974; Cotterell & Rice 1980; Hogdon & Sethna 1993). In other words, the net mode II stress intensity factor K_{II} should vanish in each location z along the crack front and any position x of the mean line, the effect of the mode III on the crack path being here neglected. To first order in $h(x, z)$, six contributions should be taken into account in the evaluation of K_{II}. The four first contributions are induced by the out-of-plane perturbations $h(z, x)$ of the crack and have already been calculated (Ball & Larralde 1995; Mochvan et al. 1998) in the limit of small perturbations $h(x, z)$. The fifth contribution arises from inevitable imperfections in the loading system or the crack alignment so that the applied loading is not in pure mode I and a small external mode II loading $K_{II}^{\text{ext}} \ll K_I^{\text{ext}}$ is applied to the experimental sample. The sixth contributions is due to the heterogeneous nature of the material and is modelled by a quenched uncorrelated random field $K_{II} = -K_I^{\text{ext}}/2\eta(z, x, h)$ written, without loss of generality, as the sum of two uncorrelated random fields $\delta K_{II}^{(1)} = -K_I^{\text{ext}}/2(\eta_{\text{q}}(z, h) + \eta_{\text{t}}(z, x))$. Finally, for a sample with Poisson's ratio ν, one gets

$$K_{II} = \frac{K_I^{\text{ext}}}{2}\frac{\partial h}{\partial x} - \frac{K_I^{\text{ext}}}{2\pi}\frac{2-3\nu}{2-\nu}\int \frac{h(x,z')-h(x,z)}{(z'-z)^2}dz' + \Delta K_{II}^{\text{memory}}$$
$$+\sqrt{\tfrac{\pi}{2}}A_I h(x, z) + K_{II}^{\text{ext}} - \frac{K_I^{\text{ext}}}{2}(\eta_{\text{q}}(z, h) + \eta_{\text{t}}(z, x)) \quad (2)$$

One can show that the third and fourth contributions are negligible (Larralde & Ball 1995; Ponson 2006d) so that the criterion of local symmetry $K_{II} = 0$ leads to

$$\frac{\partial h}{\partial x} = A_\nu \int \frac{h(z') - h(z)}{(z' - z)^2}dz' + \eta_{\text{q}}(z, h) + \eta_{\text{t}}(z, x) + F_0 \quad (3)$$

where $A_\nu = \frac{1}{\pi}\frac{2-3\nu}{2-\nu}$ and $F_0 = -2K_{II}^{\text{ext}}/K_I^{\text{ext}} \ll 1$ are constants. In other words, the morphology of the fracture surface $h(x, z)$ is given by the motion of the

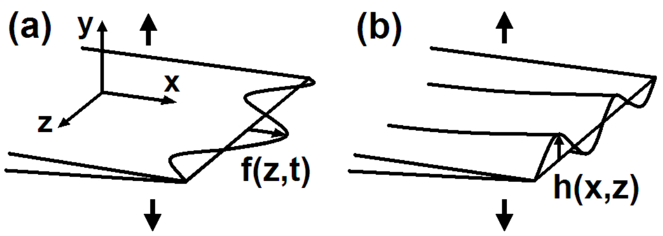

Figure 1: Geometry of perturbed cracks subject to mode I loading (large arrows indicating the direction of macroscopic loading). (a): In-plane perturbations. (b): Out-of-plane perturbations. The shape of the fracture surface is effectively the history of the out-of-plane perturbations of the crack front.

elastic string $h(z)$ that "creeps" – the x coordinate playing the role of time – within a random potential $\eta_{\mathrm{q}}(z,h)$ due to the "thermal" fluctuations $\eta_{\mathrm{t}}(z,x)$. Thus, the line – and so the fracture surface – is self-affine so that its height-height correlation function scales as $\Delta h \sim \Delta z^{\zeta}$ where the exponent ζ depends only on the range of the elastic interactions along the line. They are long-ranged, characterized by a kernel decreasing as $1/r^2$. Therefore, one gets $\zeta \simeq 0.39$ (Rosso & Krauth 2002; Vandembroucq & Roux 2004; Kolton et al. 2006). This result is in fairly good agreement with experimental observations of fracture surfaces reported for brittle glass ceramics (Ponson et al. 2006c) and sandstone (Boffa et al. 1998).

