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A simplified XFEM approach for local enrichment

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ABSTRACT: The good description of discrete cracking by XFEM brings along with it a large amount of administration and some extra computational costs due to extra degrees of freedom at the system level. In order to avoid these additional degrees of freedom necessary to describe the crack opening in the element are eliminated locally. The displacement field enrichment in the standard XFEM leads to a mismatch between the stress fields in the continuum and at the crack. Even with a linear stress-crack opening relationship the stresses in the crack will be one order higher than the stresses in the continuum. Here a triangular element with a linear strain interpolation is enriched with a linear discontinuous displacement field and thereby making the stress variations more consistent. This consistency makes it possible to directly enforce stress continuity across the crack and consequently eliminate the extra degrees of freedom at system level. The unavoidable drawback of this is the possibility of a displacement discontinuity between cracked elements and therefore the virtual work performed at these discontinuities is included in the formulation in order to reduce this nonconformity. The element is tested in a beam configuration subjected to three point bending and it performs well

1 INTRODUCTION

Since the introduction of XFEM by (Belytschko & Black 1999) and (Moës et al. 1999) it has proven to be a strong tool for modeling discrete cracks. XFEM has the ability to independently model the separated parts of the element without any coupling, which is the reason for its good numerical behavior compared to embedded formulations, see e.g. (Jirásek & Belytschko 2002). XFEM was introduced for linear elastic fracture mechanics but has since been applied to cohesive crack modeling, see e.g. (Weels & Sluys 2001), (Moës & Belytschko 2002) and (Asferg et al. 2007a). In recent years partly cracked elements have been formulated by (Zi & Belytschko 2003) and (Asferg et al. 2007b) and with the latest development (Mougaard et al. 2009) very accurate modeling can be obtained with few elements. One example is the standard three point bending test which in (Mougaard et al. 2009) was modeled with a reasonable accuracy using only 4 elements over the height of the beam.

The drawback of this good description of discrete cracking by XFEM is the heavy administration and the additional extra computational costs due to extra degrees of freedom at the system level.

In this view it is still appealing to investigate whether the additional degrees of freedom can be eliminated locally, resulting in an embedded formulation, without jeopardizing the good performance by standard XFEM.

In the standard XFEM formulation the chosen order of the displacement field is kept intact after the element has cracked. This means that the displacement jump at the crack is modeled with the same order as in the continuum. This leads to a higher order of stresses in the crack than in the continuum. Even with a linear stress-crack opening relationship the stresses in the crack will be one order higher than the stresses in the continuum. In the present formulation it is therefore suggested to use a discontinuous displacement field one order lower than for the continuous part of the displacement field. By using a lower order discontinuous displacement field the stresses in the continuum and in the crack will be more consistent. This enhanced consistency and the limited number of discontinuous variables allows for a direct enforcement of the stress continuity at the crack. The enforcement is done locally and thereby eliminates the extra degrees of freedom which would otherwise exist at the system level. The result of the present formulation is a local enrichment of the element but with no coupling to neighboring elements and therefore it is in fact an embedded formulation. It should be pointed out that in the present formulation the stress continuity is fulfilled exactly and not in a weak sense as in most other embedded formulations (Jirásek 2000).

The unavoidable drawback of embedded formulations is the possibility of a discontinuity at an interelement boundary cut by a crack because of the lack of coupling between cracked elements. As a crack progresses through elements it is possible, as in XFEM, to ensure that the crack enters the next element at the same point as it exits the first element and in this way minimize the discontinuity. In a model with a reasonable element mesh the stress discontinuity will be limited at this inter-element boundary. Since the stress continuity is enforced exactly at the crack in the adjacent elements, the displacement discontinuity at this inter-element boundary will be in the same order as the stress discontinuity between elements. Even though this displacement discontinuity at an inter-element boundary is limited, it is a non-conformity which should be kept at a minimum and therefore the virtual work performed at these discontinuities is included in the formulation.

