Use of tension softening diagrams for predicting the post-cracking behaviour of steel fibre reinforced concrete panels

A. Nour & B. Massicotte
École Polytechnique de Montréal, Montréal, Québec, Canada.

ABSTRACT: In this paper, a semi-analytical model aimed to investigate the post-cracking behaviour of FRC panels is presented. The model is based on force equilibrium in the critical cracked section and uses an arbitrary tension softening diagram as input. An additional relation that links the crack length parameter to the deflection needs to be determined. Due to the random fibre distribution and others uncertainties involved in concrete mix, deterministic approaches are not suitable for deriving the missing relationship for panels. The load deflection response is predicted using yield line theory based on the crack length parameter normalized-deflection diagram developed from the analysis of beams having the same thickness using Monte Carlo simulation (MCS) technique. The proposed approach is applied to a square panel supported on four corners with a comparison to an existing theory based on predefined parabolic softening law. A good agreement has been found between the model predictions and finite element calculations.

1 INTRODUCTION

The use of fibre reinforced concrete (FRC) as construction material has continuously grown over the last decades due to economical advantages in labour and material costs as compared to conventional reinforced concrete but also due to the enhanced mechanical characteristics of concrete. Randomly distributed fibres in concrete contribute to enhance concrete mechanical performance and durability when well proportioned and good quality FRC mixes used. Fibres significantly prevent shrinkage cracking, reduce brittleness due to impact loading, compression or in unreinforced members subjected to shear or tensile forces, control flexural cracking, improve water tightness, etc. For these reasons, FRC has been successfully used in shotcrete, precast concrete, elevated slabs, bridge decks, pavements, industrial floors, seismic resisting structures, repair, etc.

The post-cracking behaviour of FRC has been experimentally investigated based on a variety of tests. Although the actual post-cracking response of FRC is better defined by the stress-crack width $\sigma(w)$ relationship measured in a direct tension test, the vast majority of reported test results in the literature involve beams due to the ease of performing bending tests. Flexural tests have however two main disadvantages: the $\sigma(w)$ relationship cannot be obtained directly whereas tests on small FRC elements engender important scatters in results that are not representative of actual in situ conditions. The later has encouraged many investigators to propose tests on square or round panels of large dimension. Despite the better representation of concrete volumes in panel tests, the actual post-cracking behaviour of FRC in tension cannot be determined satisfactory using analytical approaches by any of the proposed tests in literature unless the softening mechanism is well understood, and the level of deformation and strain softening characteristics of the material are known.

Contributions in literature dealing with the post-cracking response determination of FRC panels using simple approaches are very limited. Marti et al. (1999) developed a simple theoretical approach that accounts for the random fibre distribution for the analysis of slabs. Their proposed model is based a priori on a predefined parabolic softening relationship which simplifies considerably the derivation of load-deflection curves from yield line theory. Tran et al. (2005) proposed an interesting formulation for determining the nonlinear load-deflection response of the ASTM C-1550 round panel. Their formulation uses yield line theory based on the flexural capacity of beams of similar composition and thickness.

For the case of FRC beams, Zhang and Stang (1998) proposed an analytical formulation which provides with satisfying accuracy the load-displacement response for an arbitrary inputted tension softening diagram. They use an additional relationship derived from fracture mechanics that links the crack mouth opening displacement (CMOD) to the external moment and a crack length parameter α (Figure 1). To the authors' knowledge, similar analytical formulation is either scarce or does not exist for the case of rectangular of round FRC panels, and such additional relationship is very difficult to derive analytically.

This study is therefore aimed to propose a simple analytical formulation to investigate the post-cracking behaviour of FRC panels of rectangular or circular geometry using an arbitrary tension softening diagram as input. The formulation assumes a symmetrical or axi-symmetrical crack pattern. In this situation, a relation that links the crack length parameter α to the deflection is missing. Due to the stochastic nature of material properties, the random fibre distribution, and others uncertainties involved in concrete mixes, deterministic approaches are not suitable for deriving that additional relationship for panels. The resort to probabilistic techniques enables modeling uncertainties and analyzing their dispersion effect. In this frame, the load deflection response is predicted using yield line theory based on the crack length parameter-normalized deflection diagram developed from the analysis of beams having the same thickness using Monte Carlo simulation (MCS) technique.

The proposed approach is applied in this paper to a square panel on four corners with a comparison to Marti's model. The model performance is also checked using EPM3D constitutive model (Massicotte et al, 2007) merged at Gauss integration point in FE computer program ABAQUS (2004).