Let us now discuss the morphology of fracture surfaces of quasi-brittle materials that are characterized by a higher roughness exponent $\zeta \simeq 0.75$. In this latter case, the material cannot be considered as linear elastic anymore. In front of the main crack, the process zone made of various microcracks modifies the main crack path and so the geometry of the resulting fracture surface. It appears natural to conjecture that the induced "porosity" *screens* the elastic interactions making the elastic kernel $1/r^{\alpha+1}$ in Equation (3) decreasing faster than the one $1/r^2$ ($\alpha = 1$) expected for sane linear elastic materials (Bonamy et al. 2006). An "arbitrary" value $\alpha \simeq 1.5 - 1.7$ would then allow to account for the value of $\zeta \simeq 0.75$ (Tanguy et al. 1998) observed for quasi-brittle materials. Understanding how damage screening can select such an effective interaction range in crack problems provides a significant challenge for future investigations.

3 MOTION EQUATION

We focus now on the dynamical properties of a single crack propagating in an heterogeneous material. At first, let us address the case of an elastic material. As mentioned in Section 2, dynamics and trajectory are decoupled according to linear elastic fracture mechanics. In other words, the time evolution of the crack is given by its in-plane deformation $f(z,t)$ while its trajectory $h(z,x)$ is set by Equation (3). To derive a motion equation of the crack, one can therefore neglect its out-of-plane perturbation so that a crack propagating through a 3D material and an interfacial crack propagation correspond to the same physical situation. The latter case has already been studied (Gao & Rice 1989) and the elastic interactions along the crack front were shown to be long-ranged (with $\alpha = 1$) so that the local stress intensity factor is given by

$$K_I(z,t) = K_I^{\text{ext}} + \frac{K_I^{\text{ext}}}{2\pi}\int \frac{f(z',t)-f(z,t)}{(z'-z)^2}dz' \quad (4)$$

Starting from the natural motion equation $v = \frac{\partial f}{\partial t} \sim K_I - K_{Ic}$, and assuming local fluctuations within the local toughness $K_{Ic} = K_{Ic}^0 - \eta(z, f(z,t))$, Equation (4) yields to

$$\begin{aligned}\frac{\partial f}{\partial t} \sim (K_I^{\text{ext}} - K_{Ic}^0) + \frac{K_I^{\text{ext}}}{2\pi}\int \frac{f(z',t)-f(z,t)}{(z'-z)^2}dz' \\ +\eta(z,f(z,t))\end{aligned} \quad (5)$$

The morphology of the crack front as well as the statistical properties of its local velocities have been widely studied both from the experimental (Schmittbuhl & Måløy 1997; Måløy et al. 2006) and the theoretical point of view (Schmittbuhl et al. 1995). According to Equation (5), the projection of the front on the mean crack plane is expected to display a self-affine geometry characterized by the roughness exponent $\zeta_{\text{dep}} = 0.39$. Until now, experimental investigations mainly performed on interfacial cracks in Plexiglas and crack front in metallic alloys have led to a larger value $\zeta \simeq 0.5 - 0.6$. However, the relevance of the linear elasticity to model crack propagation in such materials can be questioned. Here, we will focus on the averaged velocity of the crack front that is of higher interest for direct engineer applications. Indeed, from the motion equation (Eq. (5)), it is possible to derive the relation between the applied stress intensity factor K_I^{ext} and the main crack growth velocity $v_{\text{crack}} = \langle\frac{\partial f}{\partial t}\rangle_{z,t}$.