Here a triangular element of LST type is presented, with a linear strain interpolation enriched with a linear discontinuous displacement field. Most of the previous embedded formulations have been of CST type see e.g. (Oliver 1996) and (Sancho et al. 2007). In this preliminary investigation the element is formulated as a fully cracked element, realizing that this results in reduced accuracy and non-physical situations, since the element fully cracks when the element stress reaches the tensile strength in just one point. The element is tested in a beam configuration subjected to three point bending and shows reasonable accuracy.

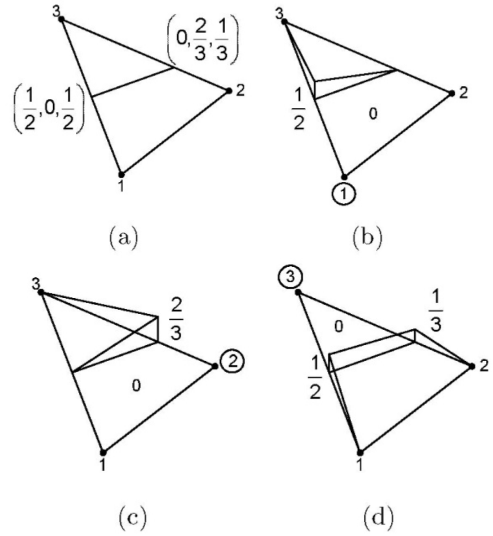

Figure 1. Linear discontinuous displacement field. (a): Geometry of crack. (b, c and d): displacement field for node 1, 2 and 3. (Asferg et al. 2007a).

The continuous shape functions used to create the discontinuous shape functions are therefore chosen to be the same as for the constant strain triangle. The discontinuous shape functions are shown in Figure 1.

By choosing the discontinuous displacements one order lower than the continuous displacements the variation of the stresses in the continuum will be linear, and the variation of the stresses bridging the crack will be approximately linear.

2 KINEMATICS

In the present formulation we adopt the partition of the displacement field into a continuous part and a discontinuous part, from XFEM. The displacement field may therefore be written as

$$\boldsymbol{u}(x,y) = \boldsymbol{N}_c(x,y)\,\boldsymbol{V}_c + \boldsymbol{N}_d(x,y)\,\boldsymbol{V}_d \tag{1}$$

where $\boldsymbol{N}_c$ and $\boldsymbol{N}_d$ are interpolation matrices, $\boldsymbol{V}_c$ and $\boldsymbol{V}_d$ are displacement vectors and c refers to the $c$ontinuous part and d refers to the d iscontinuous part. In the present element the continuous part is chosen to be the same as for the standard linear strain triangle with 12 displacement variables. The discontinuous part of the displacements can be found by multiplying a continuous shape function with the 2D Heaviside step function $H_I(x,y)$

$$\boldsymbol{N}_d(x,y) = \sum_I H_I(x,y)\tilde{\boldsymbol{N}}_c^I(x,y) \tag{2}$$

where $H_I(x,y)$ is zero on the same side of the crack as node I and one on the other side and $\tilde{\boldsymbol{N}}_c^I$ is the part of the interpolation matrix corresponding to node I.

The discontinuous part of the displacements is chosen one order lower than the continuous part.

3 VARIATIONAL AND FEM FORMULATIONS

The virtual work can be written as

$$\int_\Omega \delta\varepsilon^T \boldsymbol{\sigma}\, d\Omega + \int_\Gamma \delta[\![\boldsymbol{u}]\!]^T\, \boldsymbol{t}\, d\Gamma = \int_\Omega \delta\boldsymbol{u}^T \boldsymbol{f}\, d\Omega \tag{3}$$

where ε are the strains, $\boldsymbol{\sigma}$ are the stresses, $[\![\boldsymbol{u}]\!]$ are the displacement discontinuities or jumps, $\boldsymbol{t}$ are the tractions, $\boldsymbol{u}$ are the displacements and $\boldsymbol{f}$ are the loads.

The strains are expressed as

$$\varepsilon = \boldsymbol{B}_c\boldsymbol{V}_c + \boldsymbol{B}_d\boldsymbol{V}_d \tag{4}$$

where $\boldsymbol{B}_c$ relates to the continuous displacements and $\boldsymbol{B}_d$ relates to the discontinuous displacements.

