A PROBABILISTIC FATIGUE MODEL BASED ON THE INITIAL DISTRIBUTION TO CONSIDER FREQUENCY EFFECT IN PLAIN AND FIBER REINFORCED CONCRETE

Luis Saucedo∗, Rena C. Yu†, Arthur Medeiros†, Xiaoxin Zhang† and Gonzalo Ruiz†

∗ University of Oxford
Department of Materials, Parks Road, OX1 3PH Oxford, UK

† University of Castilla-La Mancha
Avda. Camilo José Cela s/n, 13071 Ciudad Real, Spain

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Abstract. The objective of this work is twofold. First, we aim to develop a new fatigue model valid for quasi-brittle materials like concrete, which properties have considerably larger standard deviation than metals. Having this in mind, we fit the measured strength data with a three-parameter Weibull cumulative distribution function and in turn take it as the initial distribution for an asymptotic fatigue model in concrete. Second, we endeavor to take into account the observed influence of frequency and stress ratio on the fatigue life in concrete, both plain and reinforced with fibers. The developed model is validated against fatigue tests in compression on cubic specimens for different stress ratios and loading frequencies. All the parameters have found physical meaning in the extensive experimental tests performed for two plain high strength concretes and two concretes reinforced with fibers. The secondary strain rate is found to be correlational with the number of cycles to failure. Finally, a reduced test procedure is proposed for fatigue strength characterization.

1 INTRODUCTION

Interest in the fatigue of concrete began more than a hundred years ago with the development of reinforced concrete bridges. Since then, numerous experiments have been conducted to study the influence of different fatigue parameters, for instance, see [3, 5, 6, 7, 10, 13, 15, 16, 17, 19] and the references within. These parameters are either set by the fatigue test conditions, such as the minimum stress \( \sigma_{\text{min}} \), the maximum stress \( \sigma_{\text{max}} \) and the loading frequency \( f \), or determined by the material properties, for example the static material strength \( \sigma_c \), which can be the compressive strength \( f_c \) or the tensile strength \( f_t \), or any other critical stress defined accordingly. Other parameters include the stress ratio \( R \), defined as \( \sigma_{\text{min}}/\sigma_{\text{max}} \), the stress amplitude or stress range \( \Delta \sigma \), calculated as \( \sigma_{\text{max}} - \sigma_{\text{min}} \), or the stress level \( S \), defined as \( \sigma_{\text{max}}/\sigma_c \).

For metals, the stress amplitude \( \Delta \sigma/2 \) plays an important role, and the fatigue life (the number of cycles \( N \) resisted before failure) is often described by the Wöhler curve. For concrete, however, the influence of the stress ratio, loading frequency and stress level has been observed to be important [2, 9, 13, 19]. The fatigue equation has evolved accordingly to illustrate the role of those parameters. For instance, Aas-Jakobsen [2] proposed to include the effect of the stress ratio \( R \) as follows

\[
\frac{\sigma_{\text{max}}}{\sigma_c} = 1 - (1 - R)\beta \ln N
\]
where $\beta$ is a material parameter. The same relation was confirmed by Tepfers and Kutti [6] and Tepfers [7] for fatigue strength of concrete in compression and in tension for splitting tests of cubes. Even though the influence of loading frequency (or time) has been observed as early as 1960s by Rusch [1] and confirmed by Awad and Hilsdorf [3], Sparks and Menzies [4] and Holmen [8] in 1970s, it was not included in the fatigue equation until Hsu [9], Furtak [10] improved Eq. 1 by including the loading time and frequency respectively. Zhang et al [15] further improved the equation of Furtak by redefining the stress ratio $R$ when there is stress reversal.

However, none of them considered the marked dispersion of the fatigue strength $\sigma_c$ in concrete. The first consideration of the statistical distribution of concrete strength properties for fatigue tests was by Zhao et al. [17], who considered a normal distribution as suggested in the design codes. Recently, Przybilla, Fernández-Canteli and Castillo [18] considered the statistical feature of the characterized strength for brittle materials and derived the primary three-parameter Weibull cumulative distribution function (CDF) of fracture stress from three- and four-point bending tests. Weibull distribution was also used to fit the fatigue life of concrete at various stress levels by Oh [13] and to fit the flexural fatigue life of concrete containing nano-particles by Li et al. [16].