From the Equation (5), one can show (Barabási & Stanley 1995; Kardar 1998) that, for K_I^{ext} smaller than a critical value $K_{Ic}^0 + \Delta K_{Ic}$, the crack front is pinned by the microstructural obstacles of the material while the crack front propagates with $v_{\text{crack}} > 0$ for $K_I^{\text{ext}} > K_{Ic}^0 + \Delta K_{Ic}$ (The curve $v_{\text{crack}}(K_I^{\text{ext}})$ is represented in dashed line in Figure 2). This so-called depinning transition from a pinned to a moving state only occurs at zero temperature. For $T > 0$, crack propagation is always possible even at low $K_I^{\text{ext}} - K_{Ic}^0$. In that case, the line can overcome energy barriers through thermal activation. Taking in consideration both the geometry of the line at finite temperature as well as the "energy landscape", one can show (Nattermann 1990) that the line *creeps* through the disordered media with a velocity:

$$v_{\text{crack}} \sim e^{-\frac{C_0}{T}\frac{1}{(K_I^{\text{ext}}-K_{Ic}^0)^{\mu}}} \quad (6)$$

where the exponent μ is given by (Kolton et al. 2006):

$$\mu = \frac{1-\alpha+2\zeta_{eq}}{\alpha-\zeta_{eq}} \quad (7)$$

Here, α is the range of the elastic interactions along the line ($\alpha = 1$ for a linear elastic material) and ζ_{eq} is the roughness exponent at equilibrium given by $\zeta_{\text{eq}} = \frac{\alpha}{3}$ (Nattermann & Rujan 1989). Thus, one gets:

$$\mu = \frac{3-\alpha}{2\alpha} \quad (8)$$

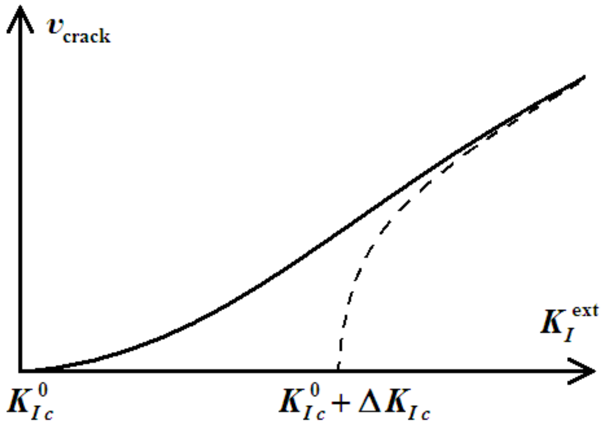

Figure 2: Variation of the mean velocity of the crack front with respect to the applied external stress intensity factor at finite temperature $T > 0$. The dashed curve correspond to $T = 0$.

For a perfectly brittle materials ($\alpha = 1$), Equation (8) yields to $\mu = 1$ so that the crack growth velocity is expected to increase as $v_{\text{crack}} \sim e^{-\frac{C_1}{K_I^{\text{ext}} - K_{Ic}^0}}$. These variations are represented in Figure 2 in solid line. Experimental works performed on brittle ceramics and rocks are currently in progress to test this relation. Let us note that this formula could have interesting engineer applications because life-time of structure are often directly linked with the growth velocities of possible cracks.

Let us now discuss the case of quasi-brittle materials. In Section 2, we have suggested that the presence of damage and microcracks ahead of the main crack could screen the elastic interactions yielding to effective elastic interactions decreasing faster along the front ($\alpha \simeq 1.5 - 1.7$) than in the elastic case ($\alpha = 1$). This effect may also be relevant to describe the role of damage on the in-plane perturbation f and thus the dynamics of the crack. Indeed, the observation of interfacial cracks with a roughness characterized by $\zeta \sim 0.5 - 0.6$ is much closer to predictions of models with shorter range interactions. In the case of screened elastic interactions, one expects therefore, according to Equation 7, lower values of $\mu \simeq 0.4 - 0.5$. In addition with its evident practical interest, measuring the variation of the crack growth velocity with the external stress intensity factor for brittle and quasi-brittle materials may be a way to probe the range of the elastic interactions along the crack front. An experimental study of the creep motion of cracks in concrete specimen is currently under progress in this direction.

