Fracture Mechanics of Concrete and Concrete Structures -
Recent Advances in Fracture Mechanics of Concrete - B. H. Oh, et al.(eds)

Energy based fatigue crack propagation model for plain concrete

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ABSTRACT: In this work, an analytical model is proposed for fatigue crack growth in concrete using the concepts of dimensional analysis and includes the following parameters – size independent fracture energy, structural size, initial crack length, loading ratio and change in energy release rate. By knowing the governed and the governing parameters of the physical problem and by using the concept of self-similarity, the relationship among involved parameters is obtained. The coefficients of the law are calibrated using available experimental data in the literature. It is shown that the proposed fatigue law can capture the size effect and agrees well with the experimental results.

1 INTRODUCTION

Repeated loading causes the crack to grow and finally leading to the failure of the structure. This phenomenon is known as fatigue fracture. Fatigue behavior is well understood for metallic structures, where it causes irreversible material damage (Paris & Erdogan 1963). But for concrete, the behavior is more complicated due to the presence of large size fracture process zone (FPZ) at the crack tip as shown in Figure 1. FPZ is a zone wherein the cement matrix is intensively cracked. Along the FPZ, there is a discontinuity in displacement but not in the stresses. Stresses are themselves a function of crack opening displacements. At the tip of the FPZ, the tensile stress is equal to tensile strength $\left(f_t^{'}\right)$ of the material and gradually reduces to zero at the tip of the true crack. It is assumed that under low cycle fatigue loading the decrease in load carrying capacity and stiffness degradation occurs primarily in the FPZ and not in the undamaged material (Foreman et al. 1967).

Development of mechanistic approaches using the concepts of fracture mechanics, for the study of crack propagation due to fatigue loading had started with the well known Paris' law (Paris & Erdogan 1963) wherein crack growth increment per load cycle is a function of applied stress intensity factor amplitude. Attempts have been made by many researchers (Swartz 1978, Perdikaris 1989 and Baluch 1989) to model crack growth of concrete by applying Paris' law (Paris & Erdogan 1963). However, one important aspect, namely - the size effect has been scarcely reported in the literature concerned with fatigue of concrete. The first attempt was made by Bazant and Xu (Bazant & Xu 1991). They combined their size effect law to the existing Paris' law and proposed a size adjusted Paris' law applicable for plain concrete, given by:

$$\frac{da}{dN} = C\left(\frac{\Delta K}{K_{IC}}\right)^n \tag{1}$$

$$\text{where} \quad K_{IC} = K_{If}\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{2}}; \quad \beta = \frac{d}{d_0} \tag{2}$$

K_{If} is the fracture toughness of an infinitely large structure, d is the characteristic dimension of the structure and d_0 is an empirical constant. It gives the crack length increment per cycle as a power function of the amplitude of a size adjusted stress intensity factor. The numerical value of the transitional size is different for fatigue loading than the monotonic one. So, to take care of fatigue loading in the computation of fatigue law coefficients, transitional size has been adjusted and the adjustment factor used for d_0 happens to be ten times the monotonic one which really has no significance. A similar law has been proposed describing the fatigue fracture of high strength concrete for predicting crack growth (Bazant & Shell 1993). However, concrete exhibits typically nonlinear fracture processes due to the large size process zone and makes this linear elastic fracture mechanics approach (LEFM) questionable.