Castillo and his coworkers [20, 21] also postulated a general probabilistic model to predict the fatigue behavior for any stress level and range based on laboratory tests for ductile materials like steel. The nine parameters involved are defined through the physical and compatibility considerations of the Weibull model. However, their model does not consider the observed influence of loading frequency in concrete.

In the current work, we first consider the entire statistical distribution given by the characterization tests and build our fatigue model from this initial distribution. Second we take into account the loading frequency based on the dynamic-response description given by the Model Code [22]. The range of application of the proposed model is below 10 Hz according to the experimental tests realized between 1/16 and 4 Hz [19]. Even Though the model itself does not limit to a given range of frequency, its application beyond 10 Hz needs further experimental collaboration.

The rest of the paper is organized as follows. A fatigue model based on an initial distribution is postulated in Section 2. The experimental program and validation of the model are given in Section 3. A reduced test procedure is proposed based on the developed model in Section 4. Finally we summarize the current work in Section 5.

2 THE FATIGUE MODEL WITH INITIAL DISTRIBUTION

As mentioned above, we aim to develop a fatigue model for concrete, taking into account the statistical distribution of the characterized strength data and the influence of loading frequency and stress ratio, the following hypotheses are assumed.

- The characterized (or experimentally measured) material property of concrete, such as the compressive or tensile strength, follows a Weibull distribution. In the current work, we focus on the compressive strength measured from cubic specimens.

- This distribution is influenced by the dynamic condition through the loading frequency. In addition, the relation given by the Model Code [22] to describe the dynamic properties of concrete is extendable to consider the influence of loading frequency according to the experimental data of Ruiz et al. [19].

- There exists a minimum stress which is the asymptote given by the zero probability of failure.

Given sufficient number of characterization tests carried out at a certain reference loading
rate $\dot{\sigma}_0$, which is considered static, where the dot \(\dot{}\) represents derivation with respect to time, denoting the measured strength data or the failure stress at one cycle as $\sigma_{f_0}$ ($f$ for failure, 0 for static loading), the probability of failure (PF) corresponding to each stress level can be fitted by a three-parameter Weibull CDF as follows

$$PF(\sigma_{f_0}) = 1 - \exp \left[ - \left( \frac{\sigma_{f_0} - \sigma_{\text{min}_0}}{\lambda} \right)^k \right]$$  (2)

where $\lambda$ and $k$ are the scale and shape parameter respectively, whereas $\sigma_{\text{min}_0}$ is the location parameter or the threshold stress below which no failure will occur, it plays the role of endurance limit. Note that, through Eq. (2) the concept of absolute failure or damage is replaced by the probability of failure, which ranges from 0 to 1. We define the distribution in Eq. (2) as the initial distribution $D_i$, which is a property of the material, and is determined necessarily through experimental characterization.

### 2.1 The influence of loading frequency

In order to relate the dynamic failure strength under compression $f_{cd}$ with its static counterpart $f_{c_0}$, we start with the empirical expression provided by the Model Code [22], which is written as follows

$$f_{cd} = \left( \frac{\dot{\sigma}_d}{\dot{\sigma}_0} \right)^\alpha$$  (3)

where $\dot{\sigma}_d$ and $\dot{\sigma}_0$ are the loading rate of the compressive fatigue test and that of the compressive characterization test respectively. The exponent $\alpha$ is fitted as 0.014 in the Model Code [22], where the effect of loading frequency is not taken into account.

