Fractals, statistics and size-scale effects on concrete strength

A. Carpinteri & S. Puzzi
Politecnico di Torino, Department of Structural Engineering and Geotechnics, Torino, Italy

ABSTRACT: The size-scale effects on concrete strength are a very important topic in engineering design. In recent years, the scientific community dedicated great efforts in order to gain a precise description of this phenomenon and to highlight the physical mechanisms that lie behind it. Aim of this paper is to review the fundamentals of the fractal approach (Carpinteri, 1994b), which has recently been at the centre of a scientific debate, and to revisit in detail its connections with statistics. The most recurrent criticisms against the fractal interpretation of the size-scale effects will be rejected and theoretical results regarding the link between fractals and statistics will be also confirmed by numerical simulations.

1 INTRODUCTION

The size-scale effect on structural strength is a very important topic in engineering design. Dealing specifically with concrete structures, it was seen that tensile strength decreases with the structural size, whereas fracture energy increases. In recent years, the scientific community dedicated significant efforts in order to gain a precise description of this phenomenon and to highlight the physical mechanisms that lie behind it. Three different approaches have been proposed and analyzed at least, i.e. the energetical, the statistical and the fractal one.

The fractal approach, originally proposed by Carpinteri (1994a,b), has been matter of intense debate, particularly in the papers by Bažant (1995, 1997b, 1998, 2000) and Bažant & Yavari (2005) and, more recently, by Saouma & Fava (2006), who question its validity and even argue that it lacks sound physical and mathematical basis. In this long standing controversy about the interpretation of scaling laws on material strength (Carpinteri & Pugno, 2005), the fractal approach has been counterposed to the energetical approach at first and to the so-called energetical-statistical one only more recently.

Aim of this paper is to review and reject most of the criticisms against the fractal interpretation of the size effects, by clarifying some aspects that have been misunderstood and confused by the above cited Authors. In particular, we will analyze the crucial critiques contained in the Appendix of the paper by Bažant & Yavari (2005), who argue that "the property that the left-size asymptote of the MFSL in a bilogarithmic plot should have the slope –1/2 must be considered as unproven by the fractal argument in [Carpinteri (1994b)]". We will critically review these criticisms, showing how they also contain some flaws and mistakes. More in detail, by analyzing a fractal distribution of micro-cracks in the framework of *Extreme Value theory* (EVT), we will show how this distribution naturally provides the slope –1/2, which corresponds to the LEFM size-scale effect.

Moreover, we will show that it is wrong to set the fractal approach to size-scale effects against the statistical one, since they are deeply connected, as shown in several papers (Carpinteri & Cornetti 2002, Carpinteri et al. 2004, 2005a,b). Eventually, the cause for the lack of specifications regarding the size scale effects in the design codes will be critically analyzed.

2 SIZE-SCALE FORMULAE

In the last two decades, several formulae have been proposed for interpreting the size-scale effects on concrete strength. Among these, two scaling laws have found broad application. The first one is the so-called SEL (Size Effect Law), originally proposed by Bažant (1984), in which LEFM and limit analysis concepts were joined together yielding:

$$\sigma_u = \frac{Bf_t}{\sqrt{1+b/b_0}} \tag{1}$$

where f_t is the plastic limit stress, B and b_0 are two constants to be determined in each case by fitting experimental data.

In the original formulation, the SEL was obtained for the three point bending geometry from energy release concepts, by introducing some simple geometrical hypotheses about the crack band and the stress relief zone. Later on, the same law was obtained by means of a much more general asymptotic analysis of the energy release (Bažant, 1997a).

A second successful size-effect formula is the so-called MFSL (Multi-Fractal Scaling Law, Carpinteri, 1994a,b), originally proposed in 1992 on the basis of fractal argumentations and in the framework of Renormalization Group Theory:

$$\sigma_u = f_t \sqrt{1 + \frac{l_{ch}}{b}} \tag{2}$$

where the asymptotic value of the nominal strength f_t, corresponding to the lowest nominal tensile strength, is reached only in the limit of infinite sizes. The transition between the LEFM asymptote at the smaller scales (fractal regime) and the horizontal asymptote (homogeneous regime) is controlled by the characteristic length l_{ch}, representing the variable influence of disorder. Both f_t and l_{ch} have to be determined in each case by fitting emperimental data.