2 MODEL DERIVATION FOR FLEXURAL ANALYSIS OF FRC PANELS

There is limited published information on analytical techniques suitable for predicting the complete flexural history of panels in bending made of strain softening FRC. Classical yield line theory (Johansen, 1972) can only predict the maximum load because it assumes that the level of resistance offered by the chosen collapse mechanism remains constant over the range of deformations associated with the introduction of load. If the resistance of a component within an assumed mechanism changes as load is introduced, the work done in resisting the external load is altered and the overall capacity changes. The magnitude of the change in load capacity can be determined only if the level of deformation and strain softening characteristics of the material are known.

The model adopted in the present study is based on the yield approach. In the initial stage the behaviour is assumed elastic until the maximum stresses reach concrete tensile strength. Beyond that point the model assumes that several fictitious cracks can develop depending on the failure pattern of the analyzed panel. Figure 1 depicts a typical stress distribution at a cracked section. The fictitious crack develops when the tensile stress reaches its ultimate value f_t and spreads along a part of the panel thickness H. After the crack has initiated, the fictitious crack progresses and the material is softened by cohesive forces in the fracture process zone where a nonlinear $\sigma(w)$ relationship is used.

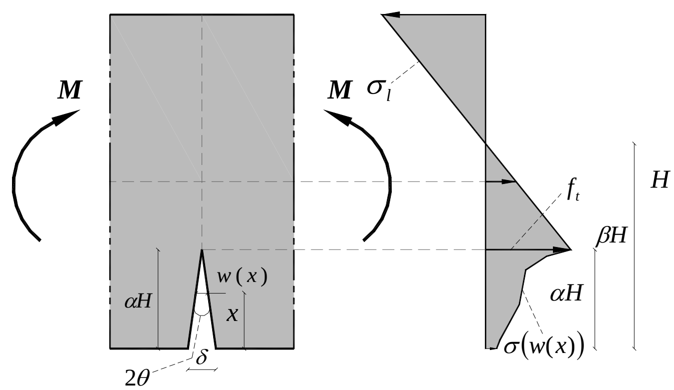


Figure 1. Stress distribution at cracked panel section.

When the crack opening displacement (COD) w reaches a critical value W_C, the stress transfer becomes zero and real crack starts to grow freely. The same cracked section proposed by Zhang and Stang (1998) for beams is adopted in the present study where it is assumed that the crack has a linear profile:

$$w = \delta\left(1 - \frac{x}{\alpha H}\right) \qquad (1)$$

for which:

$$\delta = \frac{\sum_{i=1}^{n_c} \delta_i}{n_c} \qquad (2)$$

δ is the mean CMOD, n_c is the number of cracks depending on the chosen failure pattern. Therefore, w is the mean COD at location x, and αH is the mean crack length for which the crack length parameter $\alpha \in [0, 1]$. With these assumptions, $\sigma(w)$ represents the mean softening diagram of the panel. From equilibrium conditions, we have:

$$\int_0^{\alpha H} \sigma(w(x))dx + \int_{\alpha H}^{H} \sigma_l(x)dx = 0 \quad (3.a)$$

$$\int_0^{\alpha H} \sigma(w(x))(H-x)dx + \int_{\alpha H}^{H} \sigma_l(x)(H-x)dx = M \quad (3.b)$$

M is the resisting moment per unit length along the yield line; $\sigma(w(x))$ and $\sigma_l(x)$ are the normal stress functions in the cracked (nonlinear) and the uncracked (linear) parts, respectively. $\sigma(w(x))$ is related to αH and δ using the σ-w relationship together with Eq. 1. $\sigma_l(x)$ can be related to αH, βH and δ for which βH stands for the total depth of the tensile zone with $\beta \in [0.5, 1]$.

According to the principle of virtual work, one can derive a relationship between the applied load F and the generated bending moment M as follow (Johansen, 1972):

$$\sum F \cdot \Delta = \sum M \cdot \theta \cdot L \quad (4)$$

Here Δ stands for the deflection, L is a characteristic length of the panel and θ is the corresponding crack angle of rotation (in radian) between the adjoining uncracked parts of the panel (Figure 1). By modeling the crack as a generalized plastic hinge, θ can be estimated by:

$$\theta = \frac{\delta}{\alpha \cdot H} \quad (5)$$

This approach necessitates an additional relation linking the crack length parameter to the deflection to obtain the complete solution between the external load and the deflection. In the case of beams under three point bending, Zhang and Stang (1998) used for the required additional equation a relationship derived analytically from fracture mechanics in which δ is linked to the external moment M and the crack length parameter α. The additional relationship would be difficult to derive analytically in the case of panels for which an arbitrary σ-w softening law is used.