The loading rate $\dot{\sigma}_d$ in Eq. (3) in each cycle can be roughly related to the loading frequency $f$ and the stress range $\Delta \sigma$ through

$$\dot{\sigma}_d = 2f \Delta \sigma$$  (4)

Meanwhile an expression for the exponent $\alpha$ that takes into consideration the influence of loading frequency is obtained by fitting experimental data of Ruiz et al. [19] for frequencies below 10 Hz, the function is

$$\alpha = 0.014 \exp[\gamma f]$$  (5)

where the parameter $\gamma$ needs to be determined by fitting experimental data for different loading frequencies. The coefficient 0.014 for static loading conditions is recovered for a zero frequency. As a result, the influence of frequency in a fatigue test is manifest on both the loading rate through Eq. (4) and the exponent $\alpha$ through Eq. (5). In addition, Eq. (3) allows us to shift the initial distribution $D_i$ for $f_{c_0}$ to the distribution of $f_{cd}$ in dynamic conditions.

### 3 FAILURE CURVES OF ISO-PROBABILITY

In this Section, we explore all the conditions of the failure curves in order to obtain the specific expression $\sigma_f(f, \sigma_{\text{max}}, \sigma_{\text{min}_0}, \sigma_{f_0}, R, N)$. On the one hand, each curve represents one probability of failure and intercepts the $\sigma_f$ axis at $\sigma_{f_0}$, the PF (probability of failure) of which is determined by the distribution $D_i$ defined by Eqn. (2) see Fig. 1. On the other hand, there are three limit conditions that all the failure curves of iso-probability should comply with.

$$\lim_{N \to \infty} \sigma_f = \sigma_{\text{min}_0}$$  (6)

$$\lim_{R \to 1} \sigma_f = \sigma_{f_0}$$  (7)

$$\lim_{N \to 1} \sigma_f = \sigma_{f_0}$$  (8)

It needs to be emphasized again that $\sigma_{\text{min}_0}$ is the threshold stress below which no fatigue failure will occur, whereas the $\sigma_{f_0}$ is the static strength (when stress ratio equals one or when the failure occurs after one cycle).
frequency as follows

$$a = b + c \ln(1 + f)$$  \hspace{1cm} (10)

where $b$ and $c$ are the only parameters which need to be fitted with the experimental data of fatigue tests. The logarithmic function in Eq. (10) has been inspired by earlier work of Furtak [10] and it is the function that best fits the experimental results, which we will see in the following Section.

For a fatigue test, $\sigma_{\text{max}}$, $R$ and $\ln N$ are known parameters, meanwhile, the static counterpart of $\sigma_{\text{max}}$, denominated as $\sigma_{\text{max0}}$, corresponds to the value given in the characterization tests. According to Eq. (3) they are related through the following dynamic equation

$$\sigma_{\text{max0}} = \sigma_{\text{max}} \left( \frac{\bar{\sigma}_0}{\sigma_d} \right)^{\alpha} = \sigma_{\text{max}} \left( \frac{\bar{\sigma}_0}{2f\Delta \sigma} \right)^{\alpha}$$  \hspace{1cm} (11)

By plugging in the value of $\sigma_{\text{max0}}$ for $\sigma_f$ to Eq. (9) we easily obtain $\sigma_f$ as follows

$$\sigma_f = \sigma_{\text{min0}} + (\sigma_{\text{max0}} - \sigma_{\text{min0}})N^{\alpha(1-R)}$$  \hspace{1cm} (12)

Introducing the value of $\sigma_f$ into Eq. (2) we arrive at the general expression for the cumulative probability of failure for any point of the fatigue test

$$PF(N; \sigma_{\text{max}}, f, R) = 1 - \exp \left\{ - \left[ \frac{\sigma_{\text{max}} - \sigma_{\text{min0}}}{\lambda N^{-\alpha(1-R)}} \right]^{k} \right\}$$  \hspace{1cm} (13)

but the scale parameter is now related to loading frequency, stress ratio and the number of cycles suffered. Insert Eq. (11) and Eq. (10) to Eq. (13), the following explicit CDF is obtained