Slowik et al. (Slowik et al. 1996) proposed a linear elastic fracture mechanics based fatigue crack propagation law which included parameters such as the fracture toughness, loading history, and specimen size. This law applicable to variable amplitude fatigue loading is described by

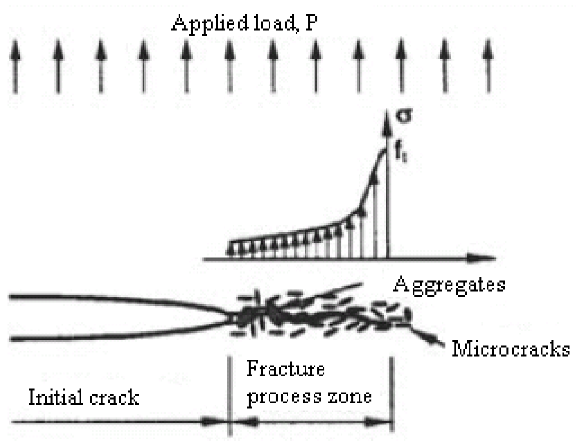

Figure 1. Fracture Process zone.

$$\frac{da}{dN} = C\frac{K_{Imax}^{m}\Delta K_{I}^{n}}{\left(K_{IC}-K_{I\sup}\right)^{p}} + F\left(a,\ \Delta\sigma\right) \tag{3}$$

where K_{Isup} is the maximum stress intensity factor ever reached by the structure in its past loading history; K_{IC} is the fracture toughness; K_{Imax} is the maximum stress intensity factor in a cycle; N is the number of cycles; $F\left(a,\ \Delta\sigma\right)$ is a function that takes into account the sudden overload on the crack propagation and m, n, p are constants for all structural sizes. Similar model based on the variation of Paris' law has been proposed by Kolluru et al. (Kolluru et al. 2000). It is observed that crack growth due to fatigue loading comprises deceleration stage followed by an acceleration stage and they developed analytical expressions for the crack growth in both the stages. But, the most important aspect of size effect has not been reported in this study.

In the recent years, the concepts of dimensional analysis and self-similarity are used by many researchers to study fatigue behavior of plain concrete. Carpinteri and Spagnoli (Carpinteri & Spagnoli 2004) have proposed a size dependent fatigue crack propagation law for concrete that expresses the crack growth rate against the stress intensity factor range. The concepts of fractal geometry were used together with a new definition of fracture energy and stress intensity factor (SIF) based on physical dimension different from the classical ones. Spagnoli (Spagnoli 2005) has derived a crack size dependent Paris' law using similarity methods and fractal concepts. The form of the fatigue law is proposed based upon the assumption of an incomplete self-similarity. The author has only shown the dependency of the fatigue parameters on growth rate and has not obtained any closed form expression for the fatigue crack propagation model. Carpinteri and Paggi (Carpinteri & Paggi 2007) have proposed an approximate relationship between the Paris' law coefficients C and m. Two independent approaches, self-similarity concepts and the condition that the Paris' law instability corresponds to the Griffith-Irwin instability at the onset of rapid crack growth have been used. Sain and Chandra Kishen (Sain & Chandra Kishen 2007) modified Slowik's law to include the effects of loading frequency and overload function. The major limitation of the Slowik's model and the modification made by Sain and Chandra Kishen (Sain & Chandra Kishen 2007) is that, these are not dimensionally homogeneous due to empirical nature of the proposed equations.

2 SCALING LAWS

Scaling laws or power-laws which describe the power-law relationship between different quantities give the evidence of a very important property of self-similarity, wherein a phenomenon reproduces itself on different time and space scales. In construction of an analytical model, it is impossible to take into account all the factors which influence the phenomenon. So, every model is based on certain idealization of the phenomenon. In constructing the idealizations, the phenomenon under study should be considered at intermediate times and distances. Therefore, every mathematical model is based on intermediate asymptotic. In fact, self-similar solutions not only describes the behavior of the physical systems under some special conditions but also describes the 'intermediate asymptotic' behavior of the solution to broader classes of problems i.e. the behavior in the regions where these solutions have ceased to depend on the details of the initial conditions and boundary conditions (Barenblatt 1996, 2004). In this work, the phenomenon of fatigue crack propagation on the basis of similarity approach is considered.

