A general static approach to size-effect in cementitious materials

G. Rosati
Professor, Dept. of Structural Engineering, Politecnico di Milano, Milan, Italy

M.P. Natali Sora
MS, Structural Engineer, Milan, Italy

ABSTRACT: Size effect is the functional dependence of the ultimate stress of a given structure (“nominal strength”) on structure size. Since size effect has to do with the limited toughness of quasi-brittle materials, concrete structures exhibit a strong size effect, that is enhanced by notches (“notch sensitivity”). Size effect appears whenever the structural behaviour shifts from ductile to brittle. Within this context, a size-effect law based on Neuber’s theory for stress concentrations around a notch is developed in this paper in the case of cementitious materials. The nominal strength σ_N is formulated as a function of the mechanical and structural parameters, assuming that a dominant cohesive crack forms at failure and that this crack is similar to a pointed notch with a small plastic area at the tip. The proposed size-effect law is shown to fit satisfactorily many test results and to agree with the size-effect laws found in the literature.

1 INTRODUCTION

As it is well known, the strength of the material is the major parameter controlling the structural capacity. However, it is equally well known that large structures exhibit a relatively lower capacity than small structures, for the same material and type of geometry (i.e. assuming perfect geometrical similitude), as if the material had a smaller strength. As a matter of fact, the nominal strength of the material - usually measured on small specimens - is never reached in large structures at failure, in the case of brittle or quasi-brittle materials (like concrete, mortar and rock). This is “size effect”, i.e. the apparent size-dependence of the strength of the material on a characteristic dimension of the structure under examination.

Since size-effect has to do with the limited toughness of quasi-brittle materials, concrete structure are prone to size effect, that is enhanced by notches, and appears also whenever the material exhibits a transition from a ductile to a brittle behavior. As for the nominal strength, it can be worked out from the applied loads and from the geometry, or can be evaluated by means of the classical de Saint Venant’s solution (Bažant 1999).

To predict size effect, deterministic and probabilistic theories have been introduced in the last thirty years by several authors (i.e. Weibull, 1939; Bažant, 1984; Carpinteri, 1994). These authors explained size-effect with the presence of randomly-dispersed defects in the materials, with fracture cohesive behavior, and with the stress concentration around notches. In materials with cohesive behaviour the energy stored elastically in the volume is dissipated on the fracture surface causing size effect. In the last five years Bažant (2004, 2005) developed and discussed a “combination” of the deterministic-energetic size-effect with statistical-energetic size-effect, by introducing (Bažant, 2002) analytical simplifications for practical uses.

This paper develops a size-effect law starting from Neuber’s theory (1958), extended to cementitious materials. This approach considers the presence of a dominant defect at failure. It can be considered a static deterministic approach to size-effect law in tension, bending and shear. Comparisons with experimental and analytical results of other authors are presented as well.

2 MODELING

The generalized size-effect law obtained from the application of Neuber’s theory (Marazzini, 1999; Schumm, 1997; Rosati, 2001; Rosati and Natali Sora, 2001) on stress concentration around a dominant defect, notch or crack, has the formulation indicated by Eq. (1). This theory assumes that a dominant crack grows at failure. This crack is considered equivalent to a pointed notch with a small plastic area at the apex (radius ρ’). This law is valid for bending, tension and shear:

$$\sigma_N = \frac{\sigma_U}{1+\frac{c_1}{\sqrt{\rho'}}\sqrt{\frac{(h-t)t}{(h-t)+c_2{}^2 t}}} \qquad (1)$$

In Eq. (1), t is the dominant crack length, h is the main dimension of the element and σ_u is the strength of the material. The values of the parameters c_1 and c_2, that depend on notch or fracture geometry and on the generalized stress, are shown in Table 1.

Table 1. Parameters of the size-effect law obtained by applying Neuber's theory.

	Tension	Bending	Shear
Single notch	$c_1 = \frac{3}{\sqrt{2}}$	$c_1 = 2$	$c_1 = \frac{1}{\sqrt{3}}$
or crack	$c_2 = \frac{3(\pi^2-8)}{8\sqrt{2}(\pi-3)}$	$c_2 = \frac{3(\pi^2-8)}{2(\pi-2)}$	$c_2 = \frac{3\pi}{4}$
Double notch	$c_1 = \frac{3}{\sqrt{2}}$		
or crack	$c_2 = \frac{3\pi}{4\sqrt{2}}$		

Similar coefficients for bending and shear with double notch or cracks or holes can be obtained.

