Effect of Stress Singularity on Scaling of Quasibrittle Fracture

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Overview of Quasibrittle Structures

Brittle heterogenous (quasibrittle) materials

-concrete
- composites
- ceramics
- rock
- poly-Si

Size-dependent failure behavior:
small size limit: quasi-plastic
large size limit: perfectly brittle
Two Types Scaling Laws of Quasibrittle Fracture

Current Understanding on Scaling of Quasibrittle Fracture

Two types of size effects:

**Type 1**

\[ \log(\sigma_N / B f_t) = \log(\sigma_N) + \log\left(\frac{D}{D_0}\right) \]

**Type 2**

\[ \log(\sigma_N) = \log(\text{Nom. Strength}) + \log\left(\frac{D}{D_0}\right) \]

*Statistical model*

*Deterministic model*

*Continuum extrapolation*

*Real structure*
Engineering Structures with Weak Stress Singularities

1. Weak stress singularities caused by geometry

2. Weak stress singularities caused by material mismatch
Theoretical Framework
Statistical Size Effect — Structures without Stress Singularities

Structures of positive geometry: peak load is reached once one representative volume element (RVE) is fully damaged — Weakest link statistical model.

\[ 1 - P_f(\sigma_N) = \prod_{k=1}^{N} \left[ 1 - P_1(\sigma_N s(x_k)) \right] \]
Multiscale Statistical Model for RVE Strength

\[ P_1(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{s_0}\right)^m\right] \]  
\[ P_1(\sigma) = P_{gr} + \frac{\tau_f}{\sqrt{2\pi\delta_G}} \int_{\sigma_{gr}}^{\sigma} e^{-(\sigma'-\mu_G)^2/2\delta_G^2} d\sigma' \]  
\( \sigma < \sigma_{gr} \)  
\( \sigma \geq \sigma_{gr} \)

Mean Size Effect Behavior

Mean structural strength:

\[ \bar{\sigma}_N = \int_0^1 \sigma_N dP_f = \int_0^\infty [1 - P_f(\sigma_N)] d\sigma_N \]

Extrapolation by cohesive crack model

\[ \sigma_N = \frac{P_{\text{max}}}{bD} \]

Small Size Asymptote (of cohesive crack)

Intermediate Asymptote (Barenblatt 1996)

Large Size Weibull Asymptote

\[ \bar{\sigma}_N = \left[ \frac{C_1}{D} + \left( \frac{C_2}{D} \right)^\frac{r_n}{m} \right]^\frac{1}{m} \]
Size effect tests on both strength histograms and mean strength of asphalt mixture at low temperature \((T = -24^\circ C)\) \((Le \ et \ al., \ 2013)\).
Energetic Size Effect — Structures with Strong Stress Singularities

1) Large-size limit:
Near-tip stress field: \( \sigma_{ij} = H r^\lambda f_{ij}(\theta) \)
Stress intensity factor: \( H = \sigma D^{-\lambda} h \)

At the large-size limit, the FPZ can be replaced by an LEFM crack (Equiv. LEFM) (Liu and Fleck, IJF, 1997):

\[
K = a H l_c^{\lambda+1/2}
\]
At the peak load, the FPZ is fully developed, i.e. the equiv. crack starts to propagate. This leads to a power-law scaling relation:

\[ \sigma_N \propto D^\lambda \]

2) Small-size limit:
The entire ligament behaves as a crack filled by a plastic glue – No size effect

Asymptotic Matching:

\[ \sigma_N = \sigma_s \left[ 1 + \left( \frac{D}{D_0} \right)^{1/\beta \gamma} \right]^{\lambda \beta \gamma} \]
Scaling Equation for Structures with Weak Stress Singularities — Generalized Weakest Link Model

Weakest link model: \[ P_f = 1 - (1 - P_{f,VI}) (1 - P_{f,VII}) \]

\[ P_{f,VI}(\sigma_N) = 1 - \prod_{i=1}^{N_1} \{ 1 - P_1[\mu(D)\sigma_N s(\mathbf{x}_i)] \} \]

where: \[ \mu(D) = \left[ 1 + \left( D/D_0 \right)^{1/\beta} \right]^{-\xi/\beta} \]

\[ P_{f,VII}(\sigma_N) = 1 - \prod_{i=1}^{N_2} \{ 1 - P_1[\sigma_N s(\mathbf{x}_i)] \} \]

Le, IJF, 2012
Generalized Weakest Link Model

RVE strength distribution:

\[
P_1(\sigma) = \begin{cases} 
1 - \exp \left[-(\sigma/s_0)^m\right] & (\sigma < \sigma_{gr}) \\
\sigma_{gr} + \frac{r_f}{\sqrt{2\pi}\delta_G} \int_{\sigma_{gr}}^{\sigma} e^{-\left(\sigma' - \mu_G\right)^2/2\delta_G^2} d\sigma' & (\sigma \geq \sigma_{gr})
\end{cases}
\]

Mean structural strength:

\[
\bar{\sigma}_N = \int_0^1 \sigma_N(P_f) dP_f = \int_0^\infty (1 - P_f) d\sigma_N
\]

Closed-form expression of \(\bar{\sigma}_N(D)\) is impossible – Approximate equation via asymptotic matching
Generalized Weakest Link Model

Large size limit:

\[
P_f(\sigma_N) = 1 - \exp \left\{ - \int_{V_I} \frac{\mu^m(D)\sigma_N^m(s(\vec{x}))^m}{s_0^m} \frac{dV(\vec{x})}{l_0^2} - \int_{V_{II}} \frac{\sigma_N^m(s(\vec{x}))^m}{s_0^m} \frac{dV(\vec{x})}{l_0^2} \right\}
\]

\[
\bar{\sigma}_N = s_0[\mu^m(D)\Psi_1 + \Psi_2]^{-1/m} \Gamma \left( 1 + \frac{1}{m} \right) \left( \frac{l_0}{D} \right)^{2/m}
\]

Small size limit:

For structures without stress singularities: \( \bar{\sigma}_N \propto (D/D_b)^{-1/r} \).