Both laws, as evidenced from the large available Literature on the subject (see e.g. the Report by Carpinteri et al., 1995), have been confirmed by results of several experimental tests in bending, tension and compression. A clearer picture emerged only recently (Carpinteri et al. 2002); it is now definitely clear that, when a strong energy-driven fracture process is activated, as in the presence of important notches in the structure, a curvature with upper convexity should be considered in the bi-log strength vs size diagram. This is exactly the case when Bažant's original Size Effect Law (SEL) applies. On the contrary, when the role of microstructural disorder and of self-similar features (i.e., fractality) dominate the damage and fracturing processes, the MFSL permits to interpolate experimental data more realistically and closer than SEL. In other words, the experimental tests in the literature strongly support the MFSL when large notches are absent.

At the beginning this fact was not acknowledged and since 1995 Bažant (1995) simply tried to oppose his 1984 SEL formula to the MFSL by Carpinteri. His opposition to eq. (2) exploited the same tactics followed to demonstrate that Weibull-type size effect is not applicable to concrete structures and concretized in a strong opposition to the existence of a finite asymptotic strength for large structural sizes, which was in contradiction with his formula.

Only later Bažant (1997a) introduced the so-called "Universal SEL", in which a horizontal asymptote in the limit of infinite sizes is present and, more importantly, the same MFSL upwards curvature may be obtained under certain values of the parameters, if the specimen is unnotched.

Nevertheless, his attempts to discredit the work by Carpinteri and co-workers and the MFSL continued (see e.g. Bažant, 1997b, 2000 and Bažant & Novák, 2000) up to the recent paper by Bažant & Yavari (2005), in which several criticisms are raised against the derivation of the MultiFractal Scaling Law presented by Carpinteri (1994b); they question its validity and even argue that it lacks sound physical and mathematical basis. Quoting from their paper (page 13, item 3): "The 'MFSL' was based on a series of hypotheses but does not follow from these hypotheses by a valid mathematical procedure."

3 THE SLOPE OF THE MFSL ASYMPTOTE AT THE SMALLER SCALES

In particular, Bažant & Yavari (2005) aim at showing that the scaling law for strength at the smaller scales, characterized by the slope –1/2 in the bi-logarithmic plot, does not follow from the statistical treatment presented in Carpinteri (1994b). They insist on this point throughout their whole paper; quoting again from:

- Page 13, item 4: "the value –1/2 is an unproven conjecture which does not follow from the fractal hypothesis";
- Page 26, item 1: "The exponent –1/2 attributed to the small-size asymptotic scaling law is supposed to be solely a consequence of a peculiar situation called 'the extreme disorder'.";
- Page 26: "the property that the left-size asymptote of the MFSL in a bi-logarithmic plot should have the slope –1/2 must be considered as unproven by the fractal argument. [...] If only the fractal viewpoint is considered, this property is merely an empirical assumption".

In this section we will reject these criticisms against the fractal interpretation of the size effects, by revisiting the statistical treatment presented by Carpitneri (1994b), clarifying some aspects that have been misunderstood by the above cited Authors and showing how these criticisms also contain some flaws and mistakes. In particular, we will reject the crucial critique against the slope –1/2 of the left-hand asymptote of the MFSL. This slope, as will be shown, not only follows from the fractal-statistical treatment, but also is explained in the framework of the Fractal Cohesive Crack Model (Carpinteri et al., 2002), that has been confirmed very convincingly by experiments (Carpinteri et al., 2002, 2003).

3.1 *The Fractal Cohesive Crack Model*

In this framework, indicating by d_σ, d_ε and $d_{\mathcal{G}}$ the fractional exponents for strength, strain and fracture energy, respectively, it has been shown that the following equation should hold:

$$d_\sigma + d_\varepsilon + d_{\mathcal{G}} = 1 \tag{3}$$

At the smaller scales, the collapse is governed by the canonical critical strain ε_c and continuum damage mechanics holds. In this case the damage is diffused (with uniform strain in the bulk) and one obtains d_ε=0. Thus, the previous relation becomes $d_\sigma+d_{\mathcal{G}}$=1. On the other hand, the maximum value for $d_{\mathcal{G}}$ is 1/2, since this value implies a fractal dimension of the dissipation domain $\Delta_{\mathcal{G}}$ = 2.5, which would correspond to the Brownian crack surface. As a consequence, d_σ = 1/2 is the limit value at the smaller scales.