Different cases can be figured out. It is possible to introduce some analytical developments of the generalized expression (1), by considering different values of the dominant crack length t :

Formulation A) The ratio t/h is kept constant. In this case the length of the fracture at failure is proportional to the main dimension of the structure. Assuming the ratio $\xi = t/h$ = constant, the expression (2) is obtained:

$$\sigma_N = \frac{\sigma_U}{1+\frac{c_1}{\sqrt{\rho'}}\sqrt{h}\sqrt{\frac{(1-\xi)\xi}{(1-\xi)+c_2{}^2\xi}}} \qquad (2)$$

The asymptotic behavior of the expression (2) is reported in (2a) and (2b):

- $h \Rightarrow 0 \quad \sigma_N \Rightarrow \sigma_U$ (2a)
- $h \Rightarrow \infty \quad \sigma_N \Rightarrow 0 \quad$ with $\frac{1}{\sqrt{h}}$ (2b)

and this form of the size-effect law corresponds to the case of a very large structure failing under a very small load.

Formulation B) Assuming for the crack length t no dependence on h, and keeping t constant, the expression (3) is obtained.

This size-effect law is characterized by a residual strength (3b) tending to infinite:

$$\sigma_N = \frac{\sigma_U}{1+\frac{c_1}{\sqrt{\rho'}}\sqrt{\frac{(h-\bar{t})\bar{t}}{(h-\bar{t})+c_2{}^2\bar{t}}}} \qquad (3)$$

$$h \Rightarrow 0 \qquad \sigma_N = \sigma_U \qquad (3a)$$

$$h \Rightarrow \infty \qquad \sigma_N = \frac{\sigma_U}{1+c_1\sqrt{\frac{\bar{t}}{\rho'}}} \qquad (3b)$$

Formulation C) The dominant defect is a function of the main dimension h (for instance $t = t_{max}(1-c/(h+c))$, where t_{max} is the maximum crack length and c is an experimentally-determined parameter). Substituting this expression into Eq. (1), the expression (4) is obtained, and for h tending to infinite, the residual strength can be evaluated (4b):

$$\sigma_N = \frac{\sigma_U}{1+\frac{c_1\sqrt{t_{max}}}{\sqrt{\rho'}}\sqrt{\frac{\left(h-t_{max}\left(1-\frac{c}{h+c}\right)\right)\left(1-\frac{c}{h+c}\right)}{h-(1-c_2{}^2)t_{max}\left(1-\frac{c}{h+c}\right)}}} \qquad (4)$$

$$h \Rightarrow 0 \qquad \sigma_N = \sigma_U \qquad (4a)$$

$$h \Rightarrow \infty \qquad \sigma_N = \frac{\sigma_U}{1+c_1\sqrt{\frac{t_{max}}{\rho'}}} \qquad (4b)$$

3 ANALYTICAL AND EXPERIMENTAL COMPARISONS

In the last thirty years several expressions of the size-effect law were introduced for cementitious material structures. The main deterministic approaches will be compared with the present formulation.

3.1 *Linear-elastic fracture mechanics*

Considering the solution of a single notch subjected to bending (for example four point bending) the expression (2) provides the nominal strength as a function of h:

$$\sigma_N = \frac{\sigma_U}{1+\frac{c_1}{\sqrt{\rho'}}\sqrt{h}\sqrt{\frac{\left(1-\frac{t}{h}\right)\frac{t}{h}}{\left(1-\frac{t}{h}\right)+c_2{}^2\frac{t}{h}}}} \qquad (5)$$

and the following expression is obtained:

$$\sigma_N = \frac{\sigma_U}{1+\frac{1}{\sqrt{\rho'}}\sqrt{h}\sqrt{\frac{c_1^{\,2}\left(1-\frac{t}{h}\right)}{\left(\frac{h}{t}-1\right)+c_2^{\,2}}}} \tag{6}$$

for h $\Rightarrow \infty$ $$\sigma_N = \frac{\sigma_U}{\frac{1}{\sqrt{\rho'}}\sqrt{h}\cdot f_N\left(\frac{t}{h}\right)} \tag{7}$$

by putting:

$$f_N\left(\frac{t}{h}\right) = \sqrt{\frac{c_1^{\,2}\left(1-\frac{t}{h}\right)}{\left(\frac{h}{t}-1\right)+c_2^{\,2}}} \tag{8}$$