Marti et al. (1999) developed a theoretical approach based a priori on a well defined parabolic softening law. Such assumption simplifies considerably the derivation of load-deflection curves from the yield line theory. Tran et al. (2005) performed a series of analysis using yield line theory to derive the nonlinear load-deflection response of the ASTM C-1550 round panels based on the flexural capacity of beams of similar composition and thickness. They did not explicitly use a σ-w softening relationship as an input in their approach. Instead they adopted the moment crack rotation angle diagrams (M-θ) of beams as input. According to Eq. 4, the post-cracking load-deflection curve can be determined by increasing the displacement at the centre of the panel and, using the resisting moment offered by beams for each corresponding crack rotation angle, find the load at equilibrium for these moments.

This study proposes a simple analytical formulation to investigate the post-cracking behaviour of FRC panels using an arbitrary σ-w diagram as input. The post-cracking behaviour of FRC panels is studied using yield line theory, where the load deflection response is predicted based on the crack length parameter-normalized deflection relationship developed from the analysis of beams having the same thickness using Monte Carlo simulation (MCS) technique. This issue is described in detail in the next section.

3 CRACK LENGTH PARAMETER-NORMALIZED DEFLECTION CURVE FOR BEAMS USING MCS TECHNIQUE

3.1 *Procedure*

To derive the additional relation relating the crack length parameter α to the deflection Δ, one follows the idea proposed by Tran et al. (2005) where the missing information is obtained from beams which is then used for analyzing panels. Because in our study any arbitrary σ-w relationship can be inputted in the numerical analysis, the α-Δ diagrams are determined from the analysis of beams. However, since panel maximum deflections are larger than beams, the crack length parameter α is obtained as a function of the normalized displacement $\bar{\Delta}$. Here, $\bar{\Delta} = \Delta / \Delta_{\max}$ where $\Delta_{\max}$ is the maximum analytical beam deflection. In the beam analysis, the following σ-w relationship is chosen for its generality and versatility:

$$\sigma(w) = f_t \cdot \left(1 - \frac{w}{W_C}\right)^N \quad (6)$$

N is the softening index. For a given value of N ($0 < N < \infty$), equation (6) covers all types of engineering materials. For instance, with $N = 0$, $\sigma(w) = f_t$ which describes the behaviour of all elastic perfectly plastic solids as it is used in Dugdale model. With $N = \infty$, $\sigma(w) = 0$, in which case Eq.(6) represents brittle materials without softening region. Expressions obtained for $0 < N < 1$, are typical of ductile metals and polymers in plane stress conditions characterized by strain hardening behaviour (Roger et al., 1986). For the materials like steel fibre reinforced concretes (SFRC), the softening index range is $1 < N < \infty$ (Ballarini and Shah, 1984).

Due to the stochastic nature of the material properties, the random fibre distribution, and others uncertainties involved in concrete mix, deterministic approaches are not suitable for deriving the α-$\overline{\Delta}$ formula for panels. The resort to probabilistic techniques enables modeling uncertainties and analyzing their dispersion effect. For this raison, a stochastic model that accounts for the randomness of the three variables (f_t, W_C and N) defining the adopted σ-w law is used. In this study, α-$\overline{\Delta}$ diagrams for panels are obtained using the analytic formulation proposed by Zhang and Stang (1998) for beams combined with the MCS technique.

3.2 *Stochastic model for σ-w diagrams*

Tensile strength f_t, critical crack opening displacement W_C and softening index N in Eq. 6 are modeled herein as random fields. For the random simulation of these properties, the chosen random variables are defined by their moments of order 1 and 2, which are respectively the mean, and the variance supposed in accordance with laboratory samples. Let T_p standing for one typical random variable, defined as a function of the deterministic function T_{0p} describing the trend, taken in practice as the mean of measured values, and also function of zero mean, unit variance Gaussian random field ΔT_p. One can write:

$$T_p = \Re\left[T_{0p} + \tau_p \ \Delta T_p\right] \tag{7}$$

$\Re$ is a transformation taking the Gaussian process ΔT_p, into the distribution appropriate for T_p and τ_p is the standard deviation. Here $p = 1$ corresponds to softening index, $p = 2$ for tensile stress, and $p = 3$ for critical crack opening displacement. The zero mean, unit variance, three-variate Gaussian random field ΔT_p, can be simulated as follow:

$$\Delta T_p = \sqrt{\frac{2}{K_p}} \cdot \sum_{l=1}^{K_P} \cos(\Omega_{l,p}) \tag{8}$$

$$mean\ (\Delta T_p) = 0; \quad p = 1,3 \tag{9}$$

Ω_l is a random phase angle distributed uniformly over the interval $[0, 2\pi]$. K is a large enough integer. A stochastic independence between f_t and the other random variables W_C and N is assumed which is preferable than assuming any erroneous correlation. In practice, there is a direct correlation between N and W_C. Small values of N are typical of high fibre dosage leading to large values of W_C, whereas large values of N are more representative of materials with small fibre dosage and therefore more brittle material, leading to small values of W_C. For this reason, the stochastic correlation is considered between N and W_C. The three properties are simulated using the stochastic formulation proposed by Nour et al. (2002). The tensile softening index is simulated using the beta distribution whereas the tensile stress and the crack opening displacement are modeled using the lognormal distribution.

