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ANALYTICAL RANDOM FIELD-BASED MODEL FOR FRACTURE IN CONCRETE

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Abstract. The paper delivers an analytical model for prediction of peak force in concrete specimens loaded in bending (both notched and unnotched). The model is capable of predicting the statistics of the peak force of beams by computing the extreme values of sliding averages of random strength field. Local strength of the specimen is modelled by a stationary isotropic random field with Gaussian distribution and squared-exponential autocorrelation function. The averaging operation represents the progressive loss in material integrity and the associated stress redistribution that takes place prior to reaching the peak load. Once the (linear) averaging process is performed analytically, the resulting random field of averaged strength is assumed to represent a series of representative volume elements (RVEs) and the global strength is found by solving for the minimum of such an effective strength field. All these operations can be written analytically and there are only four free parameters: the three dimensions of the averaging volume (RVE) and the length of the final weakest-link chain.

The model is verified using detailed numerical computations of notched and unnotched concrete beams simulated by mesoscale discrete simulations of concrete fracture performed with probabilistic distributions of model parameters. The numerical model represents material randomness both by random locations of the largest aggregates and by random fluctuations of material parameters via a homogeneous random field.

1 INTRODUCTION

Concrete fracture behavior is largely influenced by the internal structure consisting of matrix, mineral aggregates and pores. The material heterogeneity leads to complicated inelastic behavior and it must be reflected in the models either by representing the mesostructure directly (mesoscale models, e.g. [14]) or by introduction of internal length parameter in some way (e.g. the crack band model, gradient or nonlocal models). The effective width of the fracture process zone (FPZ), sometimes called the characteristic length, is estimated in [2, 3] as $l_{\rm d} = (2.7 \sim 3)d_{\rm a}$, where $d_{\rm a}$ is the maximum aggregate size of concrete.

The present paper employs a mesoscale discrete model with direct representation of the aggregate size distribution. The main advantages of this class of models are the discrete and oriented character of cracks, capability to represent transition from distributed to localized fracture or occurrence of transversal stresses. The characteristic length emerges from contribution of two sources, the constitutive relationship between discrete units and geometrical properties of them. We utilize here a simplified version of the lattice discrete particle model developed in [4–6,12]. The numerical model used

here is the same as the one presented in [9]. Therefore, only very limited space is devoted to description of the model.

The mesoscale models are incomplete unless they include also spatial variability in material properties that arise during the production process (mixing, drying, etc.) and service life. These spatial fluctuations are often modeled via random fields [9, 10, 13, 16, 17]. Even though there are several possible sources of random fluctuations, it is considered here that all of them can be approximately described by a single homogeneous random field. As argued in [16], such a random field introduces its own characteristic length scale provided in a form of its correlation length denoted l_ρ.

The model is employed to simulate three-point bending of concrete beams with and without a notch. The applied random fields are generated with various correlation lengths spanning from $l_\rho \to 0$ (independently sampled random variables) up to the infinitely long correlation length $l_\rho \to \infty$ for which the realizations are random constants over the domain and therefore the whole structure shares the same value in a single realization.

The main part of this contribution is description of analytical model capable of reproducing the strength statistics obtained from the probabilistic random discrete particle model. The classical approach to statistical strength of materials is the classical Weibull theory. The deviation of the present theory from the classical Weibull theory is twofold: (i) we allow for local stress redistribution which introduces a "deterministic length scale" similar to the nonlocal averaging operator and, (ii) we also consider spatial dependence of local strengths by introducing a "statistical length scale" in the form of autocorrelation length. Our model considers the definition of random field describing the spatial fluctuation of local random strengths and by performing the averaging operation and the subsequent computation of extremes of the averaged field, the statistics of the random flexural strength is accurately predicted.

2 DISCRETE MODEL

The model is a simplified version of [5]; the full formulation accounts also for confinement and involves more free material parameters. Our model is static, the solution proceeds in loading steps by iterations until static equilibrium is found. The location and size of the grains are randomly generated in computer based on a user-supplied sieve curve and on the total aggregate volume fraction. The discrete units are obtained by tessellation respecting the layout of grains and they are ideally rigid. Constitutive relationship is defined on contacts between them (*facets*) in terms of one normal and two mutually orthogonal tangential displacement jumps. The four governing parameters of the constitutive law are: (i) the elastic modulus, E_0, (ii) tangential/normal stiffness ratio α, (both controlling the elastic behavior), and (iii) meso-level tensile strength, f_t, and (iv) meso-level fracture energy in tension, G_F, (both controlling the inelastic behavior). Details regarding the simplified constitutive formulation are published in [8].

The basic version of the model itself provides a random response due to its random mesostructure: there is randomness in the positions and sizes of the discrete bodies (and thus in the dimensions and orientation of the facets). However, we will refer the model without random fluctuations of material parameters to as the *deterministic* model hereinafter, and the mean value and the standard deviation of the peak load provided by deterministic model will be denoted μ_d and δ_d.

The second fundamental component is an additional random spatial fluctuation of material parameters. The deterministic model combined with the randomization will be called the *probabilistic* model, and the corresponding mean and standard deviation of the peak load will be denoted μ_p and δ_p, respectively.

In the probabilistic model, the two material parameters governing fracture behavior (f_t and G_F) are considered to vary randomly in space according to single homogeneous random field $h(\boldsymbol{x})$. The probabilistic distribution function

$F(h)$ of the random field is assumed Gaussian with a left Weibullian tail [1]. The mean value is one, $\mu_h = 1$, and a the standard deviation, δ_h, is obtained from comparison with experimental data [11]. The correlation structure of the field is given by squared exponential (correlation) function. A detailed description of the probabilistic model and an efficient method for generation of the samples can be found in [9].