The approach of Linear-Elastic Fracture Mechanics gives the expression of the nominal strength in terms of stress intensity factor (K_I mode I) and ratio t/h:

$$\sigma_N = \frac{K_I}{\sqrt{h}\left(1-\frac{t}{h}\right)\cdot g\left(\frac{t}{h}\right)} \tag{9}$$

From the comparison of the two formulations, tending h to infinite expression (10) is obtained:

$$\sqrt{\rho'} = \frac{K_I}{\sigma_U} \tag{10}$$

with the condition

$$f_N\left(\frac{t}{h}\right) \cong \left(1-\frac{t}{h}\right)\cdot g\left(\frac{t}{h}\right) \tag{11}$$

Considering a coincidence in the expression (11), the functional dependence of the radius of the process zone on the critical stress intensity factor and on the strength of the material stands out clearly.

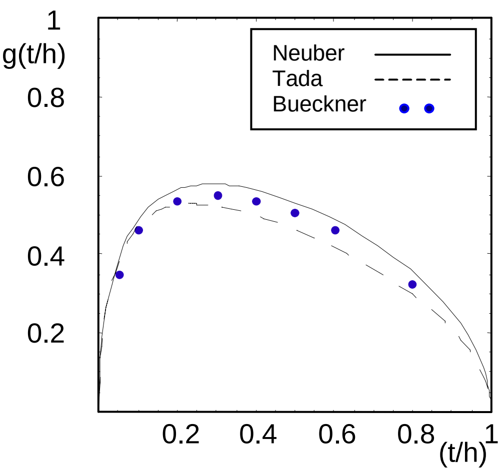

Figure 1. Stress intensity factors for notched beams (Neuber, Tada and Bueckner).

In Fig.1 the stress-intensity factor given by Eq. (11) is plotted, together with the small differences between Neuber's curve and the curves obtained by Bueckner (1960) and by Tada (1973).

3.2 *Bažant's Size-Effect Law (SEL)*

Twenty years ago, Bažant developed a size-effect law (SEL) on the basis of energy-related considerations. The SEL depends only on two empirical parameters that are determined experimentally. The formulation of this law is reported in (12), in comparison with Neuber's law Formulation A) (13):

$$\sigma_N = \frac{B\sigma_U}{\sqrt{1+\frac{h}{h_0}}} \tag{12}$$

for h $\Rightarrow 0$ $\quad \sigma_N = \sigma_U$ (12a)

for h $\Rightarrow \infty$ $\quad \sigma_N = 0$ (12b)

for h = h_0 $\quad \sigma_N = \frac{\sigma_U}{\sqrt{2}}$ (12c)

$$\sigma_N = \frac{\sigma_U}{1+\frac{1}{\sqrt{\rho'}}\sqrt{h}\sqrt{\frac{c_1^{\,2}(1-c)}{\left(\frac{1}{c}-1\right)+c_2^{\,2}}}} \tag{13}$$

for h $\Rightarrow 0$ $\quad \sigma_N = \sigma_U$ (14a)

for h $\Rightarrow \infty$ $\quad \sigma_N = 0$ (14b)

for h = h_0 $$\sigma_N = \frac{\sigma_U}{1+\frac{1}{\sqrt{\rho'}}\sqrt{h_0}\sqrt{\frac{c_1^{\,2}(1-c)}{\left(\frac{1}{c}-1\right)+c_2^{\,2}}}} \tag{14c}$$

From a physical point of view, means that the dominant crack at failure is proportional to the main dimension h of structure. The nominal strength becomes zero for h tending to infinite. It is possible to obtain SEL parameters as a function of Neuber's law Formulation A) parameters; this equivalence is expressed in (18):

$$\frac{\sigma_U}{\sqrt{2}} = \frac{\sigma_U}{1+\frac{1}{\sqrt{\rho'}}\sqrt{h_0}\sqrt{\frac{c_1^{\,2}(1-c)}{\left(\frac{1}{c}-1\right)+c_2^{\,2}}}} \tag{15}$$

$$1+\frac{1}{\sqrt{\rho'}}\sqrt{h_0}\sqrt{f(c)} = \sqrt{2} \tag{16}$$

$$\sqrt{h_0} = \frac{\left(\sqrt{2}-1\right)\sqrt{\rho'}}{\sqrt{f(c)}} \tag{17}$$

$$h_0 = \frac{\left(\sqrt{2}-1\right)^2\rho'}{f(c)} \tag{18}$$

where:

$$f(c)=\frac{c_1^{\,2}(1-c)}{\left(\frac{1}{c}-1\right)+c_2^{\,2}} \tag{19}$$

In Figs. 2 and 3 the test results by Ferro (1993, direct tension), and by Sabnis and Mirza (1979, 4-point bending) are shown, together with the best fitting deriving from Bazant's SEL and from Neuber's law Formulation A).