For a linear elastic continuum the stresses are expressed as

$$\boldsymbol{\sigma} = \boldsymbol{D}\varepsilon \tag{5}$$

where $\boldsymbol{D}$ is the constitutive matrix for a disk in either plane strain or plain stress.

The displacement discontinuity in the crack can be expressed as

$$[\![\boldsymbol{u}]\!]_{cr} = \boldsymbol{T}_{cr}(\boldsymbol{N}_d^{II} - \boldsymbol{N}_d^{I})\boldsymbol{V}_d = \boldsymbol{B}_{cr}\boldsymbol{V}_d \tag{6}$$

where $\boldsymbol{T}_{cr}$ is the transformation matrix from the global (x,y)-coordinate system to the local (n,s)-coordinate system in the crack and superscript I and II refers to the parts shown in Figure 2.

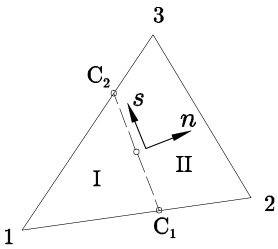

Figure 2. Element with crack from C_1 to C_2 dividing it in parts I and II .

The tractions in the crack are in general non-linearly related to the displacement discontinuities, opening and sliding, and they are defined by the constitutive model. Here this is simply stated as two functions

$$\boldsymbol{t}_{cr} = \begin{Bmatrix} \sigma_n([\![\boldsymbol{u}]\!]) \\ \tau_{ns}([\![\boldsymbol{u}]\!]) \end{Bmatrix} \tag{7}$$

The incremental relationship between tractions and the displacement discontinuities can be expressed as

$$d\boldsymbol{t}_{cr} = \boldsymbol{D}_{cr} d[\![\boldsymbol{u}]\!] \tag{8}$$

At the element sides cut by a crack the discontinuous displacements will in general lead to a discontinuity between elements. This discontinuity will be limited since the location of the crack in two adjacent elements will be the same, and with similar stress states in these two adjacent elements the constraints will lead to similar openings.

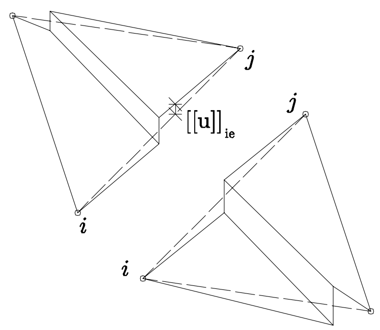

Figure 3. Inter-element discontinuity for two detached elements. The work is formulated as a contribution from each element.

In order to limit these inter-element discontinuities the work done at these boundaries is included in the internal virtual work. The work is formulated as a contribution from each element, see Figure 3 where the displacement is shown perpendicular to the element plane. If the opening is equal in the two elements the discontinuity is zero and the two terms cancel each other. This inter-element discontinuity displacement contribution from one element, measured positive as an opening, i.e. a retraction of the boundary into the element, can be expressed as

$$[\![\boldsymbol{u}]\!]_{ie} = -\boldsymbol{T}_{ie}\boldsymbol{N}_d\boldsymbol{V}_d = \boldsymbol{B}_{ie}\boldsymbol{V}_d \tag{9}$$

where $\boldsymbol{T}_{ie}$ is the transformation matrix from the global (x,y) -coordinate system to the local (n,s) -coordinate system at the boundary.

The tractions at the inter-element boundary can be expressed as

$$\boldsymbol{t}_{ie} = \boldsymbol{T}_{ie}^{\sigma}\boldsymbol{D}\boldsymbol{\varepsilon} \tag{10}$$

where $\boldsymbol{T}_{ie}^{\sigma}$ is the transformation matrix from stresses in the global (x,y)-coordinate system to tractions in the local (n,s)-coordinate system at the boundary.