3 DIMENSIONAL ANALYSIS AND SELF-SIMILARITY

In any physical problem, we try to determine the relationship among the physical quantities involved. Let us consider, there exists a relationship between a quantity a which is to be determined from experiments (governed parameter), and a set of quantities which are under experimental control (governing parameters), that can be written as

$$a = f\left(a_1, ...a_k, a_{k+1}, ...a_n\right) \tag{4}$$

where $\left(a_1......a_k\right)$ have independent physical dimensions, i.e. none of these quantities have a dimension that can be represented in terms of a product of powers of dimensions of the remaining quantities and $\left(a_{k+1}.........a_n\right)$ can be expressed as the product of powers of the dimensions of the parameters

$(a_1......a_k)$. This would mean that the dimension of the governed parameter a is determined by the dimensions of $(a_1......a_k)$. Thereby, a_{k+1} can be written as $a_{k+1}/a_1^p......a_k^r$ to make it dimensionless. Introducing the dimensionless parameters as:

$$\Pi_i = \frac{a_{k+i}}{a_1^{p_{k+i}}......a_k^{r_{k+i}}} \tag{5}$$

$$\Pi = \frac{a}{a_1^p........a_k^r} \tag{6}$$

$$\Pi = \Phi(\Pi_1,........\Pi_{n-k}) \tag{7}$$

where Φ is a function of non-dimensional terms

$$a = f(a_1,...a_k,a_{k+1},....a_n) = a_1^p......a_k^r\Phi\left(\frac{a_{k+1}}{a_1^{p_k+1}........a_k^{r_k+1}},,,,\frac{a_n}{a_1^{p_n}........a_k^{r_k+1}}\right) \tag{8}$$

Applying Buckingham Π theorem Φ turns out to be a function of (*n-k*) variables only. The quantities $\Pi_1,........\Pi_{n-k}$ are called similarity parameters, and the physical phenomenon is termed similar if the dimensionless parameters $\Pi_1,........\Pi_{n-k}$ are identical. We shall now discuss two important terms associated with dimensional analysis. (1) Self-similarity of first kind (2) Self-similarity of second kind. Let us consider the parameter a_1. This parameter is considered as non-essential if the corresponding dimensionless parameter Π_1 is too large or too small (tend to zero or infinity), giving rise to a finite non-zero value of the function Φ with the other similarity parameters remaining constant. The number of arguments can now be reduced by one and we can write

$$\Pi = \Phi_1(\Pi_2,............\Pi_{n-k}) \tag{9}$$

where Φ_1 is the limit of the function Φ as $\Pi_1 \to 0$ or $\Pi_1 \to \infty$. This is called complete self-similarity or self-similarity of first kind. On the other hand for $\Pi_1 \to 0$ or $\Pi_1 \to \infty$, if Φ tends to zero or infinity, then the quantity Π_1 becomes essential, no matter how large or small it becomes. However in some cases, the limit of the function Φ tends to zero or infinity, but the function Φ has power type asymptotic representation which can be written as,

$$\Phi \cong \Pi_1^\beta \Phi_1(\Pi_2,............\Pi_{n-k}) \tag{10}$$

where the constant β and the non-dimensional parameter Φ_1 cannot be obtained from the dimensional analysis alone. This is the case of incomplete self-similarity or self-similarity of second kind.

4 FATIGUE CRACK PROPAGATION FOR PLAIN CONCRETE AND SELF-SIMILARITY

The fatigue crack propagation phenomenon is generally analyzed in the medium amplitude range and in this range the phenomenon is characterized by an 'intermediate asymptotic' nature (Barenblatt 1996). Assuming the crack growth rate (*da*/*dN*) as the parameter to be determined in the phenomenon, which is governed by the loading parameter characterized by change in energy release rate (ΔG) and the characteristic dimension of the structure (D). Also fatigue crack growth is governed by crack length (*a*), loading ratio (*R*) defined as the ratio of minimum to maximum stress amplitude and loading frequency (ω). Material properties considered are size independent fracture energy (G_f) and tensile strength (σ_t). Now, we can write the above dependence as follows:

$$\frac{da}{dN} = \Phi\left(\Delta G,\ G_f,\ a,\ D,\ \sigma_t,\ \omega,\ t, R\right) \tag{11}$$