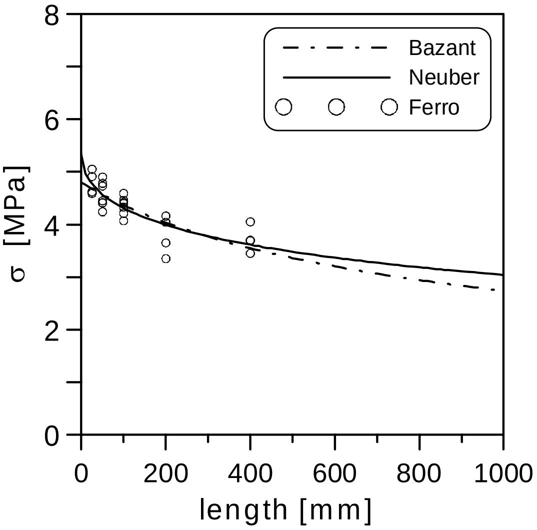

Figure 2. Best fitting of Bažant and Neuber's Formulation A) (t/h = const.), direct tension test results by Ferro.

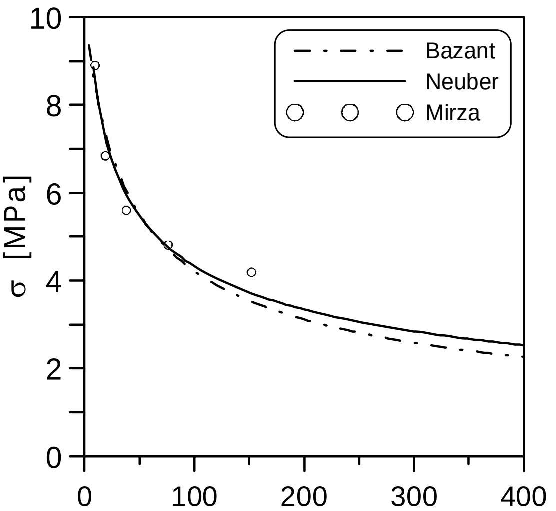

Figure 3. Best fitting of Bažant and Neuber's Formulation A) (t/h = const.), bending test results by Sabnis and Mirza.

In Figs. 4 and 5 the best fitting obtained by means of Bazant's SEL and by Neuber's law Formulation A) with the parameter h_0 from Eq.(18) are plotted, together with the test results by Ferro and by Sabnis and Mirza.

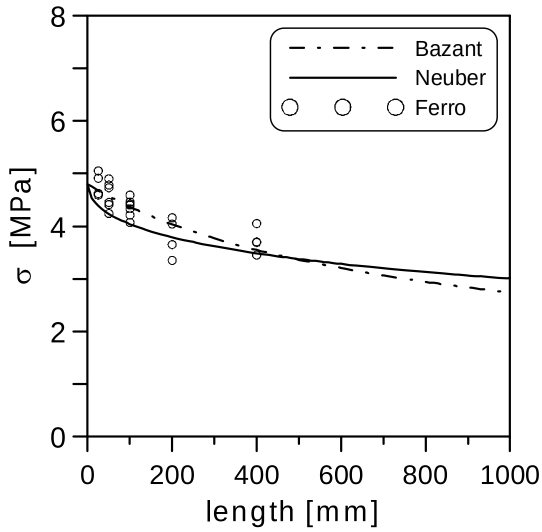

Figure 4. Best fitting by means of Bažant and Neuber's Formulation A) with parameters of equivalence (18), direct tension test results by Ferro.

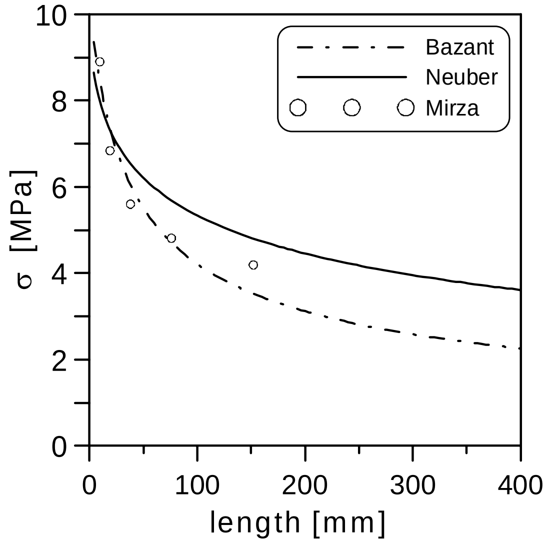

Figure 5. Best fitting by means of Bažant and Neuber's Formulation A) with parameters of equivalence (18), bending test results by Sabnis and Mirza.