$$PF(N; \sigma_{\text{max}}, f, R) = 1 - \exp \left\{ - \left[ \frac{\sigma_{\text{max}} \left( \frac{\bar{\sigma}_0}{2f\Delta \sigma} \right)^{\alpha} - \sigma_{\text{min0}}}{\lambda N^{-[b+c\ln(1+f)](1-R)}} \right]^{k} \right\}$$  \hspace{1cm} (14)
This is the output distribution $D_o$ shown in Fig. 1. It can be observed that, in Fig. 1, for a fatigue test performed at a given level of $\sigma_{max}$, the probability of failure increases with the number of cycles endured. The number of cycles $N$ resisted for a given PF under given loading conditions is also easily derived, see Eq. 15. In addition, by writing the stress range in terms of the stress ratio and the maximum stress as $\Delta \sigma = (1 - R)\sigma_{max}$ in Eq. 11, we can also predict the maximum stress for a given PF and a designed fatigue life at a given stress ratio and loading frequency, see Eq. 16.

\[
N(PF; \sigma_{max}, R, f) = \left[\frac{\lambda}{\sigma_{max} - \sigma_{min}} \sqrt{-\ln(1-PF)}\right]^{a(1-R)}
\]

\[
\sigma_{max}(PF; N; R, f) = \left[\sigma_{min} + \frac{\lambda}{\sigma_{max} - \sigma_{min}} \sqrt{-\ln(1-PF)}\right]^{\frac{1}{1-a}} \left[\frac{2f(1-R)}{\sigma_0}\right]^{\frac{1}{1-a}}
\]

where $\alpha$ is given by Eq. 5.

4 Validation against experimental data

The experimental program was designed to look into the influence of the stress ratio and loading frequency on the fatigue strength of concrete. Two groups of tests on two different types of concrete C1 and C2 were carried out for cubic specimens. Twenty compressive tests on cubes (80 mm in edge length) for C1, loaded at a rate $\dot{\sigma}_0$ of 0.25 MPa/s, and six compressive tests for cubes (100 mm in edge length) for C2, loaded at 0.2 MPa/s, were carried out for characterization purpose. The fitted Weibull distribution $D_i$ for both materials are plotted in Fig. 2, the corresponding parameters are listed in Tab. 1. Note that for both type of concretes C1 and C2, even though the scale and shape parameters $\lambda$ and $k$ are quite different, the minimum stress $\sigma_{min}$ is 3.1 MPa, which plays the role of the endurance limit. Next, the model is validated against concrete reinforced with steel fibers CF1 and polypropylene fibers CF2. Finally, after establishing the one-to-one relation between the secondary strain rate and the fatigue life from observed experimental trend, the fatigue Eq. 14 is also expressed in terms of the secondary strain rate.

Table 1: Fitting parameters for the initial distribution $D_i$ given in Fig. 2 for concrete C1 and C2.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\lambda$ [MPa]</th>
<th>$k$</th>
<th>$\sigma_{min}$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>94.7</td>
<td>12.4</td>
<td>3.1</td>
</tr>
<tr>
<td>C2</td>
<td>76.1</td>
<td>19.8</td>
<td>3.1</td>
</tr>
</tbody>
</table>

4.1 Tests on Concrete C1 for two different stress ratios

For the purpose of studying the influence of stress ratio, two series of compressive fatigue tests on cubic specimens with an edge length of 80 mm, fifteen each, were carried out at a loading frequency of 4 Hz, a maximum stress $\sigma_{max}$ of 90 MPa. Fitting the experimental data with Eq. 13, the values for $a$ and $\gamma$ shown in Tab. 2 give an error of 2.36% and 4.56% respectively. The agreement, as can be seen from Fig. 3, is remarkable. After inserting the obtained parameters $a$ and $\gamma$ to Eq. 2, we arrive at the fatigue equation shown in Eq. 17, which predicts the probability of failure for any compressive fatigue test carried out at 4 Hz, with a stress ratio $R$ and maximum stress $\sigma_{max}$ after $N$ cycles.