3.3 *Numerical analysis for beams*

The above described procedure is used to derive the crack length parameter-normalized deflection α-$\overline{\Delta}$ for beams to be used for panel analysis. Monte Carlo simulations are used to generate samples having characteristics close to specimens produced in laboratory. It is reported in literature (MacGregor et al., 1983) that the variability of f_t is roughly of the same order as for the compression strength, which is around $CV_{f_t} \approx 0.15$ to 0.25. However Bungey and Millard (1996) indicated that CV_{f_t} could be greater than 0.4 for poor concrete and for this reason CV_{f_t} is varied in this study from 0.15 up to 0.5. For the critical crack opening displacement W_C, Bazant et al. (2002) reported that the ratio between the areas under the complete σ-w curve and under the initial tangent of this curve is in the order of 2.5 with a variation coefficient around 40%. Because the complete area of the σ-w curve is controlled by W_C, a variability up to 40% for W_C seems reasonable. For the softening index N, there is practically no available information about its variability. The variation coefficient CV_N is chosen to be equal to $0.5 \cdot CV_{N,cr}$ with $CV_{N,cr} = N_0 / \tau_{N,cr}$ (Nour et al., 2002). $CV_{N,cr}$ stands for the critical variation coefficient for N, N_0 for the mean value for N, and $\tau_{N,cr}$ for the critical standard deviation for N. Hence, the following data are used:

- Mean tensile stress: μ_{f_t} = 3.5 MPa;
- Mean crack opening displacement μ_{W_C} = 8 mm;
- $N_{\min} = 1$ and $N_{\max} = 7$ with the mean value $N_0 \in [\,N_{\min}\ \ N_{\max}\,]$.
- The ratio H/S = 4 (H stands for the beam height and S for the beam clean span).

The softening index N is in direct correlation with the parameters influencing the shape of the post-cracking load-deflection response such as the type of fibres, the dosage and the mix quality. For this reason, three representative situations were considered. The first one represents specimens dominated by FRC mixes having high percentage of fibres, idealized here by samples of Monte Carlo simulations having N_0 close to $N_{\min}$ i.e. $N_0 = (3N_{\min} + N_{\max})/4 = 2.5$. The second situation considers specimens covering all possible percentage of fibres dosages, idealized here by samples of Monte Carlo simulations having N_0 equal to the central value i.e. $N_0 = (N_{\min} + N_{\max})/2 = 4$. Finally the third situation is the opposite of the first one with the majority of specimens made with low percentage of fibres, idealized here by samples of Monte Carlo simulations having N_0 close to $N_{\max}$ i.e. $N_0 = (N_{\min} + 3N_{\max})/4 = 5.5$.

Using Eq. 6, 1000 independent realizations of σ-w diagrams were randomly generated and were directly considered in Monte Carlo simulations of beam analysis. The result of this exercise is the required mean α-$\overline{\Delta}$ diagram to be used for panels. Figure 2 illustrates 25 typical realizations of σ-w diagrams for the case of $N_0 = 4$ for which a strong negative correlation between N and W_C was considered, i.e. $R_{W_C N} = -0.75$.

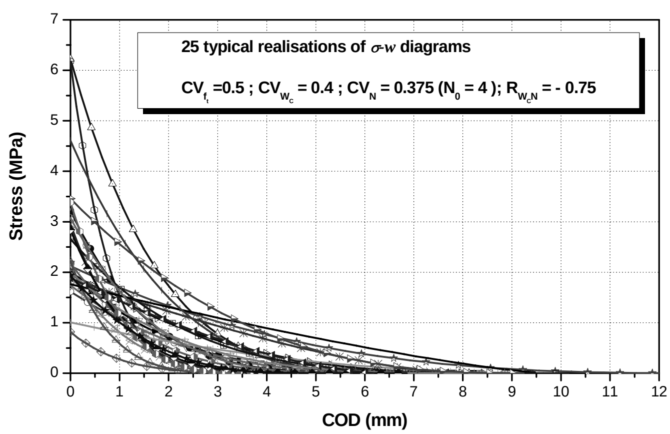


Figure 2. Typical realizations of σ-w diagrams.