3.3 *MFSL Carpinteri*

By introducing the fractal nature of fracture, Carpinteri (1994) has obtained the so-called multifractal scaling law:

$$\sigma_N=\sqrt{A_1+\frac{A_2}{h}} \tag{20}$$

The parameters of MFSL are determined by testing. For h tending to infinite the nominal strength tends to a residual asymptotic value (20a). For h = 0, the strength tends toward infinite. A good equivalence is obtained with Neuber Formulation B), by putting t = const. With the following considerations it is possible to obtain a functional dependence (22 and 25) between the two sets of parameters:

$$\sigma_N = \frac{\sigma_U}{1 + \frac{1}{\sqrt{\rho'}}\sqrt{h}\sqrt{\frac{c_1^{\,2}\left(1-\frac{t}{h}\right)}{\left(\frac{h}{t}-1\right)+c_2^{\,2}}}} \qquad (21)$$

for h $\Rightarrow \infty$ $\quad \sigma_N = \sqrt{A_1}$ (20a)

$$\sigma_N = \frac{\sigma_U\sqrt{\rho'}}{\sqrt{\rho'}+c_1\sqrt{t}} \qquad (21a)$$

$$\sqrt{A_1} = \frac{\sigma_U\sqrt{\rho'}}{\sqrt{\rho'}+c_1\sqrt{t}} \qquad (22)$$

for h = t $\quad \sigma_N = \sigma_U$ (21b)

$$\sqrt{A_1 + \frac{A_2}{t}} = \sigma_U \qquad (23)$$

$$A_1 + \frac{A_2}{t} = \pm\sigma_U^{\,2} \qquad (24)$$

$$A_2 = t\cdot\sigma_U^{\,2}\left(1-\frac{\rho'}{\left(\sqrt{\rho'}+c_1\sqrt{t}\right)^2}\right) \qquad (25)$$

In Figs. 6 and 7 the best fittings of the two independent approaches MFSL and Neuber's law Formulation B) are reported for the test results by Ferro and by Sabnis and Mirza.

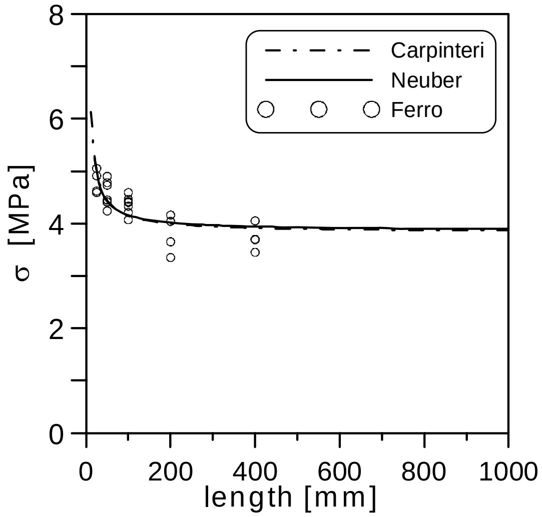

Figure 6. Best fitting of Carpinteri and Neuber's Formulation B) (t=const.), direct tension test results by Ferro.

In Figs. 8 and 9 the test results by Ferro and by Sabnis and Mirza, the best fitting by means of Carpinteri MFSL and by Neuber's law Formulation B) with the parameters obtained from equivalence (22 and 25) are reported.

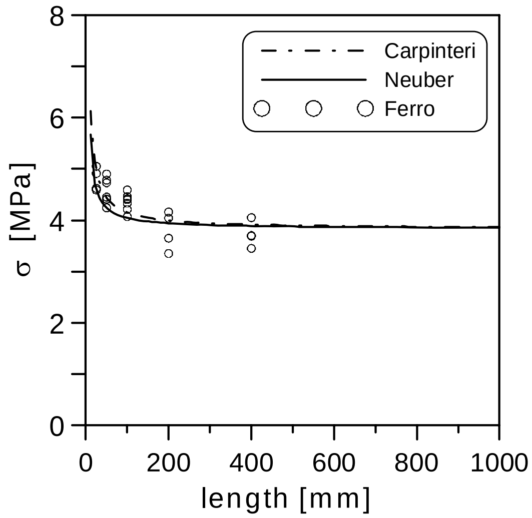

Figure 7. Best fitting of Carpinteri and Neuber's Formulation B) (t=const.), bending test results by Sabnis and Mirza.