In order to keep Irwin's characteristic length constant, we chose the following relation between random variable h and material parameters [17]

$$f_{\mathrm{t}}(\boldsymbol{x}) = \bar{f}_{\mathrm{t}} h(\boldsymbol{x}), \quad G_{\mathrm{F}}(\boldsymbol{x}) = \bar{G}_{\mathrm{F}} \left[h(\boldsymbol{x})\right]^2 \quad (1)$$

resulting in constant $l_{\mathrm{ch}}(\boldsymbol{x})$ in all facets

$$l_{\mathrm{ch}}(\boldsymbol{x}) = \frac{E G_{\mathrm{F}}(\boldsymbol{x})}{\left[f_{\mathrm{t}}(\boldsymbol{x})\right]^2} = \frac{E \bar{G}_{\mathrm{F}}}{\bar{f}_{\mathrm{t}}^2} = \bar{l}_{\mathrm{ch}} \quad (2)$$

This scaling of parameters leads to linear dependence of the structural strength on h. The sequence of events (such as damage evolution etc.) in the model is exactly the same for any positive value of h. Therefore, the interpretation of the results obtained with the probabilistic model is easier than with other relationships between random local f_{t} and G_{F}.

Identification of model parameters is based on experimental campaign described in [11] with the maximum grain diameter 10 mm. The identified parameters used in this contribution are: $E_0 = 27\,\mathrm{GPa}$, $\alpha = 0.24$, $\bar{G}_{\mathrm{F}} = 0.25\,\mathrm{J/m^2}$, $\bar{f}_{\mathrm{t}} = 2\,\mathrm{MPa}$ and, the random field is set via two parameters: $\delta_h \approx 0.14$ and $l_\rho \approx 100\,\mathrm{mm}$. The Weibull modulus is assumed to be 24 and the grafting probability is chosen $P_{\mathrm{gr}} = 10^{-3}$ (meaning that the values of h that are Weibull-distributed occupy only the far left tail up to probability 10^{-3}).

2.1 Simulated beams and load capacity

The interplay between mechanical and probabilistic components of the model was studied by performing simulations of beam loaded in three-point-bending with various realizations of the random fields. To estimate the mean value and standard deviation of the peak load, $N_{\mathrm{sim}} = 100$ realizations of every model variant is performed. The realizations differ by random mesolevel arrangements and additionally by the random field realizations (if used). The reference solution is the one with the "deterministic" model. This reference solution is compared with simulations obtained with various correlation lengths l_ρ. Since the deterministic model already has a portion of variance in the results (crack paths, peak load) due to random arrangements of numerical aggregates, we keep the sets of generated mesostructures unchanged when extending the model with random parameters of the constitutive law.

Several values of l_ρ ranging from $l_\rho \to 0$ (mutually independent random field values) to $l_\rho \to \infty$ (random values identical in the whole volume for each sample) are considered. When in between, a sample of random field with corresponding correlation length is generated and applied.

The second probabilistic parameter subjected to change is the standard deviation, δ_h, (since $\mu_h = 1$, δ_h is equal to the coefficient of variation) of the random field $h(\boldsymbol{x})$. It changes the intensity with which the randomness is applied. The basic value approximately identified on experimental data is $\delta_h = 0.14$ and three more additional values are considered in the study so that the complete list of values read: $\delta_h = \{0,\, 0.035,\, 0.07,\, 0.14,\, 0.28\}$. The $N_{\mathrm{sim}} = 100$ different samples (realizations) of random fields for each level of variability (δ_h) are similar such that they are simply re-scaled by keeping the same unit mean value.

The simulated beams with or without a central notch are loaded in three-point bending. The dimensions of the specimen are: depth $D = 150\,\mathrm{mm}$, length $L = 720\,\mathrm{mm}$, span $S = 600\,\mathrm{mm}$ and thickness $b = 40\,\mathrm{mm}$. The notched variant have a central notch of length $a_0 = 75\,\mathrm{mm}$, across half of its depth. Only the central part of the beam of size is represented by the discrete model ($450\times120\,\mathrm{mm^2}$ in the unnotched and $150\times85\,\mathrm{mm^2}$ in the notched case), the surrounding material is modeled by linear

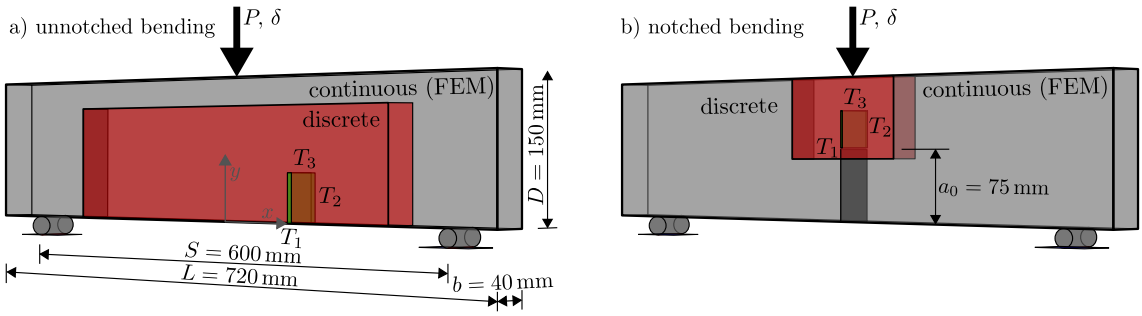

Figure 1: Beams geometry and coordinate system - a) unnotched and b) notched beams loaded in three-point-bending. The rectangular cubes of dimensions T_1, T_2 and T_3 are three length parameters of the analytical model.