\[
PF(N; \sigma_{max}, R) = 1 - \exp \left\{ - \left[\frac{0.9634}{107.4(1-R)^{0.0366}} - \frac{1}{30.5}\right]^{12.40} N^{0.75(1-R)} \right\}
\]
4.2 Tests on concrete C2 under four different loading frequencies

In order to pinpoint the influence of loading frequency, four series of compressive fatigue tests, on cubic specimens with an edge length of 100 mm, were carried out at four different frequencies for concrete C2. Fitting the data curves with Eq. [2] the adjusting parameters $b, c$ and $\gamma$ listed in Tab. [2] give an error of 3.52%, 3.10%, 1.76% and 3.19% for the frequency of 0.0625, 0.25, 1 and 4 Hz respectively, see Fig. [4]. Again the agreement is noteworthy. As a result, all the parameters in Eqs. [14][16] are determined, in turn, the fatigue life or probability of failure for concrete C2 is characterized for any given maximum stress, stress ratio or any loading frequency (below 10 Hz). Notice that the parameter $\gamma$ for the two concretes studied remains the same.

Figure 2: Initial distribution of the compressive strength fitted with the experimental measurements on cubes for concrete C1 and C2.

Figure 3: Distribution $D_o$ given by Eqn. [13], fitted through the fatigue tests with $R = 0.1$ and $R = 0.3$ for concrete C1.

Figure 4: Model and experimental data for concrete C2 for fatigue loading at four different frequencies.

4.3 Validation against concrete reinforced with fibers

In order to further validate the fatigue equation presented in Eq. [14] we also recur to the experimental results presented by Ruiz et al. [19] for concrete reinforced with steel fibers, denoted as CF1, and for concrete reinforced with polypropylene fibers, denominated as CF2. The parameters for the initial distribution $D_i$ and the fatigue equation Eq. [14] are given in Tabs. [2] and [3] We also demonstrate separately the out-
put distributions for concrete reinforced with polypropylene fibers CF2 at four loading frequencies together with the experimental data in Fig. 5, the fitting error is below 5% for all the cases considered. The distribution curves, or the curves of failure probability, at four loading frequencies are plotted together in Fig. 6 for C2 and CF1. Notice that, one the one hand, for the same failure probability, the specimen resists more cycles at a higher frequency. This is attributed to the dynamic behavior of concrete, which results an increase of the dynamic exponent $\alpha$, see Zhang et al. [23]. On the other hand, due to the effect of added steel fibers, the influence of the loading frequency on concrete CF1 is less pronounced as the frequency increases. This is consistent with the known fact that the fatigue behavior of steel is not influenced by loading frequency. In contrast, adding polypropylene fibers also alters the influence of loading frequency on curves of failure probability, such a tendency is not observed in Fig. 5.

4.4 Secondary strain rate versus fatigue life

In a fatigue test for concrete, when the deformation at the upper stress level $\sigma_{max}$ is plotted as the function of the number of cycles under testing, the resulted curve is known as the cyclic creep curve. This curve has a middle part, denominated as the secondary branch, as illustrated in Fig. 7, in which the increase of deformation per load cycle is constant. The slope of this secondary branch is called the secondary strain rate or the secondary creep rate, denominated as $\dot{\varepsilon}_{sec}$, of a specific fatigue test [11, 12].

In the current work, without causing confusion, we simply denote it as $\dot{\varepsilon}$. According to the work of Hordijk et al. [14] and of Cornelissen [11], there appears to be a strong relation between the secondary strain rate and the number of cycles to failure, with diminishing secondary strain rate, the fatigue life increases. In general, a prediction of the number of cycles to failure based on this secondary strain rate is more accurate than that based on stresses in an S-N diagram. In this Section, we attempt to find the relation between the secondary strain rate with the parameters in the fatigue equation Eq. 14.