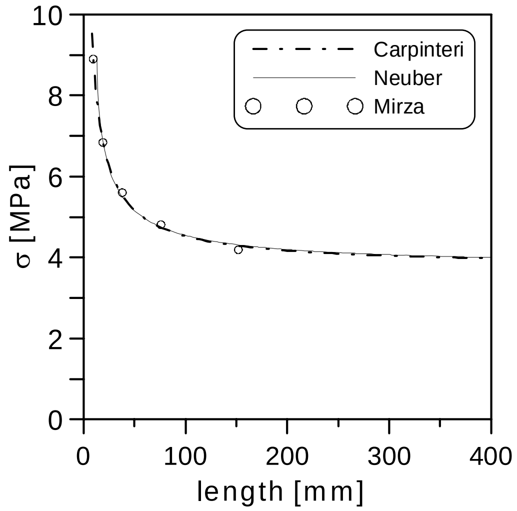

Figure 8. Best fitting by means of Carpinteri, Neuber's Formulation B) with parameters of equivalence (22 and 25), direct tension test results by Ferro.

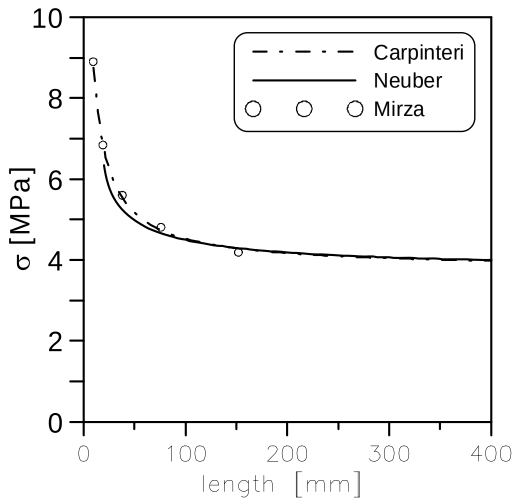

Figure 9. Best fitting by means of Carpinteri, Neuber's Formulation B) with parameters of equivalence (22 and 25), bending test results by Sabnis and Mirza.

Finally Figs.10a, 11a and 12a shown that SEL, MFSL and the Neuber's law Formulation C) satisfactorily fit the test results (in direct tension by Ferro, Fig.10a; in 4-point bending by Sabnis and Mirza, Fig.11a; and in splitting tests by Hasegawa (1985), Fig.12a).

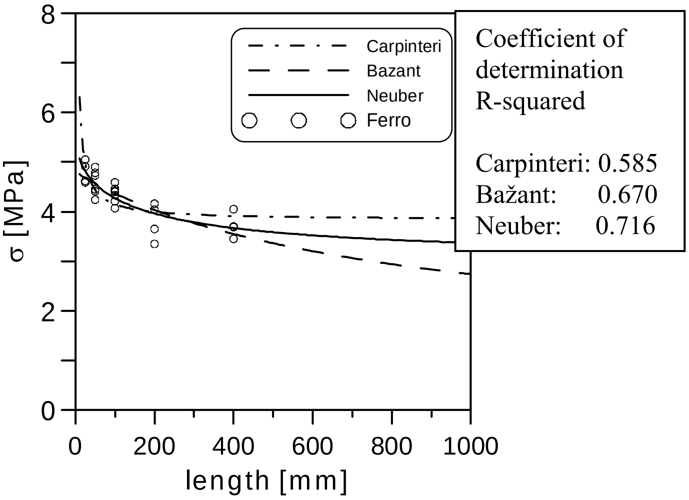

Figure 10a. Best fitting by means of Bažant, Carpinteri and Neuber's Formulation C) (t = t(h)). Direct tension test results by Ferro.

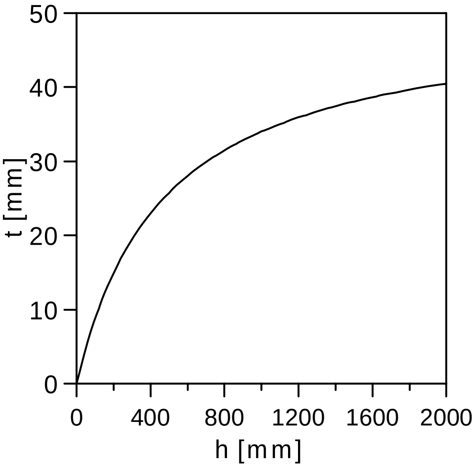

Figure 10b. t = t(h).