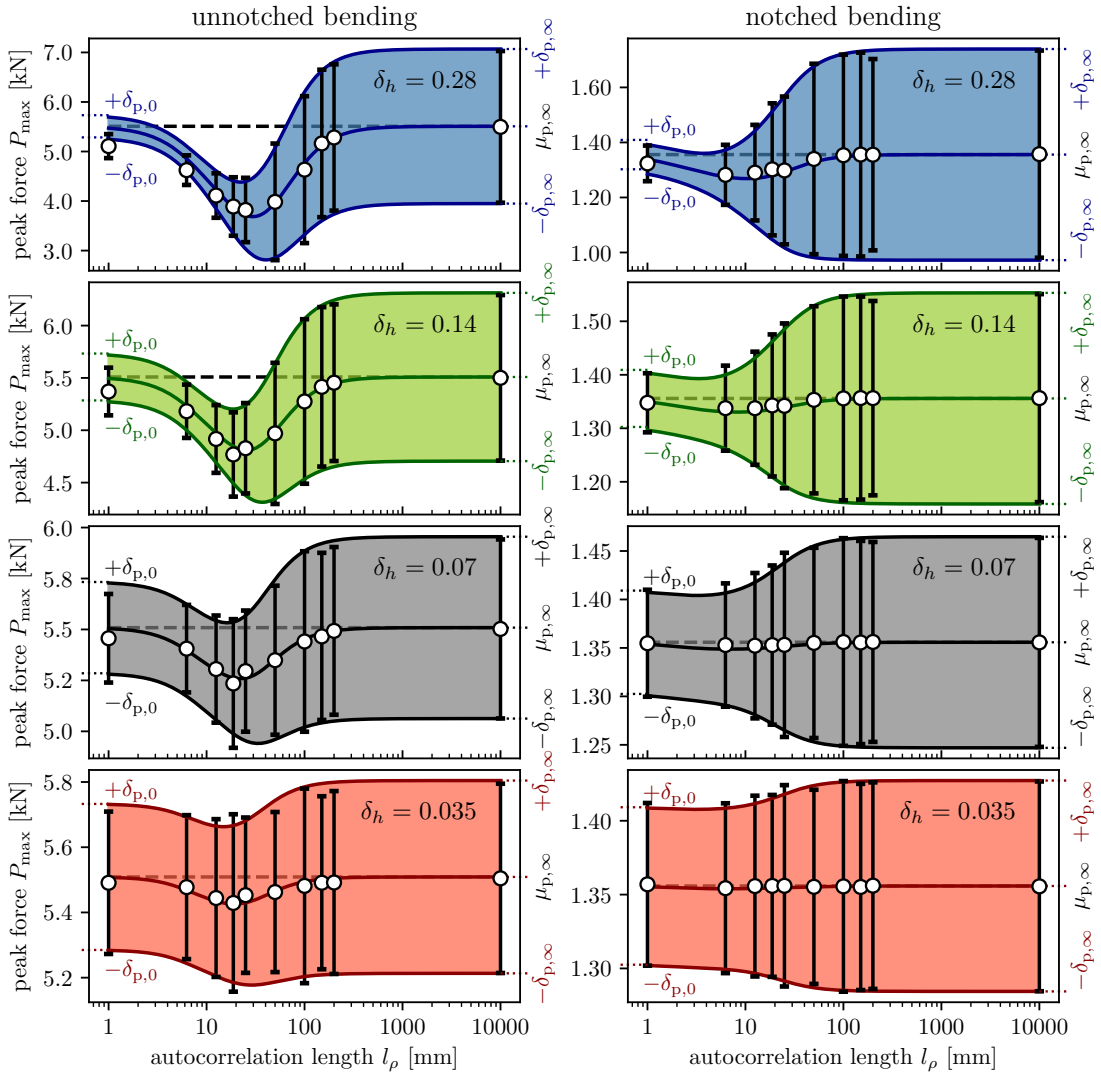

Figure 2: The mean value and standard deviation of the maximum load computed on unnotched (left) and notched (right) beams loaded in three-point-bending using the *probabilistic* discrete model, denoted by empty circles and errorbars. The curves show results obtained with the analytical model based on extremes of averaged random fields. The dotted lines on the left represent the mean value and standard deviation of peak loads obtained with the deterministic discrete model and on the right the theoretical mean and standard deviation for $l_\rho \to \infty$ according to Eq. (4), denoted as $\mu_{\mathrm{p},\infty}$ and $\delta^{\delta_h}_{\mathrm{p},\infty}$.

elastic finite elements. The beam geometry is shown in Fig. 1ab. The maximum loads cal-

culated using both the deterministic and probabilistic model with all the selected correlation lengths and variances are presented in Fig. 2 left for the unnotched beams and Fig. 2 right for the notched beams.

The dependence of the mean peak load on the correlation length, l_ρ, is different for the unnotched and notched beams, compare Figs. 2 left and right. The mean peak load is almost insensitive to l_ρ in the case of notched beams while in unnotched beams, there is a clear value of correlation length for which the peak load attains its minimum. We first present simple ideas about what governs the peak load. The sequence of cracks for an increasing deflection depends on the positions and orientations of the facets and on their material parameters. The parameters include also the multipliers – random variables that represent the discretized random field $h(\boldsymbol{x})$. The loading process involves redistribution of local forces among the surrounding contacts. Not one, but several contacts/facets within the FPZ need to be at least partially damaged before a crack may propagate. All the contacts within the current FPZ are therefore involved in certain *averaging* of their random capacities. The volume of averaging is the volume of the FPZ with size denoted here as the deterministic length, l_d. Therefore, some weighted average over the volume related to deterministic length is the governing variable determining the structural strength.