The experimental results for the materials plain concrete C2 and concrete reinforced with fibers CF1, are plotted in the space of secondary strain rate and fatigue life ($\ln \dot{\varepsilon}$ and $\ln N$) in Fig. 8. Note that, there are 43 tests for C2 and...
40 for CF1, performed for one stress ratio and four different loading frequencies. It can be observed that there is a strong correlation between the secondary strain rate and the number of cycles resisted in the log-log scale. Denominating the intercept of a straight line in Fig. 8 for loading frequency $f$ as $\ln \dot{\varepsilon}_i$, the equation for this straight line is written as

$$\ln \dot{\varepsilon} = \ln \dot{\varepsilon}_i + d \ln N \quad (18)$$

where $d$ is the slope.

Meanwhile, the relation between $\ln \dot{\varepsilon}_i$ and the loading frequency can also be fitted as follows

$$\frac{\ln \dot{\varepsilon}_i}{\ln \dot{\varepsilon}_0} = 1 - \eta \ln \left( \frac{f}{f_0} \right) \quad (19)$$

where $\eta$ is the slope, whereas $\dot{\varepsilon}_0$ is the reference secondary strain rate, corresponding to a fatigue test carried out at the reference loading frequency $f_0$. The frequency $f_0$ is an upper limit, below which, the test is considered static. This means that the following limit conditions should be satisfied for the secondary strain rate of a fatigue test.

$$\lim_{R \to 1} \ln \dot{\varepsilon} = \ln \dot{\varepsilon}_0 \quad (20)$$

$$\lim_{N \to 1} \ln \dot{\varepsilon} = \ln \dot{\varepsilon}_0 \quad (21)$$

In analog to the limiting conditions of the failure stress in Eqs. 6-8 taking into account the fact that $\ln \dot{\varepsilon}$ also depends on the stress ratio $R$ and the frequency $f$ through Eqs. 9-10, we approximate the slope $d$ as follows

$$d = d_1 + [b + c \ln(1 + f)](1 - R) \quad (22)$$

Fitting Eq. 18 with the experimental results shown in Fig. 8, $d_1$ is obtained as 1. Inserting Eq. 19 and Eq. 22 to Eq. 18, calculating the reference secondary strain rate according to linear elasticity, we have the equation below,

$$\ln \dot{\varepsilon} = \ln \left( \frac{\sigma_0}{E} \right) \left[ 1 - \eta \ln \left( \frac{f}{f_0} \right) \right] - \left\{ 1 + [b + c \ln(1 + f)](1 - R) \right\} \ln N \quad (23)$$

The solid lines in Fig. 8 are predictions according to Eq. 23. The agreement with the experimental data is extraordinary. The significance of Eq. 23 lies in the fact that, for a specific fatigue test, once the secondary strain rate is known, the number of cycles to failure ceases to be probabilistic, since there is a one-to-one correspondence between $\dot{\varepsilon}$ and the fatigue life $N$.

Moreover, we slightly modify Eq. 23 to obtain the number of cycles to failure in terms of the secondary strain rate as a function $N(\dot{\varepsilon})$, and insert it to Eq. 14. The probability of failure related to the secondary strain rate is given by the equation 24

$$PF(\dot{\varepsilon}; \sigma_{max}, f, R) = 1 - \exp \left\{ \left[ \frac{\sigma_{max} - \sigma_{min}}{2f\Delta\sigma} \right] \frac{\alpha}{\lambda N(\dot{\varepsilon})} \right\} \quad (24)$$

Comparing Eq. 14 and Eq. 24, we can conclude that for a fatigue test carried out at a given load condition -maximum stress level, stress ratio and loading frequency-, both the number of cycles to failure and the secondary strain rate are probabilistic. It cannot be overemphasized that, Eq. 23 provides the possibility to determine the fatigue life $N$ without the need of actually exhausting all the cycles.

Finally, we summarize all the parameters and the elastic modulus for the four materials in Tab. 3. It needs to be mentioned that, since fatigue tests for C1 were performed at one loading frequency of 4 Hz, it is not possible to obtain the reference loading frequency $f_0$, neither is it feasible for a separate fitting of parameters $b$ and
Instead, a unique value for the parameter \( a \) of 0.06 is obtained. In addition, the constant \( \gamma \) in the expression for the dynamic exponent \( \alpha \), which is extended to include the influence of loading frequency, is fitted as 0.24 for the two plain concretes, zero for the concrete reinforced with steel fibers and a value in between for the concrete reinforced with polypropylene fibers.