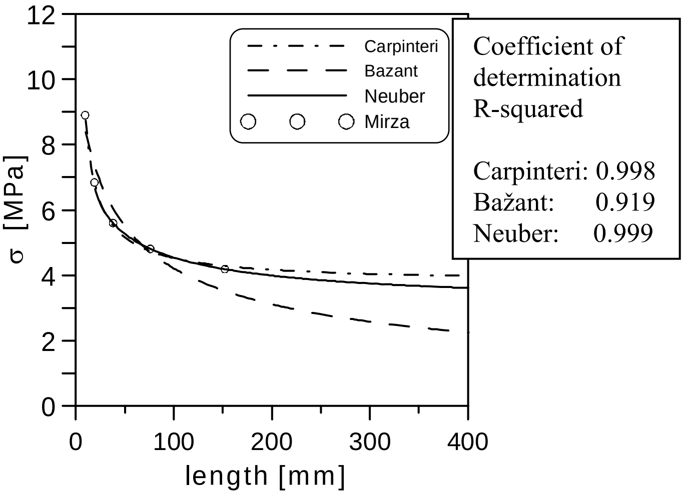

Figure 11a. Best fitting by means of Bažant, Carpinteri and Neuber's Formulation C) (t = t(h)). Bending test results by Sabnis and Mirza.

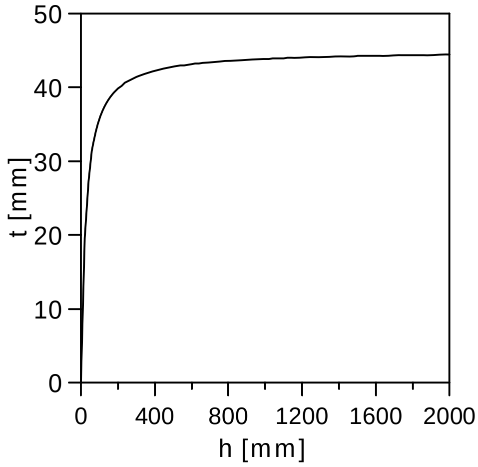

Figure 11b. t = t(h).

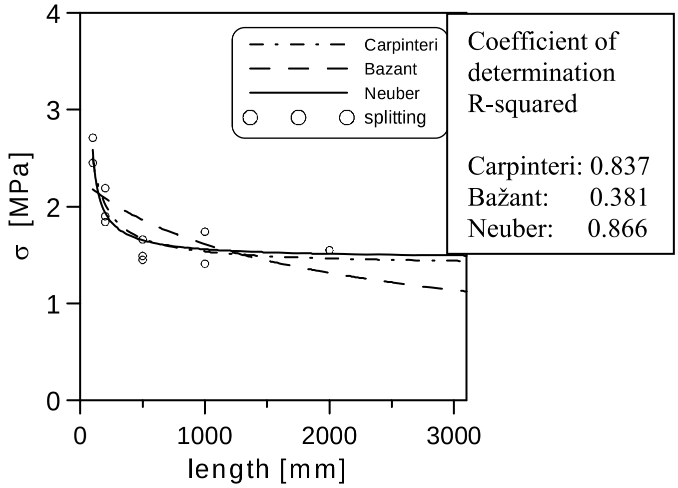

Figure 12a. Best fitting by means of Bažant, Carpinteri and Neuber's Formulation C) (t = t(h)). Splitting test results by Hasegawa.

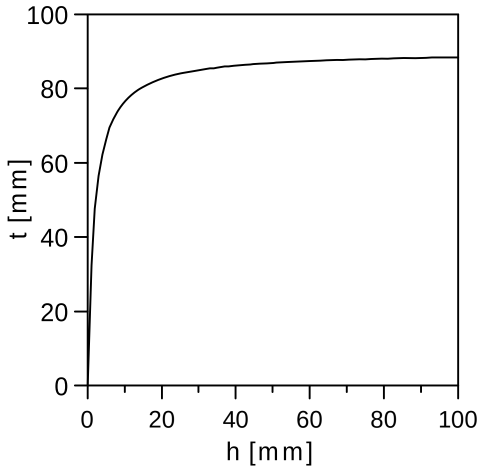

Figure 12b. t = t(h).