When $l_\rho \to \infty$, the effect of the additional variability in probabilistic models is very easy to predict. The average strengths of the deterministic models is equal to the average strengths of the probabilistic models. The variance, though, increases when using the random field. This can be analytically predicted due to the selected way of simultaneous randomization of local f_t and G_F, see Eq. (1). The application of the flat multiplier h produces loading forces that are just multiplied by a random factor h (only when constant over the whole structure: $l_\rho = \infty$). The sequence of damage evolution of individual contacts and therefore also the final crack patterns of the probabilistic models are strictly identical to the those obtained with the deterministic models (for the same grain layouts). The forces at which these events occur are, however, linearly dependent on the value of random variable $h(\boldsymbol{x}) = h$. There are therefore two *independent* sources of response variance in the probabilistic model: the variance inherent to the deterministic model and the variance due to random multiplier h. Due to the independence, one can simply calculate the mean and the standard deviation of the peak load of the probabilistic model ($\mu_{\mathrm{p},\infty}$ and $\delta_{\mathrm{p},\infty}$) as a *product of two independent random variables*: the random peak load in the deterministic model and h. The mean and standard deviation of the peak load for $l_\rho \to \infty$ therefore read

$$\mu_{\mathrm{p},\infty} = \mu_\mathrm{d}\,\mu_h = \mu_\mathrm{d} \tag{3}$$

$$\delta_{\mathrm{p},\infty} = \sqrt{\delta_\mathrm{d}^2\delta_h^2 + \mu_\mathrm{d}^2\delta_h^2 + \delta_\mathrm{d}^2\mu_h^2} \tag{4}$$

where we consider the unit mean value $\mu_h = 1$. These analytically predicted values are shown in Fig. 2 on the right hand side of each graph. The horizontal lines agree with the numerically computed values. We conclude that when FPZ size is much lower compared to correlation length ($l_\mathrm{d} \ll l_\rho$), there is no effect of the additional randomness on the evolution of the fracture process, only the forces are scaled with the random multiplier, h.

On the other extreme when the correlation length $l_\rho \to 0$, the probabilistic models deliver the mean value and standard deviation of the peak load almost identical to those obtained with the deterministic models. This is true for both, notched and unnotched bending. The unit-mean random field with basically symmetrical distribution seems to randomly modify the properties of the contacts which are variable anyway – due to random orientations of the facets. This additional randomness averages out within the FPZ. We conclude that in the case of the extremely short correlation length with respect to the deterministic characteristic length ($l_\rho \ll l_\mathrm{d}$), randomness has theoretically no effect on the average strength or its variance. The probabilistic model behavior tends to the behav-

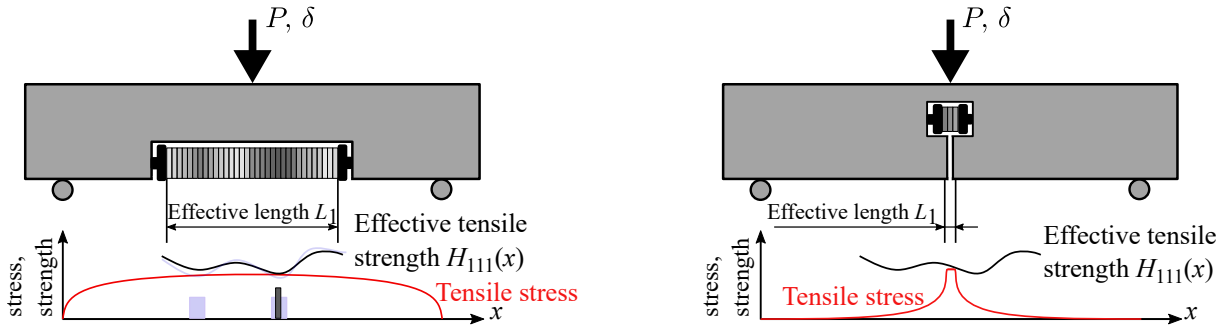

Figure 3: Illustration of the transformation of the 3D discrete model into a 1D model of a chain of random RVEs.

ior of the deterministic model and therefore

$$\mu_{\mathrm{p},0} = \mu_{\mathrm{d}}, \quad \delta_{\mathrm{p},0} = \delta_d \tag{5}$$

In our calculations, the finite size of discretization implies that the number of averaged independent variables is finite and not very high. Consequently, the averaging process does not remove randomness completely and a weak effect of the weakest-link type exists: to some extent a weaker spot may be found that leads to slight decrease in the mean value of the peak load compared to the deterministic model, see Fig. 2 left. This effect is largely restricted in notched beams, where strong stress concentrator limits possible sampling of FPZ locations.

The paper continues with the proposed analytical model capable of reproducing the dependencies of peak loads on the two parameters varied: the variance and the correlation length of the random field.

3 ANALYTICAL RANDOM FIELD-BASED MODEL FOR FRACTURE

In this section we present an analytical model capable of reproducing the results obtained with the randomized mesoscale discrete model described above. The model can replace the computationally intensive simulations and provides an insight into the elementary mechanisms; especially the role of random field in behavior of the model. We elucidate the role of the autocorrelation length and variance of the field and interaction with the inherent randomness coming from the disordered internal mesostructure.