In this study, for a given beam thickness H, the required α-$\overline{\Delta}$ diagrams are determined after achieving 1000 Monte Carlo simulations for each representative situation of N_0 (N_0= 2.5, 4 and 5.5). This allows covering a maximum range of fibre dosage. In reality 3000 samples of Monte Carlo simulations are superimposed all together in Figure 3. As shown on this figure, the randomness in f_t, W_C and N produces a scatter in α-$\overline{\Delta}$ results with an interesting trend. This permits to easily fit the obtained results with an appropriate function. The function given by Eq. 10 is chosen because it captures with fidelity the full trend observed in MCS results. Constants c_1, c_2 and c_3 are estimated using a nonlinear fitting scheme from the ensemble of realizations, whereas the assumed function satisfactory passed the Chi-square goodness of fit test.

$$\alpha = \frac{c_1 \cdot \overline{\Delta}^{\,c_2}}{1 + c_3 \cdot \overline{\Delta}^{\,c_2}} \tag{10}$$

Results reported in Figure 3 corresponds to H = 75 mm. The same routine could be easily repeated for other thicknesses.

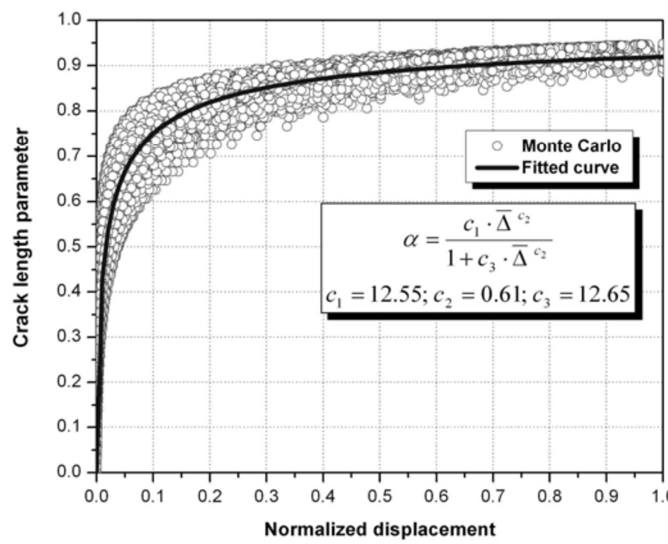


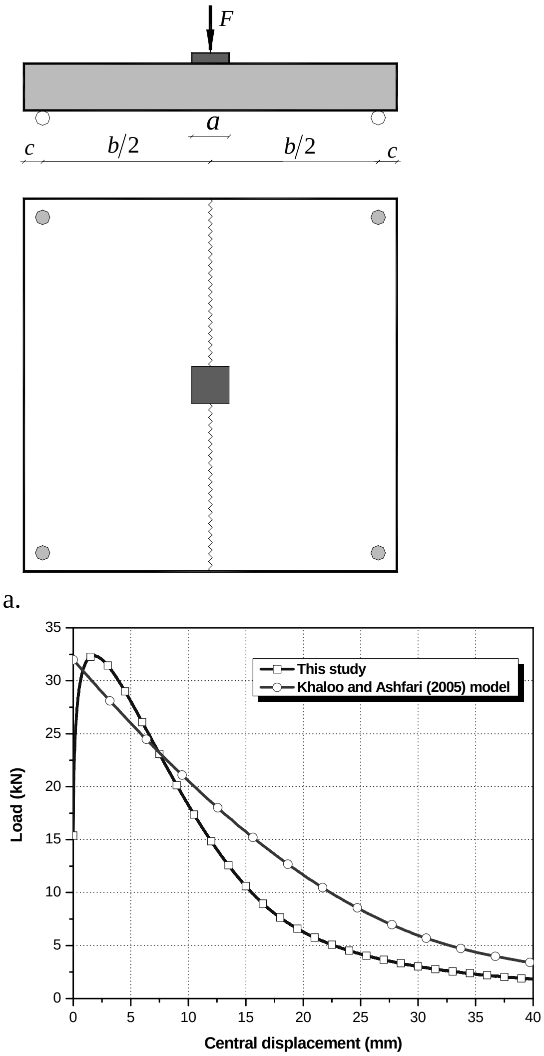
Figure 3. Crack length parameter-normalized displacement diagram for h = 75 mm.

4 APPLICATION TO A SQUARE PANEL SUPPORTED ON FOUR CORNERS

Equation 10 constitutes the additional relation that links the crack length parameter α to the deflection Δ. It is of worth to note that very limited contributions dealing with the post-cracking response for FRC panels via simple approaches are available in literature. The model proposed by Marti et al. (1999) belongs to this category; it is based on a fixed parabolic softening law and is used herein for comparison purposes. The next validations uses the following data: H = 75 mm and f_t = 3 MPa.