4 CONCLUSIONS

The behavior of a single crack propagating in a disorder material represents an interesting challenge both from the fundamental and applied point of view. Contrary to the case of an homogeneous material, the crack is observed to deviate from a straight line resulting in rough fracture surfaces and intermittent crack dynamics characterized by scaling laws. We have proposed here a description based on two decoupled equations: A *path equation* that describes the out-of-plane perturbations of the crack – and therefore to the geometry of the fracture surface – and a *motion equation* that predicts its in-plane perturbation that rules the dynamical properties of the whole crack. The case of crack propagation in brittle materials was first treated and linear elastic fracture mechanics was used to derive these two equations. The case of quasi-brittle materials was then discussed and the main physical differences with the elastic case were pointed out. The main results of this work can be summed up as the following:

(i) In brittle materials, for a slow crack propagation under mode I loading, the out-of-plane perturbations of the crack are given by a creep equation of an elastic line with long-ranged elastic interactions (Eq. (3)). The resulting roughness of fracture surfaces is self-affine, characterized by a roughness exponent $\zeta = 0.39$ in good agreement with the experimental results reported on surfaces of broken brittle materials (Boffa et al. 1998; Ponson et al. 2006c).

(ii) The in-plane perturbations of the crack are described by a pinning/depinning equation (Eq. 5) which also sets the local velocities of the front. We go further than previous analyses (Schmittbuhl et al. 1995) and derive the relation between the mean crack growth velocity v_{crack} and the external applied loading K_I^{ext}. This relation (Eq. (6)) is a stretched exponential characterized by the exponent $\mu = 1$ for the case of brittle failure.

(iii) The case of crack propagations in quasi-brittle materials has been discussed. The screening of the elastic interactions caused by the presence of damage and microcracks ahead of the main crack front is proposed to be the mechanism responsible for the differences observed with the linear elastic case. Changing the range of the interactions along the crack accounts for the higher roughness exponent $\zeta \simeq 0.75$ observed on fracture surfaces of quasi-brittle materials. Moreover, we conjecture that the relation between v_{crack} and K_I^{ext} is still valid, but with $\mu \simeq 0.4 - 0.5$ to take into account the change in the range of the elastic interactions.

An experimental study performed on both brittle sandstones and quasi-brittle concretes is currently under progress to test the proposed relation between crack growth velocity and applied loading. The experimental investigation of crack dynamics in disordered material is of great interest because it could validate the pinning/depinning description of front motion, but because also this could provide an efficient theoretical tool to predict crack growth velocity in materials and therefore life-time of structures.

REFERENCES

Ball, R. C. & Larralde, H. (1995). Three-dimensional stability analysis of planar straight cracks propagating quasistatically under type I loading. *Int. J. Frac. 71*, 365–377.

Barabási, A. L. & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press.

Boffa, J. M., Allain, C. & Hulin, J. P. (1998). Experimental analysis of fracture rugosity in granular and compact rocks. *Eur. Phys. J. Appl. Phys. 2*, 281–289.

Bonamy, D.,Ponson, L., Prades, S., Bouchaud E., & Guillot, C. (2006). Scaling Exponents for Fracture Surfaces in Homogeneous Glass and Glassy Ceramics. *Phys. Rev. Lett. 97*, 135504.

Bouchaud, E., Lapasset, G., & Planès, J. (1990). Fractal dimension of fractured surfaces: A universal value? *Europhys. Lett. 13*, 73–79.

Bouchaud, J. P., Bouchaud, E. Lapasset, G., & Planès, J. (1993). Models of fractal cracks. *Phys. Rev. Lett. 71*, 2240–2243.

Cotterell, B. & Rice, J. R. (1980). Slightly curved or kinked cracks. *Int. J. Frac. 16*, 155–169.

Gao, H. & Rice, J. R. (1989). A first-order perturbation analysis of crack trapping by arrays of obstacles. *J. Appl. Mech. 56*, 828–836.

Gol'dstein, R. V. & Salganik, R. L. (1974). Brittle fracture of solids with arbitrary cracks. *Int. J. Frac. 10*, 507–523.