The governing variables are summarized with their physical dimensions expressed in the Length-Force-Time class (LFT) in Table 1. Considering a state of no explicit time dependence and G_f, σ_t have independent physical dimensions, dimensional analysis gives

$$\frac{da}{dN} = \left(\frac{G_f}{\sigma_t}\right)\Phi\left(\frac{\Delta G}{G_f},\ \frac{\sigma_t}{G_f}a,\ \frac{\sigma_t}{G_f}D, R\right) \tag{12}$$

The non-dimensional quantities are

$$\Pi_1 = \frac{\Delta G}{G_f} \quad \Pi_2 = \frac{\sigma_t}{G_f}a \quad \Pi_3 = \frac{\sigma_t}{G_f}D \quad \Pi_4 = R$$

Now we need to analyze whether the number of arguments can be reduced further or not. Considering Π_1, it is usually small in the intermediate range of fatigue crack growth and consideration of complete self-similarity will make the crack growth independent of ΔG, so assuming incomplete self-similarity in Π_1, we can write

$$\frac{da}{dN} = \left(\frac{G_f}{\sigma_t}\right)\left(\frac{\Delta G}{G_f}\right)^{\gamma_1}\Phi_1(\Pi_2,\ \Pi_3,\ \Pi_4) \tag{13}$$

Generally, the parameter Π_2 is a large number and experimental results (Spagnoli 2005) have shown the dependence of crack growth rate on *a*. Therefore, consideration of incomplete self-similarity gives

Table 1. Governing variables of the fatigue crack growth phenomenon in plain concrete.

Variables	Definitions	Dimensions
ΔG	Stress intensity factor range	$FL^{-3/2}$
G_f	Fracture toughness	$FL^{-3/2}$
a	Initial crack length	L
D	Structural size	L
σ_t	Tensile strength	FL^{-2}
ω	Loading frequency	T^{-1}
t	Time	T
R	Loading ratio	-

$$\frac{da}{dN} = \left(\frac{G_f}{\sigma_t}\right)\left(\frac{\Delta G}{G_f}\right)^{\gamma_1}\left(\frac{\sigma_t}{G_f}a\right)^{\gamma_2}\Phi_2\left(\Pi_3,\ \Pi_4\right) \tag{14}$$

$$= G_f^{\,1-\gamma_1-\gamma_2}\Delta G^{\gamma_1}\sigma_t^{\gamma_2-1}a^{\gamma_2}\ \Phi_2\left(\Pi_3,\ \Pi_4\right)$$

Rewriting the above Equation,

$$\frac{da}{dN} = G_f^{\,m}\left(\Delta G\right)^p a^s\sigma_t^{1-s}\Phi \tag{15}$$

where $m = 1-\gamma_1-\gamma_2$, $p = \gamma_1$, $s = \gamma_2$ and $\Phi = \Phi_2$. The exponents γ_1, γ_2 and the non dimensional parameter Φ, cannot be determined from the consideration of dimensional analysis alone. These parameters can only be obtained either from a best fitting procedure on experimental results, or according to numerical simulations. In the following section, the parameters are determined through a calibration process using the experimental results available in the literature.