The sketch in Fig. 3 illustrates the idealization used in the analytical model. The unnotched beam (on the left) is transformed into a weakest-link model, i.e. it is assumed that the strength of the beam is dictated by the weakest element of (generally dependent) sub-volumes. These sub-volumes are coupled in series. They are refereed to as the *representative volume elements* (RVEs) here and they represent potential macrocracks the failure of which leads to failure of the whole beam. The weakest RVE thus corresponds to the flexural strength (peak load) of the beam. Each of such RVEs is assumed to be a rectangular cube. These cubes are assumed to have identical dimensions: the length T_1 measured along the beam span, depth T_2 measured along the vertical axis and the width T_3, see the illustration of one such RVE in Fig. 1. Since the crack front has to pass through the whole width of the beam, T_3 equals the beam width. The lengths T_1 and T_2 are two length parameters that must be obtained either by fitting of the model or, better, directly from a nonlinear analysis. They represent the width and length of the macrocrack that forms at the peak load, see Fig. 4. Numerous analyses with the discrete model suggest that these lengths are not much dependent on the parameters of the random field; they are dependent on the deterministic model and on the stress field. Values of the four model parameters identified are summarized in Table 1. The width T_1 is mostly controlled by the maximum aggregate size, $d_{\mathrm{a}} = 10$ mm. We expect also the influence of fracture energy of facets and the stress field. In the case of unnotched beams, we have found that

the crack "planes" are somewhat tortuous and the therefore the RVE thickness T_1 is wider than in the case of notched beams that are resisting to strongly localized stress fields. The depths T_2 of RVEs are similar in unnotched and notched beams.

Table 1: Four length parameters of the analytical model

Length [mm]	Unnotched	Notched
T_1 (averaging length)	10	2
T_2 (averaging depth)	50	40
T_3 (averaging width)	40 (= beam width)	
L_1 (effective length)	$30\ \delta_{\dot{H}_T}$	$1\delta_{\dot{H}_T}$

It is assumed that each RVE contains many facets (microbonds contributing to the mechanical integrity of material) whose strengths and orientations are random. Upon increasing the load on the RVE, the microbonds start to break or soften and the stress redistributes to the neighborhood. The results of this redistribution process up to the peak load is here represented simply as *stress averaging* over the RVE volume. The important information gained from the analyses performed with the probabilistic meso-scale model is that the volume within which the massive redistribution right at the peak load takes place, is almost independent of the parameters of the random field.

To reach the maximum force of each RVE, we assume that the strengths of the local microbonds are simply averaged. Therefore, we define the strengths of each RVE as the average over the volume $T_1 \times T_2 \times T_3$. As shown below, when considering the strengths of individual bonds being described by a random field, the parameters of random strength of each RVE (an averaging window) can be predicted analytically, together with spatial correlation of these averaged strengths. It is assumed that, effectivelly, the stress is constant over one RVE. Therefore, the three-dimensional problem is transformed to one-dimensional problem (a chain) with effective strength variable along the beam span, see Fig. 3.

Comparing the idealizations for the unnotched and notched beams in Fig. 3, one can notice the difference in the length of the "chain" of RVEs. This effective length, L_1. is the last parameter of the model. It is the extent of the zone within which cracks can appear in the beam. This length is dependent on the stress field, on the parameters of the strength random field and also on the RVE dimensions. The dependence on the stress field is obvious: unnotched beams have, at the peak load, very long zone of almost constant stress. This mild functions develops thanks to the redistribution capacity of the material. In the case of notched beams, the length L_1 is much shorter due to localized tensile stress field.

In both types of beams, though, the length L_1 is influenced by the random field of the effective (averaged) strength. It is because both, the autocorrelation length and the variance influence the random *gradient* of the wavy function describing the RVE strengths along the beam. Therefore, we take the effective length L_1 as proportional to the standard deviation of the first derivative of the effective random strength process, see the last row in Tab. 1. The length L_1 is therefore a function of the variance of the random field (δ_h^2), the autocorrelation length of the local random field l_ρ, and finally the averaging volume of RVE. Fig. 5 shows the computed chain lengths L_1.

Since the random fields of local strengths are almost entirely Gaussian, the effective strength of individual potential RVEs is Gaussian, too. This is due to the averaging operation that suppresses the tails and, by the virtue of the Central limit theorem, the Gaussian core spreads even wider. Therefore, it suffices to focus on averaging and extremes of Gaussian random fields. So, to describe a strength of such a structure with redistribution, one can study the purely mathematical problem of distribution of a minimum of a moving average along a line. The moving average represents effective strength of the parent 3D random field of local strength.