![Figure 7: The concept of secondary strain rate or secondary creep rate in a cyclic creep curve for a fatigue test.](image)

**Table 2:** Summary of the model parameters for the four materials considered.

<table>
<thead>
<tr>
<th>Material</th>
<th>( b )</th>
<th>( c )</th>
<th>( \eta )</th>
<th>( f_0 [Hz] )</th>
<th>( \gamma )</th>
</tr>
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<tbody>
<tr>
<td>C1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.240</td>
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<tr>
<td>C2</td>
<td>0.061</td>
<td>0.0105</td>
<td>0.081</td>
<td>0.0016</td>
<td>0.240</td>
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<tr>
<td>CF1</td>
<td>0.052</td>
<td>0.0035</td>
<td>0.086</td>
<td>0.0019</td>
<td>0.086</td>
</tr>
<tr>
<td>CF2</td>
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<td>0.0066</td>
<td>0.089</td>
<td>0.0015</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 3:** Summary of the model parameters for the four materials considered.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \lambda ) [MPa]</th>
<th>( k )</th>
<th>( \sigma_{\text{min0}} ) [MPa]</th>
<th>( E ) [GPa]</th>
</tr>
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<tbody>
<tr>
<td>C1</td>
<td>94.7</td>
<td>12.4</td>
<td>3.1</td>
<td>35</td>
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<tr>
<td>C2</td>
<td>76.1</td>
<td>19.8</td>
<td>3.1</td>
<td>34</td>
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<tr>
<td>CF1</td>
<td>68.0</td>
<td>14.0</td>
<td>4.8</td>
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</tr>
<tr>
<td>CF2</td>
<td>76.1</td>
<td>31.0</td>
<td>12.0</td>
<td>38</td>
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5 **Applications to design of reduced fatigue tests**

Since there is a direct correlation between the probability of failure (or the number of cycles resisted) and the conditions of loading in a fatigue test, such as stress ratio \( R \), loading frequency \( f \), and the fitting parameters of the corresponding characterization test, \( \lambda, k \) and \( \sigma_{\text{min0}} \) through Eqs. \([11][15]\), the probability of failure can be calculated for several stages of loading with different loading frequencies and stress ratios.

An example of seven stages of loading is given in Tab. 4 and Fig. 9. The test is designed to start with a null probability of failure (step 1), the probability of failure at the end of the step 1 is calculated according to Eqn. 13. Next the loading condition is changed to step 2. Evaluating Eq. 15 (considering the final value of \( PF \) in the step 1), we get the equivalent number of cycles for the beginning of the step 2, de-
nominated as \( N_1 \). Considering the loading cycles \( \Delta N_2 \) at step 2, we evaluate the Eq. 13 with \( N_2 = N_1 + \Delta N_2 \) to get the final value of \( PF \) on this step. Repeating this procedure for all the steps of fatigue loading in Tab. 4, fatigue failure is achieved with a reduced number of cycles (less than 20000 cycles), see Fig. 9.

![Figure 9: Damage accumulation for the different steps of loading: a reduced procedure.](image)

Table 4: An example of a reduced procedure for fatigue characterization with seven stages of loading.