5 CALIBRATION OF THE PROPOSED MODEL

In this section, the constants introduced during the formulation of the model are determined using the experimental results of Bazant & Xu (Bazant & Xu 1991). They have tested a series of geometrically similar three-point beams under fatigue loading. The dimension details and the physical properties of these specimens are shown in Table 2. Size independent fracture energy G_f was found to be 0.038 N/mm (Bazant & Pfeiffer 1988). It could be seen that the nondimensional parameter Φ is a function of $\left(\sigma_t D/G_f\right)$ and R and can be assumed to be a constant for a particular material mix (same material properties) and constant loading ratio. For finding the coefficients, the input parameters are (*da/dN*), ΔG and a. The coefficients m, p, s and Φ are determined through an optimization process using the principle of least squares which means, the sum of squared residuals has its least value, a residual being difference between the observed value and the value obtained from the model. The best suited values m and p and s found to be equal to 4.4761 and 5.4113 and 0.0648 respectively and indeed can be considered as material constants. The parameter Φ should account for specimen size and geometry. Figure 2 shows the relationship between Φ and the non dimensional parameter $\left(\sigma_t D/G_f\right)$ derived for the small, medium and large specimens. The resulting quadratic best fit is given by,

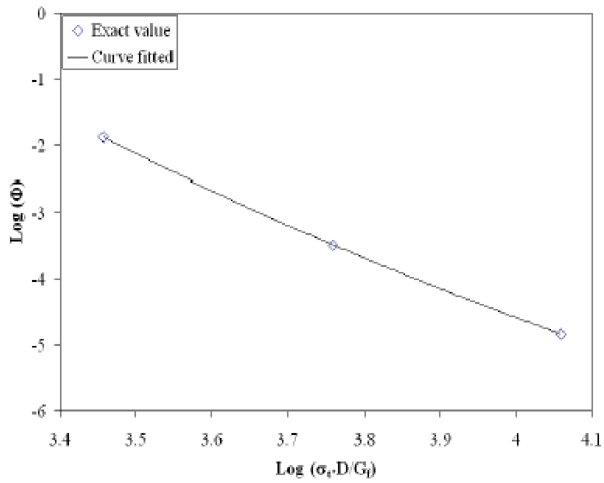

Figure 2. Relationship between Φ and the non-dimensional parameter.

$$Log\left(\Phi\right) = 1.3963\left\{Log\left(\frac{\sigma_t}{G_f}D\right)\right\}^2 - 15.399\left\{Log\left(\frac{\sigma_t}{G_f}D\right)\right\} + 34.663 \tag{16}$$

where D is the depth of the specimen. Using the above equation one can obtain the value of the parameter Φ for any size of specimen and grade of concrete.

6 VALIDATION OF THE PROPOSED FATIGUE MODEL AND DISCUSSION

In order to validate the proposed model, experimental

Table 2. Geometry and material properties of specimens.

Specimens	Depth (D) mm	Span (S) mm	Thickness (B) mm	Notch size (a) mm	Tensile strength (σt) MPa	Young's modulus (E) MPa	Fracture energy (Gf) N/mm
Beam (Bazant 1991)	38.1	95	38.1	6.35	2.86	27120	0.038
Beam (Bazant 1991)	76.2	191	38.1	12.7	2.86	27120	0.038
Beam (Bazant 1991)	152.4	381	38.1	25.4	2.86	27120	0.038
Beam (Toumi 1998)	80	420	50.0	40	4.2 ± 0.25	31600± 2200	0.05

results of Bazant & Xu (1991) and Toumi et al. (1998) have been used. Bazant & Xu have tested small, medium and large sized beam specimens in fatigue under three point bending. The specimen details are tabulated in Table 2. Figure 3 and 4 show the logarithmic plot of crack growth rate verses stress intensity factor amplitude normalized with size independent stress intensity factor (SIF) as well as size dependent SIF (K_{IC}). The experimental results reported by Bazant & Xu (1991) are plotted together with the proposed fatigue law, considering the case of constant amplitude loading.