The t-h relations utilized in best fitting process are shown in Figs.10b,11b,12b.

3.4 *Conclusions*

From a physical point of view, the simplified static deterministic theory presented here allows to obtain the empirical parameters of the classical size-effect laws.

In addition, the proposed approach makes it possible to comment and clarify the role of the mechanical and geometric parameters in the case of asymptotic behaviour (for h approaching infinite).

The comparison between the theoretical prediction and the experimental data for different generalized stresses show a good agreement. Furthermore, in Rosati and Gelmini (2005) the authors developed the closed-form solution for the extension of Neuber's theory to the notches with loaded contour with cohesive stresses (cohesive crack). Comparing these new solutions with the solutions presented here, the authors obtained a very good agreement. This comparison suggests the use of the closed form solutions proposed here in superposition with statistical approach.

REFERENCES

Bažant, Z.P. 1984. Size-effect in blunt fracture: Concrete, rock, metal. *J. Eng. Mech. ASCE 110:* 518-535.

Bažant, Z.P. 1999. Size-effect on structural strength: a review. *Archive of applied Mechanics 69*: 703-725. © Springer-Verlag.

Bažant, Z.P. & Pang S.D. & Vořechovsky M., Novák D. & Pukl R. 2004. Statistical size-effect in quasi-brittle materials: Computation and extreme value theory. *Proceedings of FraMCoS-5, 5th Int. Conf. On Fracture Mech. Of Concrete and Concrete Structures,* Vail, Colorado: 189-196.

Bueckner, H.F. 1960. Some stress singularities and their computation by means of integral equations. Boundary value problems in differential equations. University of Wisconsin Press, Madison, Wis: R.E. Langer.

Carpinteri, A. 1994. Fractal nature of material microstructure and size-effect on apparent mechanical properties. *Mech. Mat. 18*: 89-101.

Ferro, G. 1993. Effetti di scala sulla resistenza a trazione dei materiali. *Tesi per il conseguimento del titolo di Dottore di Ricerca VI Ciclo (A.A.1990/91; 1991/92; 1992/93). PhD Structural Engineering thesis Politecnico di Torino, Italy.*

Hasegawa, T. & Shioya, T. & Okada, T. 1985. Size-effect on splitting tensile strength of concrete. *Proceedings of the Japan Concrete Institute, 7th Conference*: 309-312.

Marazzini, M. & Rosati, G. 1999. Fiber-Reinforced High-Performance Concrete Beams: Material and Structural Behavior. *ACI SP 182 Structural Applications of Fiber Reinforced Concrete*: 29-52.

Neuber, H. 1958. Kerbspannungslehre (A theory for stress concentration at a notch) 226. Berlin: SpringerVerlag.

Novák D. & Bažant, Z.P. & Vítek J.L. 2002. Experimental-analytical size-dependent prediction of modulus of rupture of concrete. *NON-TRADITIONAL CEMENT & CONCRETE, Vlastimil Bilek and Zbyněk Keršer (eds), Brno:* 387-393.

Rosati, G. 2001. Influence of slenderness on the structural behavior of HPC concrete beams. *EUROMAT 2001*, Rimini – Italy, 11-14 June.

Rosati, G. & Natali Sora, M.P. 2001. Direct Tensile Tests on Concrete-Like Materials: Structural and Constitutive Behaviors. *Journal of Engineering Mechanics, ASCE 127 (4):* 364-371.

Rosati G. & Gelmini A. 2005. Effetti dimensionali in materiali coesivi: aspetti teorici ed evidenze sperimentali. *Degree thesis Department of Structural Engineering, Politecnico di Milano,* Italy.

Sabnis, G.M. & Mirza, S.M. 1979. Size-effect in model concretes?. *J. of Structural Division, ASCE*, 105: 1007-1020.

Schumm, C. E. A. 1997. Material properties and structural response of concrete in tension., *CEB Bulletin information 237 (4)- Concrete Tension and Size Effect*: 87-102.

Tada, H. & Paris, P.C. 1973. The stress analysis of cracks. Hellertown, Pennsylvania: Del Research Corporations.

Vořechovsky M. & Bažant, Z.P. & Novák D. 2005. Procedure of statistical size-effect prediction for crack initiation problems. *11th International Conference on Fracture Turin, March 20-25, paper 40-6-1:* 1-6.

Weibull, W. 1939. The phenomenon of rupture in solids. *Proc. Royal Swedish Inst. Eng. Res. (Ing-enioersvetenskaps Akad. Handl.)* 153: 1-55.