Using the virtual work Equation (3) and the definitions of generalized strains (4), (6) and (9) and stresses (5), (8) and (10) the tangential stiffness matrix stating the incremental relationship between the applied load and the displacements can be found to

$$\begin{aligned} \boldsymbol{k}_T &= \begin{bmatrix} \int_{el} \boldsymbol{B}_c^T\boldsymbol{D}\boldsymbol{B}_c & \int_{el} \boldsymbol{B}_c^T\boldsymbol{D}\boldsymbol{B}_d \\ \int_{el} \boldsymbol{B}_d^T\boldsymbol{D}\boldsymbol{B}_c & \int_{el} \boldsymbol{B}_d^T\boldsymbol{D}\boldsymbol{B}_d \end{bmatrix} \\ &+ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \int_{ie} \boldsymbol{B}_{ie}^T\boldsymbol{T}_{ie}^{\sigma}\boldsymbol{D}\boldsymbol{B}_c & \int_{ie} \boldsymbol{B}_{ie}^T\boldsymbol{T}_{ie}^{\sigma}\boldsymbol{D}\boldsymbol{B}_d \end{bmatrix} \\ &+ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \int_{cr} \boldsymbol{B}_{cr}^T\boldsymbol{D}_{cr}\boldsymbol{B}_{cr} \end{bmatrix} \\ &= \boldsymbol{k}_{el} + \boldsymbol{k}_{ie} + \boldsymbol{k}_{cr} \end{aligned} \tag{12}$$

The part from the inter-element boundary, $\boldsymbol{k}_{ie}$, makes the tangential matrix non-symmetric. This term is essential for the behavior of the element and only with this term the cracked element can pass a two element test with a simple uniaxial state of stress.

In the same manor the element nodal forces corresponding to a given displacement state can be found by

$$\boldsymbol{q} = (\boldsymbol{k}_{el} + \boldsymbol{k}_{ie})\begin{Bmatrix} V_c \\ V_d \end{Bmatrix} + \int_{cr} \boldsymbol{B}_{cr}^T\boldsymbol{t}_{cr} \tag{13}$$

The tangential stiffness matrix and the nodal force vector are now formulated on the basis of both the continuous and the discontinuous displacements.

4 CONSTRAINTS

The discontinuous displacement field is now constrained by requiring stress continuity at the crack.

At each end of the crack, points C_1 and C_2, the stress continuity is fulfilled by demanding

$$\sigma_n^{cr} = \sigma_n^{co} \qquad \tau_{ns}^{cr} = \tau_{ns}^{co} \tag{14}$$

where superscripts *co* and *cr* indicate stresses in the *co*ntinuum and in the *cr*ack, respectively.

The result is 4 constraints on the original displacement field. Instead of demanding stress continuity with both sides of the crack (resulting in a total of 8 constraints, of which 2 would be linearly dependent) the remaining 2 constraints are applied to the discontinuous part of the displacements. The discontinuous part of the displacements can model different constant stresses in the continuum at each side of the crack. Therefore, for the discontinuous part the stress continuity between these two parts is fulfilled by

$$\sigma_n^{co\mathrm{I}} = \sigma_n^{co\mathrm{II}} \qquad \tau_{ns}^{co\mathrm{I}} = \tau_{ns}^{co\mathrm{II}} \tag{15}$$

Since the discontinuous part only models constant stresses in the continuum the constraints need only to be applied at one point as indicated by the midpoint in Figure 2. In total this results in 6 constraints and thereby the 6 discontinuous displacements may be eliminated locally.

The 6 constraints may be written as

$$\boldsymbol{A}_c \boldsymbol{V}_c + \boldsymbol{A}_d \boldsymbol{V}_d = \boldsymbol{b} \tag{16}$$

Then the discontinuous displacements may then be found from the continuous displacements by

$$\boldsymbol{V}_d = \boldsymbol{A}_d^{-1}(\boldsymbol{b} - \boldsymbol{A}_c \boldsymbol{V}_c) \tag{17}$$

This relation may be used for the transformation from the continuous displacements used at the system level to the displacements used at the element level

$$\begin{Bmatrix} \boldsymbol{V}_c \\ \boldsymbol{V}_d \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A}_d^{-1}(\boldsymbol{b} - \boldsymbol{A}_c \boldsymbol{V}_c) \end{bmatrix} \boldsymbol{V}_c = \boldsymbol{T}\boldsymbol{V}_c \tag{18}$$

Applying this transformation the tangential stiffness matrix may be formulated on the basis of the continuous displacements

$$\boldsymbol{k}_T^c = \boldsymbol{T}^T \boldsymbol{k}_T \boldsymbol{T} \tag{19}$$

Similarly the resulting element nodal force vector is found by

$$\boldsymbol{q}^c = \boldsymbol{T}^T \boldsymbol{q} \tag{20}$$

Now both the tangential stiffness matrix and the nodal force vector are formulated for a cracked element, solely on the basis of the continuous displacements.