For structures with weak stress singularities:

\[
\bar{\sigma}_N \propto [\mu(D)]^{-1} (D/D_b)^{-1/r}
\]
Generalized Weakest Link Model

Asymptotic matching:

$$\bar{\sigma}_{N} = \sigma_{0} \left\{ C_{1} \left[ \mu^{m}(D) \Psi_{1} + \Psi_{2} \right]^{-r/m} \left( \frac{D + l_{s}}{l_{0}} \right)^{-2r/m} \exp\left[-\left(\frac{\lambda}{\lambda_{1}}\right)^{2}\right] + \frac{\mu^{-r}(D)D_{b}}{\exp\left[-\left(\frac{\lambda}{\lambda_{2}}\right)^{2}\right]D + l_{p}} \right\}^{1/r},$$

where the weakest link model should vanish. This transition is expected to occur in a very narrow distance due to the singular stress field. However, in the spirit of weakest link model, we should not exclude the region where the radial distance from the notch tip is less than a certain value. Therefore, when evaluating the singular stress at the V-notch tip (Baˇ zant and Xi 1991), we exclude the region where the radial distance from the notch tip is less than a certain value.

At the small-size limit, the entire structure consists of a very small number of RVEs. Therefore, it is expected that the RVEs in the singular stress zone governs the failure of the entire structure. As will be discussed later, the choice of energetic + statistical and statistical terms from the classical Weibull size asymptote of the size can be expressed as $\bar{\sigma}_{N}$.

Energetic + Statistical

Statistical

Energetic

Energetic
Computational Studies
Case 1 — Concrete Beams with a V-Notch

\[ \gamma = 0^\circ, 90^\circ, 120^\circ, 135^\circ, 170^\circ \]

\[ \lambda = -0.5, -0.4555, -0.3843, -0.3264, -0.0916. \]

Deterministic calculations:
— Size effect for structures with strong stress singularities
— Size effect in the small and intermediate size ranges for structures with weak/zero stress singularities
— Damage-plasticity model with crack band model

*Le et al., JEM, 2014*
Simulation Results — I. Load-Deflection Curves

\[ \sigma = \frac{P}{bD} \]

\[ \delta = \frac{\Delta}{D} \]
Simulation Results — II. Stress Profile and Size Effect

Stress profile along the ligament

Size effect surface

Transition from ductile failure to brittle failure

Le et al., JEM, 2014
energetic scaling term

authors only need
authors leave
notch angles,
proposed approximate size effect equation. As mentioned earlier,
the stress singularity becomes weaker, there is a clear change in the
curve for specimens without notches. At the small-size limit, the
this paper, the authors assume that the RVE size
Weibullian part can be easily determined by linear elastic analysis. In
135, and 170
aggregate


generally varies with the notch angle as

\[ \gamma = 0^\circ \]

Type-2 SE

Type-1 SE
Comparison Between Analytical Model and Numerical Simulation
Case 2 — Bimaterial Hybrid Beams

A bimaterial corner may exhibit
1) complex stress singularities;
2) two real stress singularities;
3) one real stress singularities

Assuming weak interface, we consider failure always initiates from the interface though the failure location is random for the case of weak stress singularities.

\[
\bar{\sigma}_N = \sigma_0 \left\{ C_1 [\mu^m(D)\Psi_1 + \Psi_2]^{-r/m} \left( \frac{D + l_s}{l_0} \right)^{-r/m} \exp[-(\lambda/\lambda_1)^2] + \frac{\mu^{-r}(D)D_b}{\exp[-(\lambda/\lambda_2)^2]D + l_p} \right\}^{1/r}
\]

\[
\mu(D) = \left\{ 1 + [(D/D_1)^{-2\lambda_1} + (D/D_2)^{-2\lambda_2} + (D/D_3)^{-\lambda_1-\lambda_2}]^\beta \right\}^{1/2\beta}
\]
Stochastic Simulation of Fracture of Bimaterial Hybrid Beam

Weak interface — stochastic mixed-mode cohesive model

\[ B_i(x) = \zeta(x) \bar{B}_i \]

Steel/unidirectional carbon-epoxy composite

Mixed-mode cohesive element
Comparison Between Analytical Model and Numerical Simulation

Le and Xue., EFM, 2013
Conclusions

1. For structures with strong stress singularities, the size effect is energetic (deterministic). The large-size asymptote is governed by the order of the dominant stress singularity.

2. For structures without stress singularities, the size effect can be explained by the weakest link statistical model.

3. For structures with weak stress singularities, the size effect consists of both energetic and statistical components.

Thank You