3.2 *The fractal-statistical explanation*

In this section we will revisit the statistical treatment presented by Carpinteri (1994b), which provides a different explanation for the slope of the left-hand asymptote of the MFSL, and which has been criticized by Bažant & Yavari (2005). Before revisiting this explanation of the MFSL asymptotic slope, let us start from the critiques.

At page 26 of their paper, Bažant & Yavari affirm that "defects of maximum size $a_{\max}$ cannot have the same probability distribution of a as the ensemble of all defects, but could have only one of the three possible extreme value distributions (Fréchet, Weibull or Gumbel) of which only the Weibull distribution would be realistic here because a non-negative threshold on a exists".

The second part of the statement is definitely wrong: the existence of a (presumably upper) "non-negative threshold on a" is merely speculative and, in any case, unproven. Moreover, the limit distribution for an heavy tailed distribution, such as Pareto, or Cauchy, is not the Weibull, as erroneously stated, but the Fréchet one (this result was already used by Freudenthal (1968) almost 40 years ago).

This is not the crucial point, however: in the first part of the statement, Bažant & Yavari (2005) affirm that $a_{\max}$ cannot have a power-law (fractal) distribution and, consequently, that the assumption ($a_{\max}$ /b)=const. (b being the structural size) is unjustified. This is the key point, since from this hypothesis follows the –1/2 (LEFM) slope of the left-hand asymptote.

However, this critique will be rejected: although formally correct, the cited statement misses the fundamental nature of the extremal behaviour of the defect size a. If we consider the probability distribution of self-similarity:

$$p(a) = \frac{C}{a^{N+1}} \tag{4}$$

with exponent N = 3, we may demonstrate that ($a_{\max}/b$) is constant on average. This result can be obtained rigorously in the framework of EVT, by assuming the above distribution of defect size for the ensemble of all defects (as already done by Carpinteri, 1994b).

To show this, and reject the critique by Bažant & Yavari, we need to introduce some fundamentals of EVT. Let us suppose we have a sequence of random variables X_1, X_2,… who are i.i.d., i.e. independent and identically distributed. Let $P(x)$ be their common cumulative distribution function:

$$P(x) = \Pr\{X_i \le x\} \tag{5}$$

Also let M_n = max(X_1, … , X_n) be the n-th sample maximum of the process. Then, it is obvious that:

$$\Pr\{M_n \le x\} = \left[P(x)\right]^n \tag{6}$$

This results is trivial, since for each value x such that $P(x) < 1$, we have $\Pr\{M_n \le x\} \rightarrow 0$ for $n \rightarrow \infty$. For non-trivial limit results, it is necessary to find two sequences of real numbers $a_n > 0$, b_n, such that:

$$\Pr\left\{\frac{M_n - b_n}{a_n} \le x\right\} = \left[P\left(a_n x + b_n\right)\right]^n \rightarrow H \tag{7}$$

The Three Types Theorem (Fischer & Tippett, 1928) affirms that, if H exists, it should be one of the three extreme value distributions (Fréchet, Weibull or Gumbel). These three distributions may be written in a single form, which is usually referred to as the *Generalised Extreme Value* distribution (GEV):

$$H(x) = \exp\left\{-\left(1 + \xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}\right\} \tag{8}$$

where ξ is a shape parameter, μ a location parameter, $\sigma > 0$ is a scale parameter and $1+\xi(x-\mu)/\sigma > 0$. The sign and value of ξ identifies the distribution type: ξ>0 corresponds to the Fréchet distribution, ξ<0 to the Weibull distribution, whilst the limit $\xi\rightarrow 0$ corresponds to the Gumbel one (Gumbel, 1958).

