VIII International Conference on Fracture Mechanics of Concrete and Concrete Structures
FraMCoS-8
J.G.M. Van Mier, G. Ruiz, C. Andrade, R.C. Yu and X.X. Zhang (Eds)

COMPLEXITY OF STRUCTURES: A POSSIBLE MEASURE AND THE ROLE FOR ROBUSTNESS

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Key words: Structural complexity, Graph theory, Information theory, Robustness

Abstract. Complexity is a concept widely applied in many fields of science and humanities. Although the word is extensively used in construction engineering, no definition and measure has been already given. As a practical rule, a structure subjected to loads is considered simple if load paths are easily identifiable. On the contrary, complex schemes are those in which the structural behavior is difficult to catch.

The previous generalized concepts are inserted in a more precise framework. Using information theory, a definition of structural complexity is given. A simple example is made in order to describe the new measure. Some issues of the approach are presented and discussed. The references to algebraic graph theory, required for reducing the computational effort, are briefly presented.

Then, the complexity measure is applied for evaluating the "damage tolerance" of a frame, i.e. its robustness. This concept, as introduced by Lind, can be defined as the capacity of the system to sustain local damage without failure. Robustness is a prerequisite in the design of a structure and represents a modern research topic in the field of structural engineering.

1 INTRODUCTION

In structural engineering, the word "complex" is often employed to define something that is difficult to understand or to solve. There are many ideas that surround the concept of complexity: the size, the presence of elements with different functions, or the difficulties in modeling and calculus. In general, a complex structure is the one that cannot be reduced to a simple scheme without missing important aspects of its structural behavior.

The concepts linked to complexity are extensively used in many disciplines, e.g. biology, game theory, communication, computer science, etc. Lloyd [1] found more than 30 different definitions, which can be substantially grouped into two categories. On one side, there are the measures that capture the randomness, the information content or the description of a process, e.g. periodical systems are less complex than random ones. On the other side, complexity depends upon the size of the process: the larger the system the greater the complexity. Both categories interfere in structural engineering.

2 STRUCTURAL COMPLEXITY

Although the concept of structural complexity is extensively employed in civil engineering practice, a proper definition has not been formulated yet. Extending the general definition given by Simon [2] a complex structure can be defined as the one made up by a large number of parts that interact in a non-simple way. The whole system is not simply the mere sum of the single resisting mechanisms, it takes into account the interaction that the different mechanisms have with one another [3].

In another way, let us imagine to create a duplicate of the structure to define the complexity. Rather than creating a perfect copy of the original, we suppose to formulate a scheme which (1) is able to behave similarly to the original structure under the same set of external loads, and (2) has the minimum possible number of elements. The simpler the structure, the lower the number of elements in the copy. As limit case, the simplest structure is the one that can be condensed into a single element. On the contrary, if the duplicate has the same number of elements which absolve to the same functions, the structure is considered complex.

Therefore, for a given set of external forces acting on a scheme, the complexity of a system can be calculated by considering the amount of information carried in it, i.e. the possibility of reducing the size of the scheme without changing the overall behavior. The approach is based on the concept of entropy introduced by Shannon [4] that captures the amount of information within a sequence. It can also be found in other approaches to complexity [5, 6]. In statistical mechanics, entropy is essentially a measure of the number of ways in which a system may be arranged, often taken to be a measure of disorder: the higher the entropy, the higher the disorder. The entropy is proportional to the logarithm of the number of possible microscopic configurations of the individual atoms and molecules of the system (micro-states), which could give rise to the observed macroscopic state (macro-state) of the system, e.g. through Boltzmann's constant of proportionality.