4.1 *Comparison with Marti's model*

Khaloo and Afshari (2005) performed several tests to determine the flexural strength, load-deflection curve and energy absorption of small concrete slabs. Figure 4 shows the test principle and typical crack pattern for the adopted square panel supported on four corners under a central point loading. Using yield line theory, Khaloo and Afshari (2005) estimated the resisting moment M according to Marti et al. (1999) model, and developed the equations defining the crack angle rotation θ and the load F function of the central displacement Δ and the moment M, respectively:



a.

b.

Figure 4. Failure mechanism and load displacement curves for a square panel on four corners.

$$\theta = \frac{4\Delta}{b-a} \tag{11.a}$$

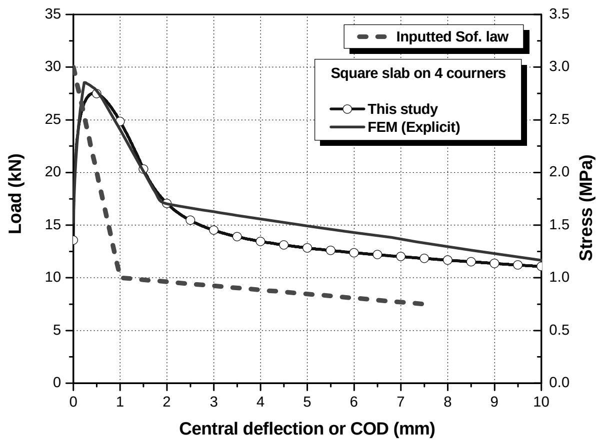
$$F = \frac{4(b+2c)}{b-a} M \tag{11.b}$$

For the numerical application, one considers b = 680 mm, a = 80 and c = 70 mm, N = 2 (parabolic softening law) and W_C = 15 mm. Figure 4.b illustrates the comparison between the proposed approach and the adopted Khaloo and Afshari (2005) theory. In the proposed approach, three regions describe the load–deflection curves. The first region has an ascending slope covers the response form the onset of concrete cracking up to the maximum load. Marti et al. (1999) neglected the contribution of the elastic part of the sound ligament in tension. Therefore their theory predicts only the behaviour of FRC after ultimate. The second region begins at ultimate load and ends at a point where the tensile forces are resisted totally by bond between fibres and concrete. This corresponds to the steepest softening portion of the curve. In the third region, the slope of the curve reduces accompanied by an asymptotic residual load. One sees that both approaches predict practically the same ultimate value, but they exhibit different behaviour in the softening region. The results indicate that the theoretical predictions from Khaloo and Afshari (2005) are not conservative and present higher energy absorption compared to the proposed approach. Their model predicts also higher energy absorption than obtained experimentally.

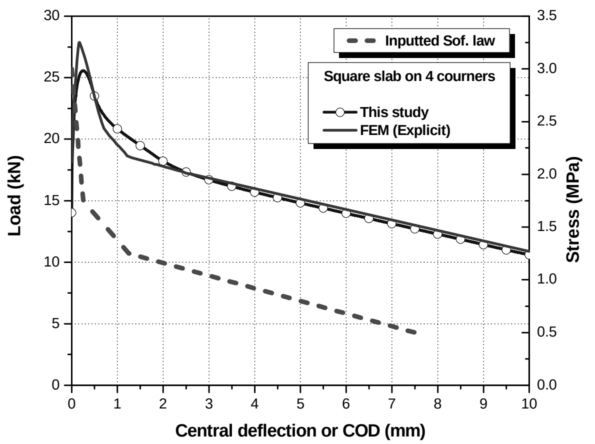
4.2 *Comparison with finite element method*

In this section, the proposed approach is compared to the finite element method. To this end, EPM3D concrete model is used for the analysis. This model was originally developed by Bouzaiene and Massicotte (1997) and was recently merged for standard and explicit computations in ABAQUS (2004) by Ben Ftima and Massicotte (2004). It was also used for modelling concrete structures reinforced with internal and external FRP (Nour et al., 2006). In compression, the concrete model follows a three-dimensional hypoelastic approach which accounts for anisotropy and inelastic volume expansion using a compression scalar damage parameter λ (Bouzaiene and Massicotte, 1997). Degradation of material properties due to cracks propagation is described by means of a scalar parameter which enables coupling the compressive and tension damage parameters for the tensile residual stress calculation. The model is based on smeared crack approach with cracks spread over the elements, so the analyses are performed without introducing to the finite element model any crack pattern.