Let us consider, as done by Carpinteri (1994b), that the defects are distributed according to a probability density function $p(a)$ characterized by a fractal tail, such as that of Eq. (4). The corresponding cumulative distribution function $P(a)$ is thus:

$$P(a) = 1 - Ca^{-N} \tag{9}$$

where C>0 and N>0 are constants. This form corresponds to a Pareto-type tail, sometimes referred to as a Cauchy-type (Freudenthal, 1968). In order to perform the renormalization of eq. (7), we choose $a_n = (nC)^{1/N}$ and $b_n = 0$. By doing so, we obtain:

$$\left[P\left(a_n x + b_n\right)\right]^n \Longrightarrow \exp\left(-x^{-N}\right) \quad (10)$$

Comparing this equation with eq. (8), it is clear that the limit distribution for $a_{\max}$ is not the Weibull, as erroneously stated by Bažant & Yavari (2005), but the Fréchet one.

If we now assume that interaction between the flaws is negligible (weakest link hypothesis, Weibull, 1939), the strength of a solid made of the material is determined by the size of the largest flaw. The relation between the strength of the material and the flaw size may be defined by the LEFM equation (Griffith, 1921):

$$\sigma\sqrt{a} = k \quad (11)$$

k being a constant value, which depends on the values of the elastic constants: E and ν, on the fracture toughness K_{IC} and on the specimen geometry. If we consider the cumulative probability $P_\sigma(\sigma)$ of strength and that of maximum flaw size $P_{a\max}(a)$, they are in relation through the following equality chain:

$$\begin{aligned} P_\sigma(\sigma) &= \Pr\{\Sigma \le \sigma\} = \Pr\{A_{\max} \ge k^2/\sigma^2\} = \\ &= 1 - P_{a_{\max}}\left(k^2/\sigma^2\right) \end{aligned} \quad (12)$$

As already shown, the cumulative distribution for $a_{\max}$ is of the Fréchet type:

$$P_{a_{\max}}(a) = \exp\left[-\left(a/u\right)^{-N}\right] \quad (13)$$

and by substituting the latter relation into eq. (12), we obtain:

$$F_\sigma(\sigma) = 1 - \exp\left[-\left(\sigma/\sigma_0\right)^{2N}\right] \quad (14)$$

where $\sigma_0 = u/k^2$. Thus, the distribution of strength σ is a Weibull one, and the Weibull modulus is in relation with the exponent N of the flaw distribution: $m = 2N$ (Freudenthal, 1968).

So far, Bažant is right when he claims that the demonstration in Carpinteri (1994b) is not rigorous, since the passage from $p_a(a)$ to $p_{a\max}(a)$ (passage from eq. (34) to eq. (35)) is formally not correct, but he is wrong when he pretends that this imprecision invalidates the conclusions about the small-size asymptotic scaling law and particularly about the exponent –1/2. Bažant, by observing that $p_{a\max}(a)$ cannot be a power-law (Pareto) distribution, concludes that the initial hypothesis ($a_{\max}/b$) = const, which is the cause for the slope –1/2, is unjustified.

On the contrary, it could be shown that this hypothesis is true in a mean sense (provided that the probability density function $p_a(a)$ of the ensemble of all defects is Pareto, see eq. (4)) and this conclusion may be shown to be valid in a strictly and rigorous way, by recurring again to *Extreme Value Theory*, more precisely to the analysis of exceedances over thresholds. Let us consider the distribution of the stochastic variable X conditionally, on exceeding some high threshold z, so that we may define another random variable $Y = X - z$:

$$P_z(y) = \Pr\{Y \le y \mid Y > 0\} = \frac{P(z+y) - P(z)}{1 - P(z)} \quad (15)$$

In the limit for $z \rightarrow z^* = \sup\{x: P(x) < 1\}$, the intensity $P_z(y)$ may converge to a limit:

$$P_z(y) = G(y, \sigma, \varepsilon) \quad (16)$$

where G is the *Generalized Pareto Distribution* (GPD):

$$G(y) = 1 - \left(1 + \xi\frac{y}{\sigma}\right)^{-1/\xi} \quad (17)$$

with $1+\xi y/\sigma > 0$. The GPD is closely related to the GEV distribution (see eq. (8)), as shown by Pickands (1975). He demonstrated that, for any given cumulative distribution function P, a GPD approximation exists if and only if the limit of eq. (8) exists. In that case, if H is written in the GEV form of eq. (8), then the shape parameter ξ is the same as the corresponding parameter in eq. (17).