2.1 Information theory

In information theory, the entropy is the amount of information required to describe the state of the system. Shannon [4], in a general theory of communication, introduced a metric for measuring the entropy. Supposing to have a set of possible n outcomes to which a set of probabilities $(p_1, p_2, ..., p_n)$ is assigned, i.e. $\sum_{i=1}^{n} p_i = 1$, a measure of how much uncertain is the choice (or how much chance is involved in the event) can be expressed as

$$H = -K \sum_{i=1}^{n} p_i \log p_i, \quad (1)$$

where K is a positive constant that depends upon the unit of measure, as well as the base of the logarithm. This quantity H is called information-entropy and is the only function that satisfies the following axioms:

1. $H = 0$ if and only if all the p_i are zero, except one having unit value. The entropy is null if the outcome is certain.
2. If all p_i are equal, i.e. $p_i = \frac{1}{n}$, then H is a monotonically increasing function of n. This means that with equally likely events there is more choice, or uncertainty, when there are more possible events.
3. For a given number of outcomes n, H achieves its maximum, $\log n$ (for $K = 1$), when all the events have equal probability to occur:
$$p_1 = p_2 = ... = p_n = \frac{1}{n}.$$
This situation corresponds to the maximum uncertainty.

In graph theory, this concept has been extended by substituting the outcomes with the decompositions that can be made on the graph, and the probabilities with the ratios between a functional assigned to each decomposition, f_i, and the sum of all the functionals extended to all the possible decompositions, i.e.

$$H_f = -\sum_i \frac{f_i}{\sum_j f_j} \log \left(\frac{f_i}{\sum_j f_j} \right). \quad (2)$$

As an example, consider the two schemes of Figure 1. Both structures support horizontal forces. The difference between the two is the size of the central column and, therefore, the relative stiffness between the elements. Easily, one recognizes in structure (b) the path of loads from the elevation to the foundation, i.e. structure (b) is simpler than structure (a).

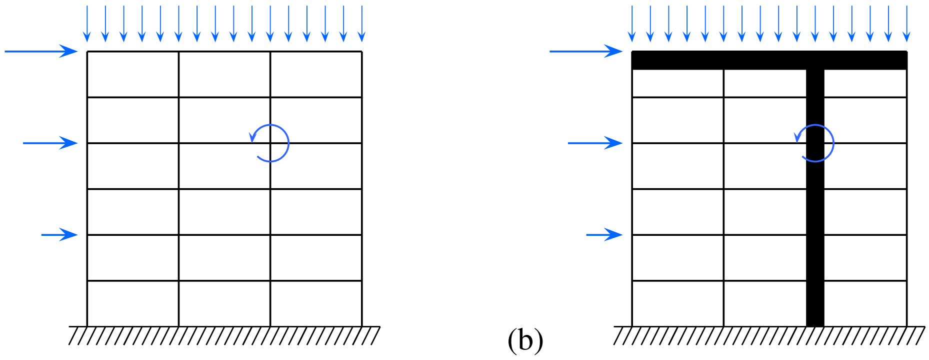


Figure 1: Two similarly connected and loaded structures. The one on the left-hand side (a) is composed by elements that have comparable stiffness, while the other (b) is composed by certain elements with higher stiffness [8].

2.2 Decomposition of the structure

In order to follow the same procedure adopted in graph theory and previously illustrated, the original statically indeterminate structure is decomposed. To explain the idea at the base of decomposition, consider the structure as set of edges, the beams, and vertices, the nodes of the structure. To each edge, some properties are assigned. In this manner, the structure is turned into a graph; the nodes, or the vertices, belong to one of the following classes:

- loaded nodes: the nodes of the structure, on which an external set of forces acts;
- foundation nodes: the nodes which have restrictions in displacement;
- connection nodes: the nodes which connect the elements, with no loads directly acting on them.

The foundation node has to be unique [7]: it represents a unique point to which all the loads are transferred.

The decomposition procedure consists in identifying all the possible paths between the loaded nodes and the unique foundation node. The paths have not to create closed circuits. In frames, the set of the decomposed structures, called *fundamental structures*, is the set of the rooted trees which contains all the loaded nodes [8, 9]. This is possible if the loads act only on nodes (not distributed on the elements), and if no internal hinges are present.

2.3 Choice of the functional

Many authors dealt with the choice of a proper descriptor of the structure. For example, Biondini [10] used the properties of stiffness matrix as an indicator of the performance of the structure. Unfortunately, the stiffness matrix contains quantities with different physical meaning and different physical units. Moreover, the stiffness matrix contains information on the connections between the elements, but it does not take into account the direction and the magnitude of loads on the structure. Kinematical quantities may synthesize both the stiffness properties and the external action. Anyway they suffer the same problem of the components of stiffness matrix since the physical meaning is different for displacements and rotations. At the same time, it is necessary to identify a quantity which summarizes the behavior of all nodes and elements.