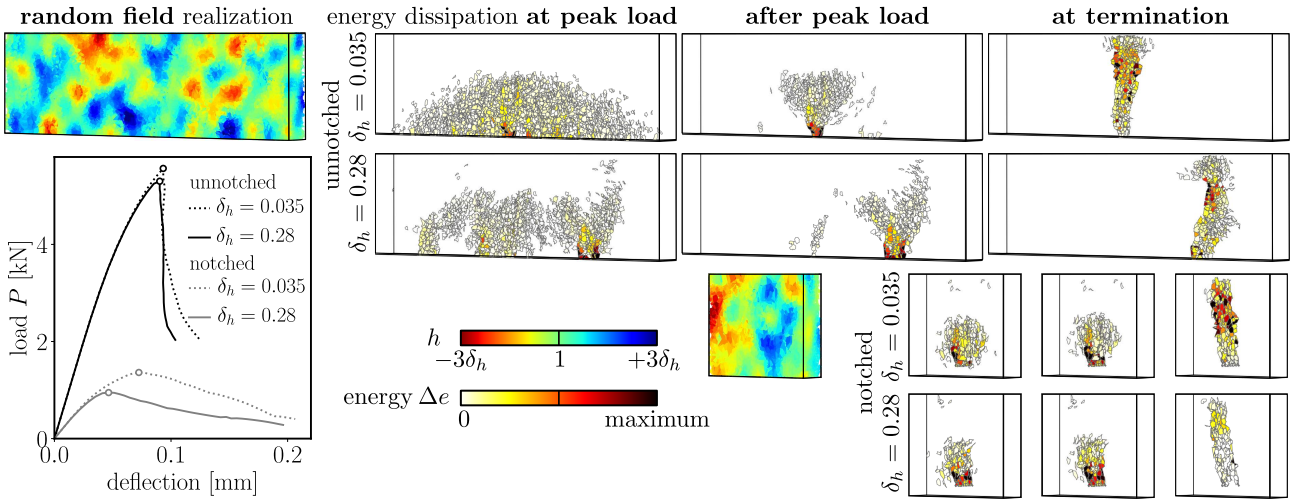

Figure 4: Distribution of energy dissipation within single computational step at inter-particle contacts of one realization of the *probabilistic* model ($l_\rho = 25$ mm) at the peak load, one step after the peak load and at termination of unnotched and notched beam. Two variants of δ_h are shown in two rows, the top row was computed with $\delta_h = 0.035$ while the bottom row with $\delta_h = 0.28$.

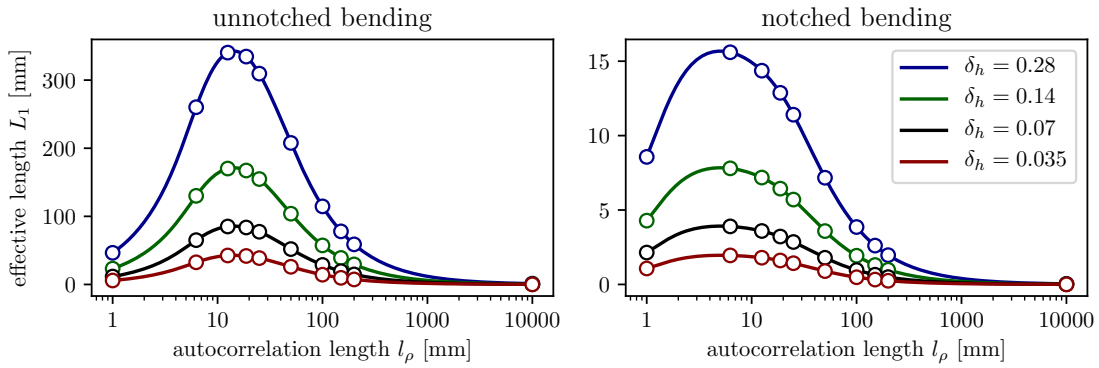

Figure 5: Effective length L_1 as a function of autocorrelation length and variance.

3.1 Local strength as a random field

Consider a Gaussian random field $h(\boldsymbol{x})$ where the spatial coordinate $\boldsymbol{x}$ is defined in three dimension (the beam volume). It is a random function depending on the spatial coordinate, $\boldsymbol{x}$. We consider h to be a stationary random field in the strong sense. Thus we deal with a random field fully defined by the constant mean value $\mu_h = 1$, standard deviation δ_h and autocorrelation function $\rho(\tau; l_\rho)$, where τ is the lag (distance between two spatial points). This function is considered to be separable isotropic Gaussian (squared-exponential) autocorrelation function

$$\rho(\tau; l_\rho) = \exp\left[-\left(\frac{\tau}{l_\rho}\right)^2\right], \quad |\tau| \geq 0 \qquad (6)$$

The separability means that the correlation between two different random variables $h(\boldsymbol{x}_1)$ and $h(\boldsymbol{x}_2)$ is a product of autocorrelations that depend solely on distances (projections) along individual dimension of $\boldsymbol{x}$.

3.2 Moving average of a random process

Assume now a one-dimensional random field, a random process $h(x)$. Let us also consider another random field, $H_T(x)$, a *moving average* of the random process $h(x)$, defined as

$$H_T(x) = \frac{1}{T}\int_{x-T/2}^{x+T/2} h(u)\mathrm{d}u \qquad (7)$$

where T denotes the averaging length. The relationship between the original random field $h(x)$

and the averaged $H_T(x)$ is illustrated in Fig. 6 left using a single realization.

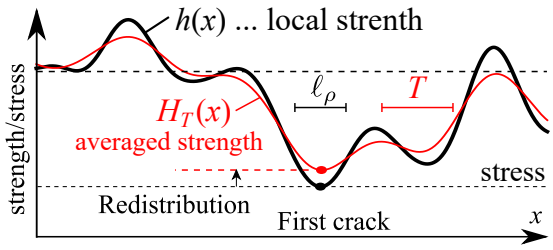

Figure 6: Effect of moving average on tensile strength.