The generalized Pareto distribution is often used to model the tail of any given distribution. It has three basic forms (corresponding to the three forms of the GEV distribution). Distributions whose tails decrease as a polynomial, such as the Pareto, lead to a positive shape parameter, $\xi>0$. Distributions whose tails are finite, such as the beta, lead to a negative shape parameter $\xi<0$, and distributions whose tails decrease exponentially, such as the normal, lead to a generalized Pareto shape parameter $\xi \rightarrow 0$.

Thus, the Pareto distribution assumed for the defect size, described by eq. (9), is a particular case of the GPD, and obviously the limit for its exceedances is given by a distribution of the same form. To prove this, consider the conditional distribution $P_z(y)$ (see eq. (16)) for some ky, with $k>0$ constant, and with P defined by eq. (9):

$$P_z(ky) = \frac{Cz^{-N} - C(z+ky)^{-N}}{Cz^{-N}} \quad (18)$$

Now, if we assume $k = k^* z$, the result is straightforward:

$$P_z(ky) = 1 - (1 + k^* y)^{-N} \quad (19)$$

Thus, let us now reason in terms of the exceedance distributions, rather than in terms of the probability density functions. As a first step in our reasoning, let us consider that, in a body of characteristic linear size b, a threshold $\bar{a}$ can be defined, such that, on average, one defect only (i.e. the largest) exceeds it. Let the material be uniform, so that we may define ρ as the mean (volumetric) density of defects. With this notation we obtain:

$$\Pr\{a \geq \bar{a}\}\rho b^3 \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi = 1 \tag{20}$$

ϕ and θ being the longitude and the latitude of the defect orientation. The factor 1/4π pertains to all imperfections, since all orientations are equally probable. As already stated, one defect only is expected to exceed the threshold $\bar{a}$ in a body of linear size b; its dimension, however, is still random, since we only know that this defect (which is obviously the largest defect in the body) is larger than the threshold: $a_{\max} \geq \bar{a}$.

If now a geometrical similar body of characteristic size kb is considered, we might want to evaluate the number of defects exceeding the size $k\bar{a}$; then we can impose the condition that this number is equal to one: one defect only exceeds the threshold $k\bar{a}$:

$$\Pr\{a \geq k\bar{a}\}\rho (kb)^3 \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi = 1 \tag{21}$$

Equating eqs. (20) and (21), the following relation is obtained:

$$\Pr\{a \geq \bar{a}\} = \Pr\{a \geq k\bar{a}\}k^3 \tag{22}$$

It is easy to see that the cumulative probability $P_a(a)$ of eq. (9) satisfies this condition, if N = 3. More generally, the conditional probability for a exceeding some high threshold $\bar{a}$ is of the form of eq. (19) with exponent N = 3. Let us observe how, in this case, the distribution of self-similiarity has been obtained not from the hypothesis ($a_{\max}/b$) = const, but from the equality of the number of defects exceeding a given size $\bar{a}$, so that $\bar{a}$ /b is constant. Note also the difference with the treatment in Carpinteri (1994b), where the distribution of self-similarity is characterized by the exponent N = 2.

Now, in order to conclude the demonstration that the slope –1/2 still holds, we need to evaluate how these defects behave. It is easy to show that they have a mean value proportional to the threshold $\bar{a}$:

$$E\left(a \middle| a \geq \bar{a}\right) \propto \frac{N}{N-1}\bar{a} \tag{23}$$

This is, on average, the value of the maximum defect size $a_{\max}$; therefore, we can conclude that ($a_{\max}/b$)= const, but on average only (otherwise, the probability density of $a_{\max}$ should be Pareto and this is not possible, since it must be Fréchet, as previously shown).

Summarizing, the statement by Bažant & Yavari (2005) that the hypothesis ($a_{\max}$/b) = const is not justified in the framework of EVT, so that the slope –1/2 for the size-scale asymptote is unproven, misses the fundamental nature of the extremal behaviour of the defect size a if a fractal distribution is considered.