It has been found that the work of internal deformation can represent an efficient parameter able to represent both the distribution of stiffnesses across the structure and the intensity and direction of the external loads. Moreover, internal deformation work can be computed easily by means of Clapeyron's Theorem, if linear elasticity is supposed as material property.

2.4 The Index of Structural Complexity

The Authors already demonstrated that the internal deformation work in each fundamental structure is greater or equal to the deformation work in the original statically indeterminate structure [8]. In other words,

$$\psi_i = \frac{W_{in}}{W_{\mathcal{S},i}} \leq 1 \quad (3)$$

where W_{in} is the deformation work in the original statically indeterminate structure and $W_{\mathcal{S},i}$ is the deformation work in the i-th fundamental structure. This ratio can be used to compute the performance of each fundamental structure, or possible load path. Depending on the value of ψ_i, the performance of the fundamental structure is compared with the one of the original structure. In other words it is possible to asses if the removed internal restrains do not give a significant contribution to the overall behavior.

The ratio has been called *performance ratio*, to highlight its capacity to identify predominant load paths. In summary:

- $\psi_i \approx 1$ the fundamental and the original structures have similar deformation works, i.e. the load path is representative of the behavior of the original structure;
- $\psi_i \approx 0$ the deformation work in the fundamental structure is larger than the corresponding in the original structure. The load path is not representative of the original structure;
- intermediate values are possible.

Since the ψ_i-values give an information about the way the loads are transferred from the elevation to the foundation, it can be used as the functional f_i of eqn. (2). The study of the entropy of the distribution of the values of the ratio gives a measure of the complexity of the structure. Applying eqn. (2), the *Structural Complexity Index* is defined as

$$SCI = -\sum_{i=1}^{s} \frac{\psi_i}{\sum_{j=1}^{s} \psi_j} \log_2 \left(\frac{\psi_i}{\sum_{j=1}^{s} \psi_j} \right), \quad (4)$$

where s is the number of fundamental structures, or load paths. In order to compare different structures with different number of fundamental structures. This can be done by dividing eqn. (4) by the maximum value of complexity for a structure with s load paths, i.e. when all the load paths have performance ratio equal to $1/s$, as stated in the introduction. The *Normalized Structural Complexity Index* is, thus,

$$NSCI = \frac{SCI}{-\log(1/s)} = \frac{SCI}{\log(s)}. \quad (5)$$

Values of the $NSCI$ close to one indicate that the scheme is complex, on the contrary values near to zero imply that the scheme is simple.

3 EXAMPLE

In this section, an example of the metric described above is presented. Consider the frame structure sketched in Figure 2. It is composed of 10 nodes –9 nodes in elevation, one foundation node– named with capital letters A,. . . , J, and 15 beams. Beam dimensions are 30×60 cm, column dimensions are 40×40 cm. The frame is made of concrete (with Young's modulus equal to 25 GPa). Linear elasticity is supposed for material behavior.

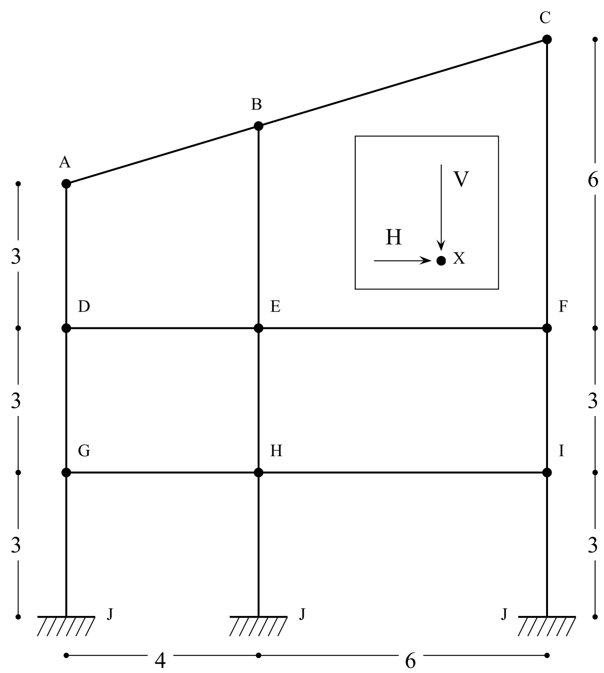


Figure 2: Sketch of the frame. The measures are in meters. The foundation node is considered unique, as reported in the text.