Finite element analyses of FRC panels involve modeling severe nonlinearities. Strategy solution using standard computations through ABAQUS leads to serious converging difficulties resulting in a large number of iterations, which complicates exploring the post-cracking behaviour of FRC panels. In this case, the analyses are more efficient using explicit computations. Whereas ABAQUS standard must iterate to determine the solution to a nonlinear problem, ABAQUS explicit determines the solution by explicitly advancing the kinematic state from the previous increment. The explicit procedure does not require any iteration and no global tangent stiffness matrix. For these reasons, the explicit solution strategy is adopted in this study. However, a special caution is required for choosing the time duration of the analysis, because the explicit solution method is a truly dynamic procedure and the inertia forces should not in any case play a dominant role in the solution.



a.



b.

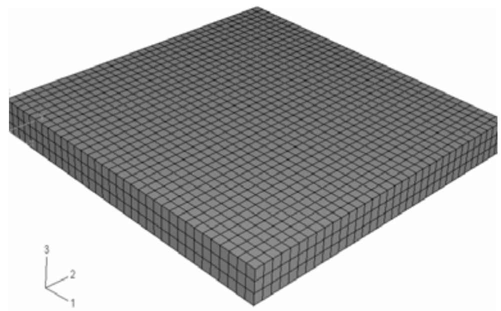
Figure 5. Load displacement curves for a square panel on four corners. a. bilinear softening diagram. b. tri-linear softening diagram.

Furthermore, the energy balance check is employed to evaluate whether or not a simulation is in accordance with a quasi-static response. For the numerical analysis, one considers $b = 680$ mm, $a = 0$ and $c = 70$ mm. The analyses are carried out considering the bilinear and the tri-linear softening laws shown in Figure 5. In EPM3D, any polylinear softening diagram defined by a series of points $(\sigma_i, w_i;$ $i = 1, \ldots, N_p)$ can be used for the analysis according to the following expression:

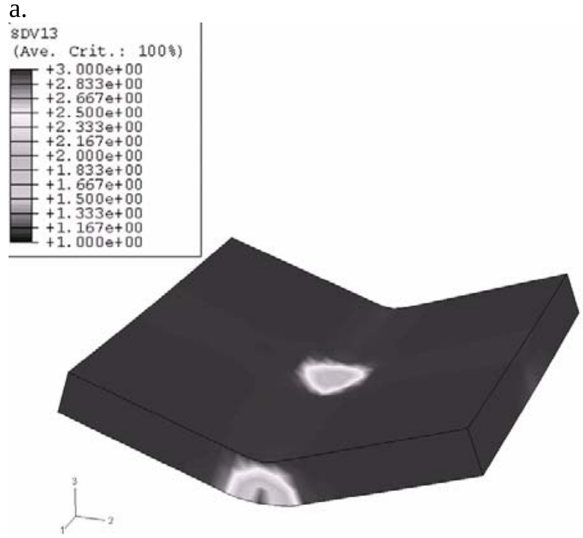
$$\sigma(w) = (1 - 1.25\lambda) \sum_{i=1}^{N_p - 1} \sigma_i + \frac{\sigma_{i+1} - \sigma_i}{w_{i+1} - w_i}(w - w_i) \qquad (12.a)$$

$$w = \frac{\varepsilon}{L_{cr}} \qquad (12.b)$$

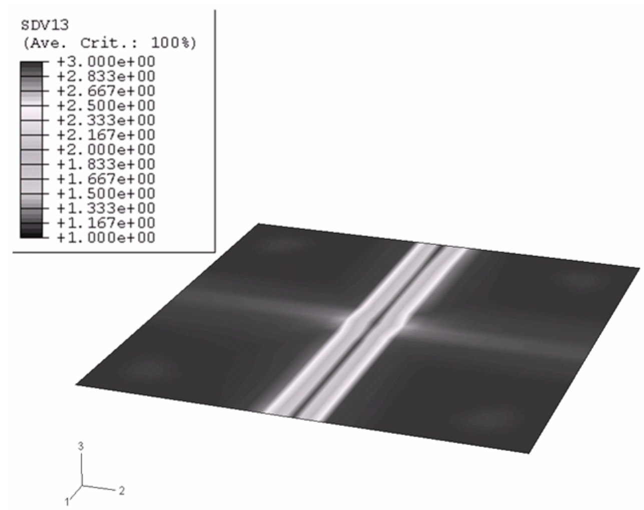
Here N_p is the number of points in the softening diagram, ε is the tensile strain and L_{cr} is the critical length transforming the crack opening in tensile strain, assumed in this study equal to 0.5*H*. The comparison results are clearly illustrated in Figure 5.



a.



b.



c.