3.3 *Monte Carlo numerical simulations*

The result proved in the previous section may be checked numerically by performing Monte Carlo numerical simulations. Let us consider a homogeneous material, characterized by a constant volumetric defect density ρ, whose defects are distributed according to the self-similarity distribution of eq. (4) with N = 3. For any given scale b, the mean number of defects inside a cube of side b is equal to $n = \rho b^3$. By generating virtual samples with n defects sampled from the self-similarity distribution, we may evaluate how the size of the maximum defect $a_{\max}$ behaves. For the numerical experiments, the chosen scale range is 1:256 (the largest scale corresponds to 2^{24} defects); for each size, 1000 specimens are generated.

The mean value of $a_{\max}$ is plotted against the size b in Figure 1 in the case with N=3; a_0 and b_0 are reference quantities for normalization. As could be seen in the bi-logarithmic plot, a power-law emerges with slope approximately equal to 1. This result supports the conclusion of the previous section, i.e. ($a_{\max}/b$) = const, so that the small-size asymptote of the MFSL is characterized by the LEFM exponent –1/2.

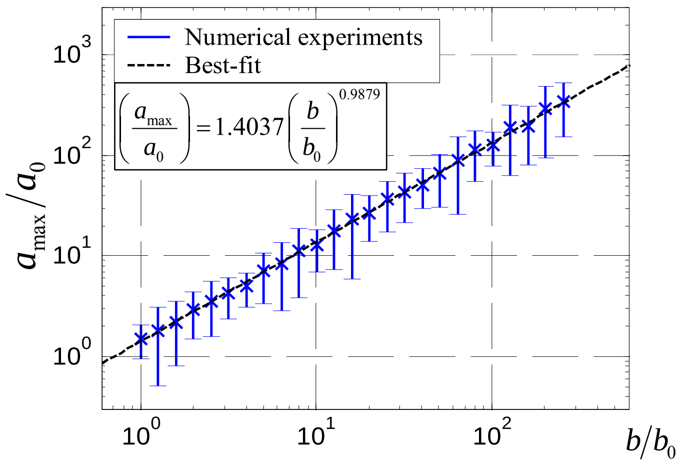

Figure 1. Numerical assessment of the power-law relationship between $a_{\max}$ and b in the case of N=3. The slope very close to 1 of the best-fitting power law confirms that, on average, ($a_{\max}/b$) = const.

Accordingly, the critiques in the Appendix of the paper by Bažant & Yavari (2005) are rejected. It is not true that “the maximum size of defects is simply assumed to scale up with the body size b”; as shown before, this assumption is well justified both theoretically and numerically. Even the critique about the fact that in Carpinteri (1994b) “the maximum defect size a_{max} is treated as nonrandom when the scaling is considered, although in reality it should more properly be considered as a randomly distributed" can be rejected. Indeed a_{max} is random, but on average it may be assumed to scale up with the body size, as already pinpointed.

3.4 *Further minor remarks*

The above results give a strong and, in our opinion, definitive support to the fractal-statistical explanation of the left-hand asymptote slope of the MFSL. Nevertheless, some further minor remarks have to be added. The first remark concerns the scaling of a_{max} when $N \neq 3$. In this case, as stated by Carpinteri (1994b), the scaling of the maximum defect size for geometrical similar bodies in a scale ratio k>1 may be written as:

$$a_{max}\left(kb\right) = k^{\beta} a_{max}\left(b\right) \tag{24}$$

The value of β, however, is not β=3/(N+1), but it must be β=3/N, as could be easily shown. If we consider that $a_{max} \propto \bar{a}$ (see eq. (23)), and introduce the scaling law of eq. (24) into eq. (22) we obtain:

$$\left[a_{max}(b)\right]^{-N} = \left[k^{\beta} a_{max}(b)\right]^{-N} k^{3} \tag{25}$$

from which follows that:

$$\beta = 3/N \tag{26}$$

Once again, this result has been confirmed by Monte Carlo numerical simulations, as could be seen in Figure 2. The numerical results for β are clearly in good agreement with the theoretical values given by Eq. (26).