<table>
<thead>
<tr>
<th>Step</th>
<th>( \sigma_{\text{max}} ) [MPa]</th>
<th>( \sigma_{\text{min}} ) [MPa]</th>
<th>( f ) [Hz]</th>
<th>( \dot{\varepsilon}_{\text{d}} ) [MPa/s]</th>
<th>( \Delta N )</th>
<th>( PF_i )</th>
<th>( PF_f )</th>
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<tr>
<td>1</td>
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<td>10</td>
<td>1</td>
<td>110</td>
<td>2000</td>
<td>0</td>
<td>0.2137</td>
</tr>
<tr>
<td>2</td>
<td>70</td>
<td>10</td>
<td>0.001</td>
<td>0.12</td>
<td>350</td>
<td>0.2137</td>
<td>0.4988</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
<td>5</td>
<td>9</td>
<td>1890</td>
<td>3000</td>
<td>0.4988</td>
<td>0.5008</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>15</td>
<td>2</td>
<td>260</td>
<td>500</td>
<td>0.5008</td>
<td>0.6886</td>
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<td>100</td>
<td>25</td>
<td>7</td>
<td>1050</td>
<td>4500</td>
<td>0.6886</td>
<td>0.7086</td>
</tr>
<tr>
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<td>78</td>
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<td>5</td>
<td>5</td>
<td>850</td>
<td>9000</td>
<td>0.8499</td>
<td>1</td>
</tr>
</tbody>
</table>

6 Conclusions

By taking into consideration the dynamic properties of concrete, fitting the results of material characterization tests with a Weibull distribution and assuming it as the initial distribution, which can be shifted along the failure axis, we have developed a fatigue model which is capable of dealing with different frequencies and stress ratios for two plain concretes and two concretes reinforced with fibers. Since only two adjusting parameters are needed, the rest are related with material properties and test conditions, reduced fatigue test procedure can be designed for fatigue life prediction.

The model is validated against a total of 153 fatigue tests for two plain high strength concrete and two concretes reinforced with steel or polypropylene fibers, performed at two different stress ratios and four different loading frequencies. In addition, we have shown the failure is probabilistic in terms of the number of cycles \( N \) or the secondary strain rate \( \dot{\varepsilon} \), but there is a one-to-one relation between \( N \) and \( \dot{\varepsilon} \). In practical terms, this provides the possibility of determining the fatigue life \( N \) without actually exhausting all the cycles.

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Nomenclature

- $\sigma_{\text{max}}$: Maximum stress.
- $\sigma_{\text{min}}$: Minimum stress.
- $\Delta \sigma$: Stress range $\sigma_{\text{max}} - \sigma_{\text{min}}$.
- $R$: Stress ratio defined as $\sigma_{\text{min}} / \sigma_{\text{max}}$.
- $\sigma_{\text{max}0}$: Static equivalence of $\sigma_{\text{max}}$.
- $\sigma_f$: Failure stress.
- $\sigma_{f0}$: Intercept of the IPFC with the $\sigma_f$-axis.
- $f_c$: Dynamic compressive failure stress or the dynamic strength under compression.
- $f_{c0}$: Static failure stress or static strength under compression.
- $\sigma_{\text{min}0}$: Horizontal asymptote which determines the lower stress value.
- $\dot{\sigma}_0$: Loading rate of the compressive test.
- $\dot{\sigma}_d$: Loading rate of the fatigue test.
- $\sigma_c$: Critical stress, can be the compressive strength $f_c$ or the tensile strength $f_t$.
- $N$: Number of cycles to failure.
- $\lambda, k$: Scale and shape parameter of the Weibull distribution.
- $PF$: Probability of failure in any point of the domain $\sigma_f - \ln N$.
- $f$: Loading frequency of a fatigue test.
- $f_0$: Reference loading frequency.
- $a, b$ and $c$: Parameter that adjusts the relation between $\ln N$ and $f, R$.
- $\dot{\varepsilon}$: Secondary strain rate in a fatigue test.
- $\dot{\varepsilon}_0$: Reference secondary strain rate calculated as $\dot{\sigma}_0/E$.
- $\dot{\varepsilon}_i$: Intercept of the $\ln[\dot{\varepsilon}] - \ln[f]$ curve with the $\dot{\varepsilon}$-axis.
- $\alpha$: Exponent that measures the amplification of dynamic strength.
- $\gamma$: Coefficient that takes into consideration of loading frequency for $\alpha$.
- $\eta$: Slope of the $\ln \dot{\varepsilon}_i$-$\ln f$ curve.
- $\beta$: Material parameter for the fatigue equation proposed in [2].

REFERENCES