Figure 6. Finite element model and residual tensile stress distribution. a. Finite element mesh. b. Predicted failure mode. c. Tensile stress distribution at the slab bottom surface.

If one considers finite element solution as reference, one sees that the proposed method captures with fidelity the global trend in the load-deflection curve. With the proposed approach, the predicted maximum load is roughly 3 % less than finite element result for the bilinear softening law and is 8.5 % less for the tri-linear law; also the observed difference in post-peak is acceptable for both softening laws. The finite element model is shown in Figure 6.a along with the predicted failure mode (Figure 6.b). As shown in Figures 6.b and 6.c, the failure pattern takes the form of a straight line and a flexural hinge divides the slab into two rigid parts. In term of failure mode, EPM3D predictions are in accordance with experimental observations of Khaloo and Afshari (2005). One sees that the tensile stress decreases significantly in the vicinity of the middle straight line of the slab as well as around the applied load.

Despite EPM3D is based on a smeared crack approach, it allows to determine the crack pattern for subsequent analysis of complicated slab geometries using yield line theory, and the complete load-deflection response can be easily obtained via the proposed approach.

5 CONCLUSIONS

In this paper, a semi-analytical model aimed to investigate the post-cracking behaviour of FRC panels has been presented. The load deflection response is predicted using yield line theory based on the crack length parameter-normalized deflection diagram developed from the analysis of beams having the same thickness using Monte Carlo simulation (MCS) technique. The proposed model has been applied to a square panel supported on four corners with a comparison to an existing theory based on predefined parabolic softening law. Also, a good agreement has been obtained between the model predictions and finite element calculations. This study indicates that it is possible to obtain satisfactory predictions of the post-cracking load-deflection response for panels with independently obtained experimental/analytical data for the stress-crack width relationship using this present simple model.

REFERENCES

ABAQUS. (2004). User’s Manual Version 6.5. Hibbit, Karlsson & Sorensen, Inc., USA.

Ballarini, R., Shah, S. P., and Keer L. M. (1984). Crack growth in cement-based composites. *Engineering Fracture Mechanics*, 20(3), 433-445.

Bazant, Z. P., Qiang, Y. U., and Goangseup, Z. (2002). Choice of standard fracture test for concrete and its statistical evaluation. *International journal of fracture*, 118, 303-337.

Ben Ftima, M., and Massicotte, B. (2004). Rapport interne sur l’introduction du model de béton a Abaqus.

Bouzaiene, A., and Massicotte, B. 1997. Hypoelastic tridimensional model for non proportional loading of plain concrete. Journal of Engineering Mechanics, ASCE, 123(11): 1111-1120.

Bungey, J. H., and Millard, S. G. (1996). *Testing of concrete structures*. Blackie Academic and Professional, Glasgow, UK.

Johansen, K. W. (1972). *Yield line theory*, Cement and Concrete Association, U.K.

Khaloo, A. R., and Afsari M. (2005). Flexural behaviour of small steel fibre reinforced slabs. *Cement and concrete composites*, 27, 141-149.

MacGregor, J. G., Mirza, S. A., and Ellingwood, B. (1983). Statistical analysis of resistance of reinforced and prestressed concrete members. *Journal of the American Concrete Institute*, 80(3), 167-176.

Marti, P., Pfyl, T., Sigrist, V., and Ulaga, T. (1999). Harmonized test procedures for steel fiber-reinforced concrete. *ACI Materials Journal* November–December, 676–685.

Massicotte, B., Nour, A., Ben Ftima, M. and Yildiz, E. (2007). EPM3D – A user-supplied constitutive model for the nonlinear finite element analysis of reinforced concrete structures. Report EPM/GCS-2007-01, École Polytechnique de Montréal.

Nour, A., Slimani, A., and Laouami, N. (2002) Foundation settlement statistics via finite element analysis. *Journal Computers and Geotechnics*, 29(8), 641-672.

Nour, A., Massicotte, B., Yildiz, E., Koval, K. (2006) Finite element modeling of concrete structures reinforced with internal and external FRP. *Accepted for publication in Canadian civil engineering journal*.

Roger M., Foote, L., Mai., Y. W., and Cotterell., B. (1986). Crack growth resistance in strain-softening materials. *Jour. Mech. Phys. Solids*, 34(6), 593-607.

Tran, V. N. G., Bernard, E. S., and Beasley, A. J. (2005). Constitutive modeling of fibre reinforced shortcrete panels. *Journal of Engineering Mechanics*, 131(5), 512-521.

Zhang, J., and Stang, H. (1998). Application of stress crack width relationship in predicting the flexural behaviour of fibre-reinforced concrete. *Journal Cement and Concrete Research*, 28(3), 439-452.