We now analyze the properties of the averaged random field. The mean value is not affected by the averaging operation: $\mu_T = \mu_h$. The covariance function is affected such that the point variance is reduced

$$\mathrm{Var}[H_T] \equiv \delta_T^2 = \gamma(T)\,\delta_h^2 \tag{8}$$

where $\gamma(T)$ is a non-negative *variance function* of $h(x)$, i.e. a function that measures the point variance reduction. It is related to the autocorrelation function $\rho(\tau)$ as [15]

$$\begin{aligned}\gamma(T) &= \frac{1}{T^2}\int_0^T\int_0^T \rho(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{T}\int_0^T \left(1-\frac{\tau}{T}\right)\rho(\tau)\mathrm{d}\tau\end{aligned} \tag{9}$$

Substituting the selected autocorrelation function (Eq. 6) into Eq. (9) yields a closed-form expression for the variance reduction function

$$\begin{aligned}\gamma(l_\rho,T) = \left(\frac{l_\rho}{T}\right)^2\Bigg\{&\sqrt{\pi}\frac{T}{l_\rho}\mathrm{Erf}\left(\frac{T}{l_\rho}\right) \\ &+\exp\left[-\left(\frac{T}{l_\rho}\right)^2\right]-1\Bigg\}, \quad l_{\mathrm{ch}}\geq 0\end{aligned} \tag{10}$$

where Erf is the Error function. This variance function is a smooth transition between a constant for no averaging ($\gamma(l_\rho,0)=1$) and tends to zero as $T\to\infty$ (ergodicity in the mean). For very large values of averaging length T, the variance reduction function becomes approximately inversely proportional to T: $\gamma(l_\rho, T\to\infty)\to\sqrt{\pi}l_\rho/T$. The crossover averaging length at which the transition from a constant asymptote towards this asymptotic inverse proportionality takes place is defined as the position of the intersection of the two asymptotes

$$T^\star = \sqrt{\pi}l_\rho \tag{11}$$

The averaging operation decreases the local variance. This reduction is compensated by an increase in the *scale of fluctuation* is such a way as to keep their product invariant. The scale of fluctuation is important when determining the "equivalent number of independent observations", $L/\ell_{\rho,\mathrm{eff}}$, contained in a sampling interval L. Indeed, $\ell_{\rho,\mathrm{eff}}$ plays the role of an effective autocorrelation length of the averaged random field. The scale of fluctuation of H_T reads

$$\theta_{\mathrm{ch}}(l_\rho,T) = \ell_{\rho,\mathrm{eff}} = \frac{l_\rho}{\gamma(l_\rho,T)} \tag{12}$$

3.3 3D averaged random field

We now extend the concept into three dimensions. The local strength is a stationary Gaussian random field $h(\boldsymbol{x})$ with a constant mean value μ_h, standard deviation δ_h and separable isotropic autocorrelation function

$$\rho(\tau_1,\tau_2,\tau_3) = \Pi_{i=1}^3\rho(\tau_i) \tag{13}$$

where τ_i is the projection of the distance of two points onto axis x_i.

Consider now the 3D-averaged random field

$$\begin{aligned}H_{1,1,1}(\boldsymbol{x}) = &\frac{1}{T_1\cdot T_2\cdot T_3} \\ &\int_{x_1-T_1/2}^{x_1+T_1/2}\int_{x_2-T_2/2}^{x_2+T_2/2}\int_{x_3-T_3/2}^{x_3+T_3/2} h(\boldsymbol{t})\mathrm{d}\boldsymbol{t}\end{aligned} \tag{14}$$

where the averaging window is the RVE box $T_1\times T_2\times T_3$ discussed above. The resulting random field $H_{1,1,1}(\boldsymbol{x})$ is Gaussian with the (intact) unit mean value, and its (point) standard deviation is reduced to

$$\delta_{1,1,1} = \sqrt{\gamma_{1,1,1}(T_1,T_2,T_3)}\,\delta_h \tag{15}$$

where the variance function is a product of individual variance functions obtained by averaging

over individual dimensions (recall the separability property of autocorrelation in Eq. (13)

$$\gamma_{1,1,1}(T_1, T_2, T_3) = \gamma(T_1) \cdot \gamma(T_2) \cdot \gamma(T_3) \quad (16)$$

The autocorrelation length of $H_{1,1,1}(\boldsymbol{x})$ along x_1 can be calculated using $T = T_1$ in Eq. (12).

One might be interested in analyzing the *derivative of the averaged random process* $H_{1,1,1}(\boldsymbol{x})$ along the first axis, x_1, denoted as $\dot{H}_T(\boldsymbol{x})$. Since $H_{1,1,1}(\boldsymbol{x})$ is stationary, any random variables $H_{1,1,1}(\boldsymbol{x}_0)$ and $\dot{H}_T(\boldsymbol{x}_0)$ are uncorrelated. The mean value $\mathrm{E}[\dot{H}_T()] = 0$. The variance of the derivative is a constant function along axis x_1 and its value reads

$$\delta^2_{\dot{H}_T} = \frac{2\delta^2_{1,1,1}}{T_1^2 \gamma(l_\rho, T_1)} \left[1 - \rho(T_1; l_\rho)\right] \quad (17)$$

Finally, the standard deviation of the derivative of the averaged process is simply

$$\delta_{\dot{H}_T} = \sqrt{\delta^2_{\dot{H}_T}} \quad (18)$$

The averaged random field that serves as the effective strength multiplier of each RVE is now fully characterized.