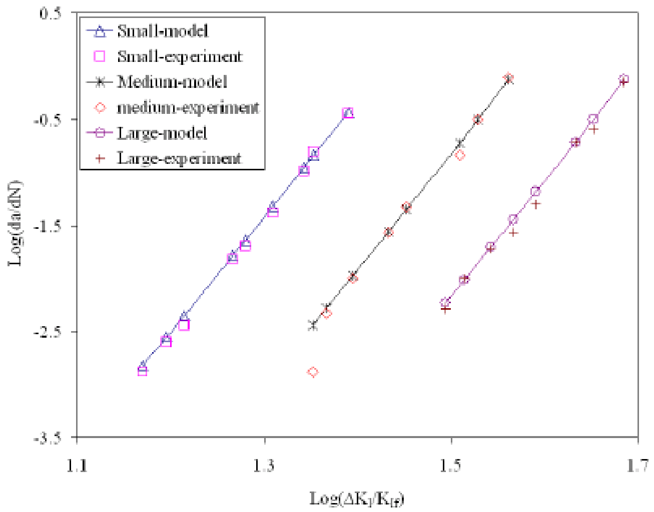

Figure 3. Logarithmic plot of crack length increment per cycle verses the stress intensity factor amplitude normalized with K_{If}.

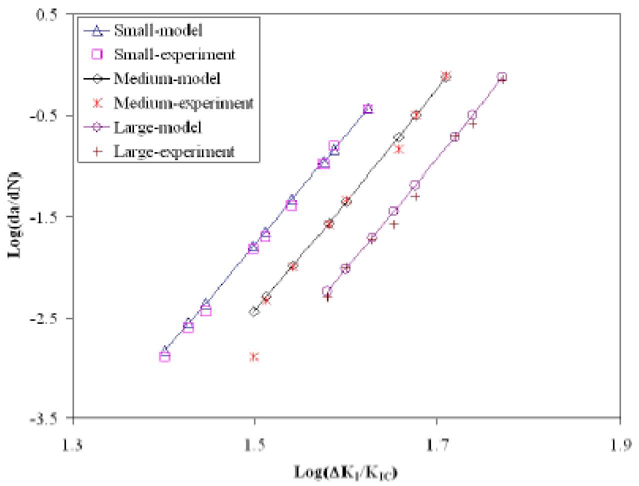

Figure 4. Logarithmic plot of crack length increment per cycle verses the stress intensity factor amplitude normalized with K_{IC}.

From this plot, the Paris' constants C (vertical axis intercept) and m (slope of the straight line) may be determined. Table 3 shows the values of Log C and m obtained from the proposed model and experimentally by Bazant & Xu. The agreement between the experimental data and the model predictions is noticeably good. In Figure 5 the relative effective crack length a, has been plotted as a function of number of load cycles for the proposed fatigue law. In the experimental study, the small, medium and large specimens failed at N = 974, 850 and 882 cycles respectively.

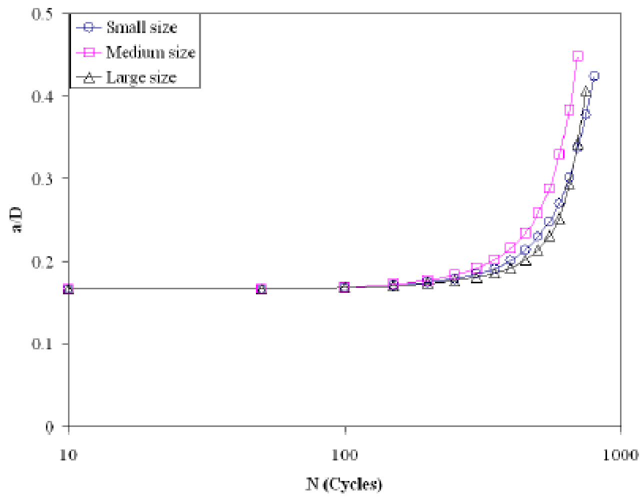

Figure 5. Calculated growth of relative crack length with the number of cycles.

Table 3. Fatigue law coefficients.

Specimens (Beam)	Proposed model Log C	m	Experiment Log C	m
Small	-18.289	11.09	-16.7	11.78
Medium	-18.993	10.50	-18.2	9.97
Large	-19.594	10.90	-18.2	9.27

Table 4. Validation with experimental data.