Hogdon, J. & Sethna, J. P. (1993). Deviation of a general three-dimensional crack propagation law: A generalization of the principle of local symmetry. *Phys. Rev. B 47*, 4831–4840.

Irwin, G. R. (1958). Fracture. In *Handbuch der Physik*, Volume 6, pp. 551. Springer-Verlag.

Kardar, M. (1998). Nonequilibrium dynamics of interfaces and lines. *Phys. Reports 301*, 85–112.

Kolton, A., Rosso, A., Giamarchi, T., & Krauth, W. (2006). Dynamics below the depinning threshold in disordered elastic systems. *Phys. Rev. Lett. 97*, 057001.

Larralde, H. & Ball, R. C. (1995). The shape of slowly growing cracks. *Europhys. Lett. 30*, 87–92.

Måløy, K. J., Hansen, A., Hinrichsen, E. L., & Roux, S. (1992). Experimental measurements of the roughness of brittle cracks. *Phys. Rev. Lett. 68*, 213–215.

Måløy, K. J., Santucci, S., Schmittbuhl, J., & Toussaint, R. (2006). Local waiting time fluctuations along a randomly pinned crack front. *Phys. Rev. Lett. 96*, 045501.

Mochvan, A. B., Gao, H., & Willis, J. R. (2000). On perturbations of plane cracks. *Int. J. Solids Struct. 35*, 3419–3453.

Morel, S., Schmittbuhl, J., Lopez, J. M., & Valentin, G. (1998). Anomalous roughening of wood fractured surfaces. *Phys. Rev. E 58*, 6999–7005.

Mourot, G., Morel, S., Bouchaud, E., & Valentin, G. (2005). Anomalous scaling of mortar fracture surfaces. *Phys. Rev. E 71*, 016136.

Nattermann, T. (1990). Scaling approach to pinning: Charge-density waves and giant flux creep in superconductors. *Phys. Rev. Lett. 64*, 2454-2457.

Nattermann, T. & Rujan, P. (1989). Random field and other systems dominated by disorder fluctuations. *Int. J. Mod. Phys. B 3*, 1597.

Ponson, L., Bonamy, D., & Bouchaud, E. (2006). Two-dimensional scaling properties of experimental fracture surfaces. *Phys. Rev. Lett. 96*, 035506.

Ponson, L., Bonamy, D., Auradou, H., Mourot, G., Morel, S., Bouchaud, E., Guillot, C., & Hulin, J. P. (2006). Anisotropic self-affine properties of experimental fracture surfaces. *Int. J. Frac. 140*, 27-36.

Ponson, L., Auradou, H., Vié, P., & Hulin, J. P. (2006). Low self-affine exponents of fractured glass ceramics surfaces. *Phys. Rev. Lett. 97*, 125501.

Ponson, L. (2006). *Crack propagation in disordered materials: How to decipher fracture surfaces*. Ph. D. thesis, Ecole Polytechnique.

Rosso, A. & Krauth, W. (2002). Roughness at the depinning threshold for a long-range elastic string. *Phys. Rev. E 65*, 025101(R).

Roux, S. & Herrmann H. (1990). *Statistical Models for the Fracture of Disordered Media*. Elsevier.

Schmittbuhl, J. & Måløy, K. J. (1997). Direct observation of a self-affine crack propagation. *Phys. Rev. Lett. 78*, 3888–3891.

Schmittbuhl, J., Roux, S., Vilotte, J. P., & Måløy, K. J. (1995). Interfacial crack pinning: effect of nonlocal interactions. *Phys. Rev. Lett. 74*, 1787–1790.

Schmittbuhl, J., Schmitt, F., & Scholz, C. (1995). Scaling invariance of crack surfaces. *J. Geophys. Res. 100*, 5953–5973.

Tanguy, A., Gounelle, M., & Roux, S. (1998). From individual to collective pinning: effect of long-range elastic interactions. *Phys. Rev. E 58*, 1577–1590.

Vandembroucq, D. & Roux, S. (2004). Large-scale numerical simulations of ultrametric long-range depinning. *Phys. Rev. E 70*, 026103.