As a first example, the following nodal forces are applied on all elevation nodes (i.e. A,..., I)

$$\begin{aligned} V &= 100 \text{ kN} \\ H &= 100 \text{ kN}. \end{aligned} \quad (6)$$

First, the frame is converted into a graph and, by means of algebraic graph procedures, the fundamental structures are defined, see the Appendix for details on the procedure. The number s of fundamental structures is 1183 and the complexity parameters are computed with eqns. (4) and (5).

$$\begin{aligned} SCI &= 9.5849 \\ NSCI &= 0.9389. \end{aligned} \quad (7)$$

The previous results illustrate that, in the domain of the information related to load paths, the loaded structure is complex. The performance ratios, computed with eqn. (3), indicate the load paths in the structure and its efficiency. The highest value of the performance ratio is 0.2064 and relates to the fundamental structure depicted in left-hand side of Figure 3.

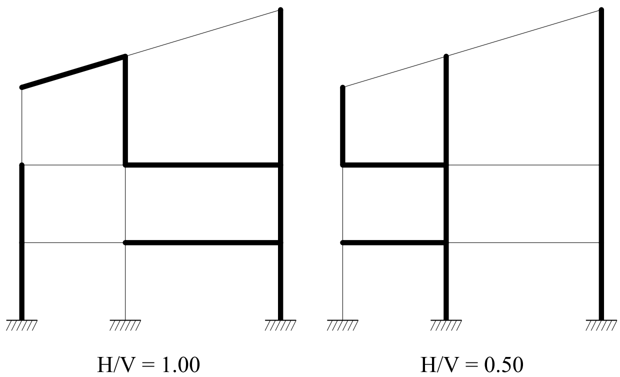


Figure 3: Fundamental structures for $H/V = 1.00$, on left-hand side, and $H/V = 0.50$, on right hand side. The fundamental structures are represented by thick lines.

We perform now a parametric analysis controlling the ratio between the horizontal and the vertical forces acting on the structure. In particular we fix the vertical force on each node in 100 kN, and we reduce the magnitude of the horizontal force from 100 kN to zero.

For example, we suppose a ratio $H/V = 0.50$, that means that $H = 50$ kN. In this case, the values of the complexity parameters reduce, showing that the structural behavior is turning into a simpler one. In particular, we get

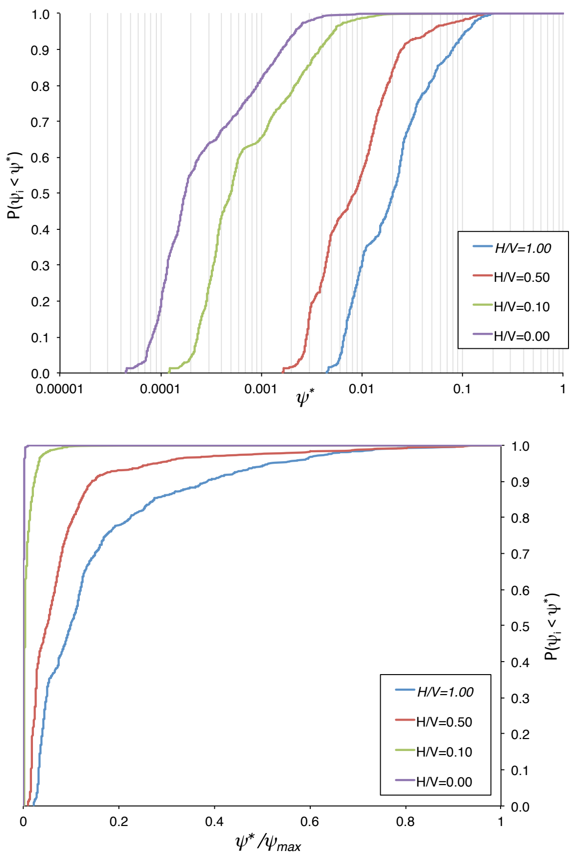
$$\begin{aligned} SCI &= 9.3588 \\ NSCI &= 0.9168. \end{aligned} \quad (8)$$