3.4 Minimimum of a stationary Gaussian random process over a finite interval

Consider a Gaussian one-dimensional random field $H_{1,1,1}(x)$ (a random process defined over a continuous time/distance x) that is stationary in the strict sense, with nonzero mean, $\mu_{1,1,1}$, and standard deviation, $\delta_{1,1,1}$. We are interested in the distribution of extremes of realizations for a closed interval of $x_1 \equiv x$. In this work, we build our model on an elegant approach expressed via Theorem 2 in [7].The distribution function of the minimum is the probability that the sample function remains below a given threshold k over the interval of length L_1

$$G(k, L_1) = 1 - \left[1 - \Phi(u)\right] \exp\left(-\frac{\varphi(u)}{1 - \Phi(u)} \lambda L_1\right) \quad (19)$$

where $\Phi(k) = \int_{-\infty}^{k} \varphi(t)\mathrm{d}t$ is the standard Gaussian cumulative distribution, $\varphi(t) = \exp[-(t^2/2)]/\sqrt{2\pi}$ is the standard Gaussian density and $u = (k - \mu_{1,1,1})/\delta_{1,1,1}$. The corresponding probability density function g reads

$$g(k, L_1) = \frac{1}{\sigma}\left[\varphi(u)\left(1 - \lambda L_1 u\right) + \lambda L_1 \frac{\varphi^2(u)}{1 - \Phi(u)}\right] \exp\left(-\frac{\varphi(u)}{1 - \Phi(u)} \lambda L_1\right) \quad (20)$$

Parameter λ has the dimension of inverse distance and depends on the second derivative of the autocorrelation function of the random field at zero lag

$$\lambda = \sqrt{\frac{-\rho''(0)}{2\pi}} = \frac{1}{l_\rho \sqrt{\pi}} \quad (21)$$

According to our extensive numerical simulations, the above approximation to the minimum provides highly accurate results even for short intervals, L_1, and thresholds u close to the mean value of the stationary process $H_{1,1,1}(x)$.

Finally, given the density function $g(k, L_1)$, the mean value and standard deviation, $\mu_{\min}$ and $\sigma_{\min}$ of the random minimum can be evaluated simply using their definitions

$$\mu_{\min} = \int_{-\infty}^{\infty} k\, g\,(k, L_1)\,\mathrm{d}k \quad (22)$$

$$\sigma^2_{\min} = \int_{-\infty}^{\infty} k^2\, g\,(k, L_1)\,\mathrm{d}k - \mu^2_{\min} \quad (23)$$

3.5 Application to the random discrete model

The above analysis assumed that the local strength variability is dictated just by the random field, i.e. that the "deterministic" model has no variance. If this is true, the above-described model of averaged random field can readily be applied and $\mu_{\min}$ and $\sigma_{\min}$ are the mean value and standard deviation of beam strength.

In this paper, however, we used the unit-mean random field $h(\boldsymbol{x})$ only as *multiplier* of local bond strengths. Therefore, the two moments in Eqs. (22) and (23) are multipliers of

strengths obtained using the deterministic discrete model. Both independent models deliver random strengths and so the final strength must be treated as a product of two independent random variables. Therefore, the resulting mean value and standard deviation of flexural strength is computed using

$$\mu_{\mathrm{p},\rho} = \mu_{\mathrm{d}}\,\mu_{\min} \tag{24}$$

$$\delta_{\mathrm{p},\rho} = \sqrt{\delta_{\mathrm{d}}^2\sigma_{\min}^2 + \mu_{\mathrm{d}}^2\sigma_{\min}^2 + \delta_{\mathrm{d}}^2\mu_{\min}^2} \tag{25}$$

It is easy to check that this result has Eqs. (3, 4 and 5) as the asymptotic solutions for both, the zero autocorrelation length (independent strengths) and infinite autocorrelation length (sample paths are random constants). Fig. 2 presents comparisons of these analytical predictions with the results obtained with the probabilistic discrete models. Both the mean values and standard deviations are matched very well.

In *notched* simulations, the average peak load is found to be almost insensitive to the spatial variability in material parameters. The reason is that the stress concentration is so severe that the crack is forced to propagate from one specific location (the notch tip) and the spatial variability in material parameters is not sufficient to change the location of dissipative processes. However, the standard deviation of the peak load decreases with the decrease in the autocorrelation length due to averaging of the fluctuations within the active zone. The size and shape of the active zone is independent of the applied autocorrelation length.

In *unnotched* beams, the average values of the peak load in probabilistic models with both short and long autocorrelation lengths are approximately equal to the one obtained with deterministic model. However, the peak load exhibits a clear downtrend when the autocorrelation length approximately equals the size of the FPZ (or the active zone). The drop in the average strength depends on the standard deviation of the random field. The minimum of the mean strength occurs when the autocorrelation length is close to the internal material length. Such an autocorrelation length enables the structure to sample the position of fracture process zone inside the weakest spot but averaging within the fracture process zone is not yet so severe to filter the randomness out. The standard deviation follows the same trend as in the notched case.

4 CONCLUSIONS

The paper delivers an analytical probabilistic model in which the local strength of material is modelled by a stationary isotropic random field with Gaussian distribution and squared-exponential autocorrelation function. The mean value is kept one and there were two parameters of the random field we have varied: the variance of the field and the autocorrelation length.

The proposed model takes into account progressive loss in material integrity and the associated stress redistribution that takes place prior to reaching the peak load. This stress redistribution is influenced by both the material parameters and the specimen geometry and loading that dictate the stress field. These nonlinear processes are considered in the analytical model by simply averaging the local strength within a volume representing the decisive region. The mechanics is thus reduced to merely averaging the local random strengths within all possible RVEs by performing a moving average operation. Once the (linear) averaging process is performed analytically, the resulting random field of local strength is assumed to represent a series of RVEs (the weakest-link model) and the global strength is found by solving for the minimum of such an effective random field. All these operations can be written analytically and there are only four model parameters: the three dimensions of the averaging volume (RVE, one parameter is the beam width) and the length of the final weakest-link chain.

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