5 NUMERICS

With the formulation of the tangential stiffness matrix and the nodal force vector, the basis for solving the non-linear Equations is established. Since the extra degrees of freedom necessary to describe the crack opening is eliminated locally, no extra variables are needed at the system level.

By assembly the global nodal force vector $\boldsymbol{Q}$ can be established and the virtual work Equation stated as

$$\boldsymbol{Q}(\boldsymbol{V}) = \boldsymbol{R} \tag{21}$$

where $\boldsymbol{R}$ is the global nodal load vector. This non-linear Equation is solved iteratively applying the tangential relation

$$\boldsymbol{K}_T d\boldsymbol{V} = d\boldsymbol{R} \tag{22}$$

where $\boldsymbol{K}_T$ is the global tangent stiffness matrix assembled from element matrices, $d\boldsymbol{V}$ is the global incremental displacement vector and $d\boldsymbol{R}$ is the global incremental load vector.

The element presented in the previous section is based on the LST, however, it allows for the formation of a displacement discontinuity or a crack. Thus, we have named the element "dLST". The dLST has six nodes and twelve degrees of freedom, two at each node describing the displacement vector. The actual values of the discontinuous displacements are calculated at the element level and therefore no global degrees of freedom are needed for the discontinuity description.

A crack is formed if the principal stress in a point in the element exceeds the tensile strength. This may result in unrealistic situations since the element is either uncracked or fully cracked. Therefore, when the tensile strength is exceeded in only a small part of the element and the element fully cracks, a part of the crack may close. In the present work the behavior of the crack is extrapolated and small negative openings are accepted at the crack tip.

In principle the crack is set to grow perpendicularly to the principal stress direction, and, if possible, to be continuous between elements; however in the present work the direction is fixed.

The nonlinear equilibrium Equations may be solved by standard FEM procedures such as arch-length procedures. Here, however, a simple Newton-Raphson algorithm with a sign change on the incremental load when the direction of the path changes, has proven sufficient. The convergence criterion is based on the energy in the residual divided by the elastic energy of the first load step.

6 EXAMPLE

To illustrate the capabilities of the suggested element a standard test namely the RILEM three-point bending test (Vandervalle 2000) is considered. A side view of the test setup is shown in Figure 4. The beam has a square cross-section and a 25 mm deep notch cut perpendicular to the bottom face in the mid-section plane; it is simply supported at the ends and loaded by a load acting vertically downwards at mid-span.

The data for the concrete beam is given in Table 1 and in Figure 5. The constitutive model for the crack is very simple and considers only Mode I behavior. The present example is governed by the opening of the crack and the Mode II behavior can therefore be disregarded.

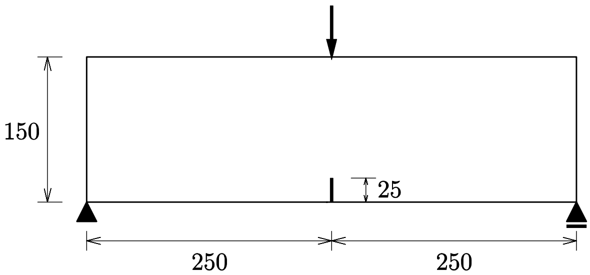

Figure 4. Geometry of the RILEM test beam with a 25 mm notch.

For numerical reasons the shear stiffness of the crack is given a large artificial value.

Table 1. Concrete material parameters.