A second remark concerns the fact that the self-similarity distribution (with N=3) corresponds to the maximum disorder, as stated by Carpinteri (1994b). Thus, the slope –1/2 of the left-hand asymptote is a theoretical upper bound, provided by three concurrent conditions (Carpinteri, 1994b):

– linear elastic material;
– Griffith cracks (singularity of order –1/2);
– maximum disorder (N=3).

To show this, the same demonstration already presented in Carpinteri (1994b) may be followed. Rewriting Eq. (24) with the correct value of β provides:

$$a_{max}(kb) = k^{3/N} a_{max}(b) \tag{27}$$

whereas the characteristic size of the body increases linearly with k:

$$kb = k\frac{a_{max}(b)}{\xi} \tag{28}$$

The maximum defect is larger than the body itself, which is clearly meaningless, for $a_{max}(kb) \geq kb$, from which, considering Eqs. (27) and (28), follows:

$$k \geq \xi^{\frac{3-N}{N}} > 1 \tag{29}$$

It is possible therefore to conclude that N=3 (self-similarity distribution) is the minimum value, corresponding to the maximum disorder. Once again, the conclusions contained in Carpinteri (1994b) are preserved, with a small correction: N=3 (and not N=2) corresponds to the self-similarity distribution.

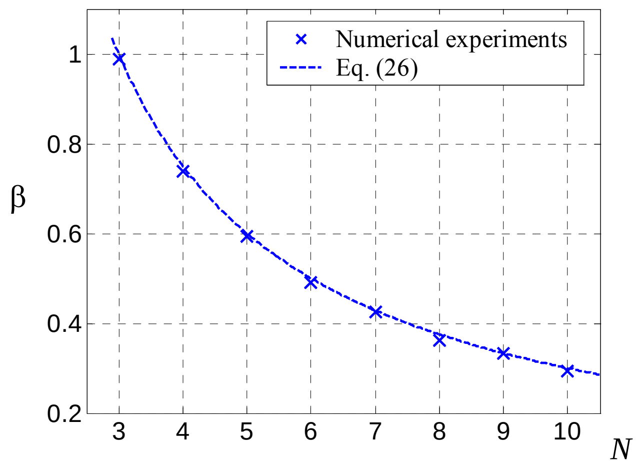

Figure 2. Scaling exponent β as a function of N. Results from Monte Carlo numerical simulations confirming Eq. (26)

4 SIZE EFFECTS AND THE DESIGN CODES

Until the end of the 1980s, no size effect was taken into account in concrete design codes. A possible reason for this was probably, as quoted by Bažant et al. (2004), that "whenever a size effect was detected in tests, it was automatically assumed to be statistical, and thus its study was supposed to belong to statisticians".

More recently, however, Bažant (2002) attributed the lack of size-effect specifications in the design codes to "the variety of formulae and the underlying (or absent) concepts". Even more explicit are Saouma & Fava (2006), who state that "the duality of contradictory models [SEL and MFLS] is becoming an impediment to modernization of the ACI code". We cannot agree with such opinions. The large variety of formulae and the underlying concepts simply show that this field is far from having come to a complete definition of the physical phenomena which subtend all the size-scale effects.

In addition, we must mention that code-making committees would probably never include too complicate formulae or theories, which pretend to be universal, into code of practice formulae. On the contrary, they are usually looking for simple effective formulae with a clear and limited field of use, most often restricted to a single application or geometry. That is, in our opinion, the reason why the MFSL, which is a two-parameter formula, has been adopted for the shear strength in reinforced concrete by the CEB (CEB-FIB Model Code, 1991), whereas the "Universal SEL", which is a six-parameter formula (see Eq. (46) in Bažant, 1997a), has not.

5 CONCLUSIONS

In this paper we reconsidered the fractal-statistical treatment by Carpinteri (1994b), which has been criticized by Bažant in several papers. As shown, this treatment should be considered valid: the exponent –1/2 attributed to the left-hand asymptote of the MFSL is definitely not an assumption, or "solely a consequence of a peculiar situation called 'the extreme disorder'". Rather, it is the direct consequence of LEFM when a fractal distribution of defects (with power-law tail) describes the flaw distribution inside the tested material. In addition, this result is confirmed by the Fractal Cohesive Crack Model and strengthened by Monte Carlo numerical simulations.

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