As stated before, the analysis of the performance ratios gives an idea on how the loads are transmitted to the foundation. In this case, differently from before, the fundamental structure which exhibits the highest value of ψ (0.1716) is illustrated in the right-hand side of Figure 3. The main differences refer to the upper-left part in which the contribution of the columns becomes more relevant as much as the horizontal force reduces, i.e. the resultant nodal force tends to be vertical. As a limit situation, the case $H = 0$ kN is considered. In this sense, the $NSCI$ is equal to 0.4527 and the fundamental structure with higher performance index is the one constituted by the loaded columns alone. The associated ψ-value is, as expected, 0.9999.

Table 1: Number of fundamental structures with performance ratio greater than a given percentage of the maximum value, ψ_{max}. Values are cumulative.

	H/V			
ψ^*/ψ_{max}	1.00	0.50	0.10	0.00
≥ 0.90	6	5	1	1
≥ 0.80	9	9	1	1
≥ 0.50	69	27	1	1
≥ 0.20	262	82	2	1
≥ 0.10	581	256	4	1

In order to study the contribution of different fundamental structures, we ordered the ψ-values obtained from the analysis of the 1183 statically determinate structures in ascending order. To control easily if there are more than one mechanisms with relevant ψ, we normalized each order position by dividing it by the number of fundamental structures. Plotting these data on a graph like the one of Figure 4(a), it is possible to asses the percentage of mechanisms with performance ratio smaller than a given value ψ^*. The loading cases considered refer to H/V values equal to 1.00, 0.50, 0.10 and 0.00. Referring to cases 1.00 and 0.50, there are many mechanisms with relatively high performance indexes. In Figure 4(b) the values of ψ^* are normalized to the maximum value. There are six fundamental structures in the range $0.90 - 1.00\,\psi_{max}$ in the case $H/V = 1.00$, which reduces to five in the case $H/V = 0.50$. There is only one in the cases $H/V = 0.10$ and 0.00. These data are reported in Table 1. The representativeness of a particular load path in the case of $H/V = 0.00$ is clearly visible. This aspect makes the corresponding $NSCI$ low.



(a) (b)

Figure 4: Cumulative probability plots. Probabilities of occurrence of mechanisms with performance ratio smaller than ψ^*, i.e. $P(\psi_i \leq \psi^*)$. On plot (a), the values of ψ^* are absolute, on plot (b) the values are normalized to the maximum ψ-value of each load case. The considered load cases are $H/V = 1.00$, 0.50, 0.10 and 0.00.

4 ROBUSTNESS

4.1 Basic concepts

Structural robustness plays a fundamental role in the design of structures [11]. This fact is confirmed by the requirements that design codes have given after serious accidents in the past (for example Ronan Point Apartment Building in 1968). For the EuroCodes, the structures should be robust in the sense that the consequences of structural failure should not be disproportional to the effect causing the failure [12]. Lind [13] introduced the concept of *damage tolerance* and Masoero and others [14] recently proposed the analogy of structural robustness with material's toughness, a well-known concept of material science and fracture mechanics.

In practice, structural robustness can be achieved in different ways, such as alternating the load paths or compartmentalizing the structure. As reported in the previous sections, frame behavior exhibits a sort of redundancy that can be intended as the capacity to redistribute the loads and to vary the load paths.

Moreover, recent research trends show that the events with the smallest occurrence probabilities are the ones that have the worst effects on the structure [15]. This is the case of so-called "black-swan" events, i.e. events whose existence was not known (or easily predicable) before the fist occurrence. In this sense, it is more convenient to approach the problem of structural safety with the so called *consequence based design*, as already done in other design fields such as natural hazards, aerospace and material engineering.