Parameter	Value
Young's modulus, E_c	37.4 GPa
Poisson's ratio, ν_c	0.2
Tensile strength, f_t	3.5 MPa
Fracture energy, G_f	160 J/m^2

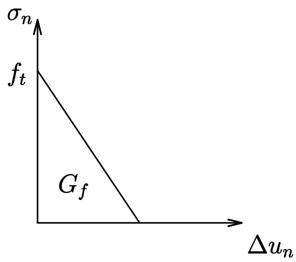

Figure 5. Linear tension softening curve.

The dLST element is tested against a benchmark calculation using interface elements (Asferg et al. 2007a) which has been performed using the commercial FEM code DIANA (Diana 2003).

The suggested element with only local enrichment can model several cracks but in the present example the crack growth of a single crack at the mid plane is considered.

The reference beam was modeled in a regular mesh with 48 elements over the height of the beam.

For the dLST element regular meshes with 6 and 12 divisions of the height of the beam are tested.

The resulting load deflection curves can be seen in Figure 6. The deflection is measured at the center of the beam relative to a point $\frac{1}{2}h$ above the support.

The result for the coarse mesh captures the softening with a slight underestimation of the maximum load but with a non smooth load displacement path. This is due to the small number of elements and to the fact that the element cannot crack partially. For the finer mesh the maximum load level is captured somewhat better and due to the larger number of elements the load displacement path is clearly smoother. Considering the crude assumption that each element cracks at once when the tensile strength is exceeded in just one point the response is captured quite well.

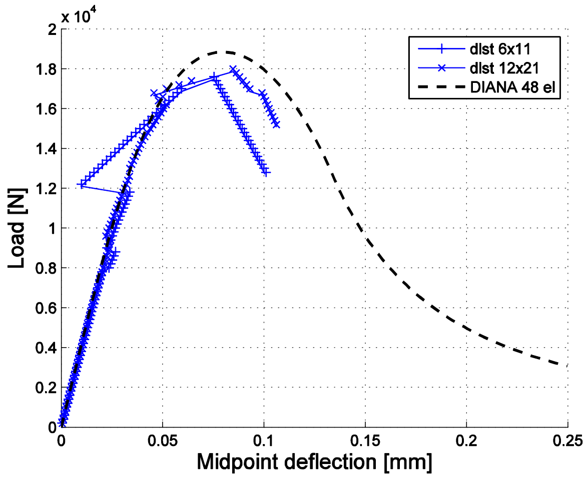

Figure 6. Load-deflection curves of the RILEM test beam with a 25 mm notch. Comparison between a DIANA model with 48 interface elements over the ligament height, and two dCST models with unstructured meshes having 14 and 28 elements over the ligament height, respectively.

In Figure 7 the crack opening at a late stage for the 12x21 model is shown. The figure shows a rather smooth crack closure and some minor non-conforming displacements at inter-element boundaries.

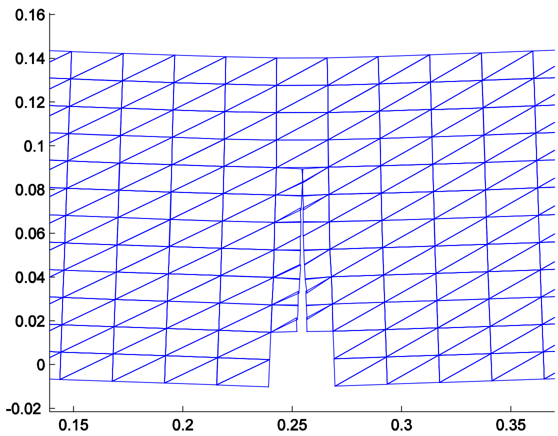

Figure 7. Element mesh for dLST model close to the notch, and a scaled deformation illustrating the crack opening at a late stage.

7 CONCLUSION

An element called dLST with a linear strain distribution in the continuum and a possible linear opening of the crack is formulated based on XFEM shape functions. The extra discontinuous displacements are directly eliminated by demanding stress continuity at the crack and therefore the element is only locally enriched and in fact an embedded crack element.

In the setup of the virtual work, the work done by the unavoidable inter-element discontinuities is included in order to suppress this non-conformity.

The element is fairly simple and has the possibility of modeling multiple cracks.

The preliminary results for the element presented here seem promising and a more detailed analysis awaits.

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