Stress ratios (Beam)	$Log\left(\frac{\Delta K}{K_{IC}}\right)$	$Log\left(\frac{da}{dN}\right)$ Experiment	$Log\left(\frac{da}{dN}\right)$ Model
$F_{max}/F_u = 0.87$	-0.136	-3.72	-3.97
	-0.104	-3.37	-3.74
	-0.072	-3.03	-3.57
$F_{max}/F_u = 0.81$	-0.16	-3.98	-3.94
	-0.136	-3.72	-3.71
	-0.104	-3.37	-3.65
$F_{max}/F_u = 0.76$	-0.216	-4.58	-4.14
	-0.208	-4.49	-4.08
	-0.2	-4.41	-3.97
	-0.123	-3.57	-3.82
$F_{max}/F_u = 0.70$	-0.242	-4.20	-3.7
	-0.196	-3.75	-3.4
	-0.168	-3.75	-3.14
	-0.104	-2.83	-3.02
	-0.069	-2.46	-2.91

In the proposed model, the specimens fail at N = 880, 810 and 800 cycles respectively. A good agreement is seen between the experimental results and the proposed fatigue model, thereby validating the same.

To verify the effectiveness of the proposed model further, the experimental data reported by Toumi et al. (Toumi et al. 1998) is used and a comparative study is carried out. The specimen details such as dimensions and material properties are tabulated in Table 2. In the above experimental study, fatigue tests were performed at different values of upper limit of cyclic load F_{max} ($0.87F_u$, $0.81F_u$, $0.76F_u$, $0.70F_u$), keeping the lower limit load F_{min} constant for all tests i.e. $0.23F_u$, where F_u is the peak load during static tests. Four various loading ratios (R) are incorporated into Equation 16, leading to

$$Log(\Phi) = 1.3963\left\{Log\left(\frac{\sigma_t}{G_f}D\right)\right\}^2 - 15.399\left\{Log\left(\frac{\sigma_t}{G_f}D\right)\right\} + 34.663 + 2.5R \quad (17)$$

The crack growth rate is plotted against the variation in the stress intensity factor for different stress ratios and compared with the experimental data points as shown in Figure 6. Table 4 shows the results of the crack growth rates with varying normalized stress intensity factor. It is seen that there is good agreement between the experimental result and the proposed model.

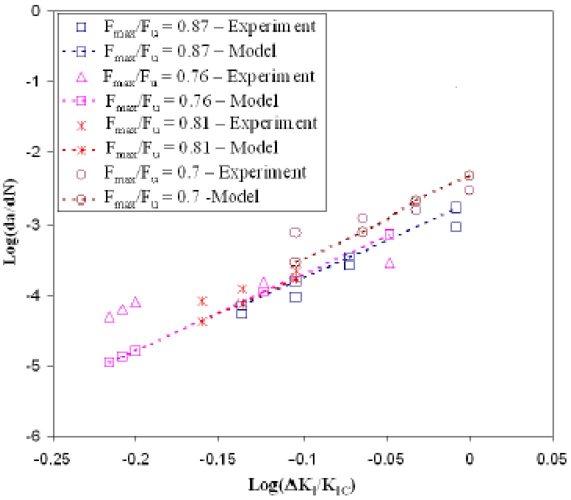

Figure 6. Logarithmic plots of crack growth rate verses stress intensity factor amplitude.

7 CONCLUSIONS

In this study, a fatigue crack propagation model for plain concrete is developed using the concept of dimensional analysis. An important aspect of the proposed model is that it is dimensionally homogeneous as compared to existing empirical models. The proposed model takes into account a number of parameters, such as, the tensile strength, fracture toughness, loading ratio and most importantly the structural size. In most cases, the crack growth rate is not only a function of stress intensity range but also the mean stress level which is considered in the fatigue analysis through stress ratio. The analytical prediction through the present model matches closely with experimental results for plain concrete specimens.

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