4.2 Complexity and Robustness

In order to analyze the implications of the proposed complexity metric on structural robustness, we propose to remove column **JH** from the frame sketched in Figure 2. This event can be caused e.g. by an explosion or by a local failure, e.g. during and earthquake. We take the increment of deformation work after the removal as a parameter which gives an idea about the effects of the removal. The greater the ratio of deformation work after/before the removal, the largest the effects on the structure. The deformation work is computed by means of Clapeyron's Theorem. The values of this ratio are reported in the second line of Table 2. It is possible to notice that, as much as the structure is "simple" (following the definition previously introduced), the ratio increases. The effects of column removal in the case of purely vertically loaded structure ($H/V = 0.00$), the deformation work is more than 17 times greater. Similar results are given in case of removal of column **JG**, see the first line of Table 2. In this case, the increments of deformation work are greater for the structures which are "simple" and snaller for the complex ones.

Table 2: Ratio between the deformation work after and before column removal

Removed element	H/V			
	1.00	0.50	0.10	0.00
JG	1.39	1.16	11.21	44.37
JH	1.63	2.10	8.44	17.27
mean	1.51	1.63	9.82	30.82

The average value of the ratio between the deformation work after and before the removal is then computed, see the bottom line of Table 2. The trend of the mean values can be compared with the value of the $NSCI$: as much as the complexity index decreases, the ratio tends to increase. This aspect has to be analyzed in relation to the importance the removed element has in the original structure. Since in case of purely vertically loaded structures, the columns play the fundamental role in the transfer of loads from the elevation to the foundation, any beam removal would not cause substantial changes in the behavior of the structure (in other words the deformation work does not increase significantly).

As a simple explanation of the results found in the numerical simulations, consider the effects of the removal of element **JG** in the cases $H/V = 1.00$ and 0.50. In the first situation, the ratio between the deformation work after and before the removal is 1.39, in the second case it is 1.16. This result may be in contrast with the fact that as much as the complexity reduces the ratio increases. In order to explain the result, and to stress the efficacy of the approach, consider that element **JG** is part of the fundamental structure which has highest performance ratio, ψ, for the case $H/V = 1.00$. On the contrary, it is not an element of the fundamental structures with the highest ψ-value in the case $H/V = 0.50$. In our opinion, this aspect explains the fact that the removal of elements that are not part of the dominant load path causes limited effects on the structure.

A final and important consideration focuses on the fact that a structure can be either complex or simple depending upon the loading scheme acting on it. In this sense, element removal can have irrelevant or disproportional consequences in different cases.

5 CONCLUSIONS

A definition of structural complexity based on information theory has been given. In order to estimate the amount of complexity, the scheme is divided into a set of statically determinate structures, called fundamental structures, to which a weighting parameter is given. Differently from previous approaches, the deformation work has been herein considered as the parameter giving the amount of importance of a fundamental structure. The information content of the structure, i.e. the capacity of the scheme to be described with the lower amount of fundamental structures, is used as a measure of structural complexity.

An example on a 15-elements frame showed how the loads can affect the complexity of structural bahavior, and the fact that the two entities (stiffnesses distribution and external actions) are intimately linked in the evaluation of the overall structural behavior.

Moreover, the links between complexity and robustness have been investigated. The numerical experiments conduced on the example frame consisted in the removal of two columns and in the analysis of the variation of the internal deformation work. As a first result, the variation from undamaged to damaged frame increases as much as the Normalized Structural Complexity Index reduces. In parallel, if the removed element is part of the fundamental structure with highest performance ratio, the effects are more relevant. These preliminary aspects have to be taken into account in the design of structures in which damage tolerance is required.

REFERENCES

[1] Lloyd, S., 2001. Measures of complexity: a nonexhaustive list. *Control Systems IEEE* **4**:7–8.

[2] Simon, H. A., 1962. The Architecture of Complexity. *Proc. Of The Am. Phil. Soc.* **106**:467–482.

[3] Ay, N., Olbrich, E., Bertschinger, N. and Jost, J., 2006. A Unifying Framework for Complexity Measures of Finite Systems. *Complexity* **06-08-028**:1–15.

[4] Shannon, C.E., 1948. A Mathematical Theory of Communication. *Bell System Technical Journal* **27**:379–423.

[5] Grassberger, P., 1986. Toward a quantitative theory of self-generated complexity. *Int. J. of Th. Ph.* **25**:907–938.

[6] Dehmer, M. and Mowshowitz, A., 2011. A history of graph entropy measures. *Information Sciences* **181**:57–78.

[7] Henderson, J.C. and Bickley, W.G., 1955. Statical Indeterminacy of a Structure: A Study of a Necessary Topological Condition Relating to the Degree of Statical Indeterminacy of a Skeletal Structure. *Aircraft Engineering and Aerospace Technology* **27**:400-402.

[8] De Biagi, V. and Chiaia, B., 2012. Complexity and robustness of frame structures. Submitted to *Int. J. of Sol. and Struct.*.

[9] Diestel, R., 2010. *Graph theory*. Springer.

[10] Biondini, F., Frangopol, D.M. and Restelli, S., 2008. On Structural Robustness, Redundancy and Static Indeterminacy. *Crossing Borders*, p. 237.

[11] Starossek, U. and Haberland, M., 2008. Approaches to measures of structural robustness. *Structure and Infrastructure Engineering* **7**:625–631.

[12] CEN, 1994. *Eurocode 1: basis of design and actions on structures*.

[13] Lind N.C., 1996. Vulnerability of damage-accumulating systems. *Reliability Engineering and System Safety* **53**:217–219.

[14] Masoero, E. Wittel, F. K., Herrmann, H. J, and Chiaia, B., 2012. Progressive Collapse Mechanisms of Brittle and Ductile Framed Structures. *Journal of Engineering Mechanics* **136**:987–995.

[15] Nafday, A.M., 2011. Consequence-based structural design approach for black swan events, *Structural Safety* **33**:108–114.

[16] Godsil, C. and Royle, G., 2001. *Algebraic graph theory*. Graduate texts in mathematics, Springer.

A COMPUTATIONAL ASPECTS FOR THE DEFINITION OF THE FUNDAMENTAL STRUCTURES

In order to compute the number of fundamental structures which can be found in a frame structure, graph theoretical aspects are taken into account. First, the structure, originally seen as an ensemble of elements, with geometrical and mechanical properties, linked together in joints, with defined coordinates, is turned into a set of vertices and edges, i.e. the associated graph G. From a topological point of view, any fundamental structure can be considered as a spanning subgraph of G, [9].

The static determinacy of the fundamental structures implies that any spanning subgraph will be a rooted tree of G, i.e. a spanning tree [8, 9]. Mathematically, the search of the set of rooted tree of a graph coincides with the extraction of the set of all possible fundamental structures of the scheme, since the elements are jointed at the nodes. This search is made by means of algebraic graph theory. The number of spanning trees in a graph G is determined by its Laplacian, which is found from the incidence matrix of the graph [16]. As a lemma of Kirchhoff's Theorem, let n be the number of vertices of G and let $\lambda_1, ..., \lambda_n$ be the ordered eigenvalues of the Laplacian of G. The number of spanning trees, s, is defined as

$$s = \frac{1}{n} \prod_{i=2}^{n} \lambda_i. \tag{9}$$

The Cyclomatic Number by Henderson and Bickley [7] associates the First Betti Number of the frame associated graph to the indeterminacy number. For a graph with n vertices and e edges, the Cyclomatic Number $\mathcal{C}$ is equal to

$$\mathcal{C} = e - n + 1 \tag{10}$$

The degree of static indeterminacy (Γ) of the frame is given by

$$\Gamma = 3 \times \mathcal{C} = 3\left(e - n + 1\right).$$

A graph can be turned into a tree by removing $\mathcal{C}$ edges. All the possible combinations of e elements taken as groups of $\mathcal{C}$ elements indicate the elements of the structural scheme which have to be temporarily ignored. Clearly, the number of combinations

$$\binom{e}{\mathcal{C}} \geq s. \tag{11}$$

This is due to the fact that the removals indicated in the set include also the cases in which parts of the scheme are totally separated. In this sense, the Laplacian of the subgraph is written and it is controlled if the the number of spanning trees of the graph associated to the fundamental structure is one, i.e.

$$\frac{1}{n} \prod_{j=2}^{n} \lambda_j = 1. \tag{12}$$

If eqn. (12) is equal to 0, the subgraph is composed by two or more components and cannot be considered as a fundamental structure.