Fracture Mechanics of Concrete Structures,
Proceedings FRAMCOS-2, edited by Folker H. Wittmann,
AEDIFICATIO Publishers, D-79104 Freiburg (1995)

ON AN EFFICIENT NEW NUMERICAL METHOD FOR RESOLUTION OF THE FRICTIONLESS CONTACT PROBLEMS WITH A VARIATIONAL INEQUALITY APPROACH

A. Noune, R.M. Courtade and H. Benkhira
Laboratoire Mécanique et Matériaux, IUTA Génie Civil
Villeurbanne, France.

Abstract
This work presents a numerical solution for frictionless contact problems and its application to a practical example. The numerical solution is obtained by the finite element discretisation of the variational inequality. This inequality is related to the extremum principle. The numerical method is applied for any contact body geometry. The presented example illustrates the method efficiency. This method allows to detect the contact area and the pressure acting on it.

1 Introduction

Recently, the mathematical formulation dealing with the structural problems including the unilateral stresses was extensively developed by several authors. Especially, an important category of these problems is the contact without friction between two elastic bodies.

In 1881, the french engineer C.A. Coulomb published his "Théorie des machines simples" in which he presented his well known law of friction. The classical Coulomb law of static dry friction, of course, asserts that relative sliding between two bodies in contact along plane surfaces will occur when the shear force parallel to the plane reaches a critical value proportional to the normal force pressing the two bodies together. The constant of proportionality is called the coefficient of friction.

In 1933, A. Signorini studied the general problem of equilibrium of an elastic body in contact without friction with a rigid foundation. Also, in 1964, G. Fichera presented a rigourous analysis of this phenomenon, and he introduced for the first time the variational formulation of such a problem.

Therefore, this study is focused on the resolution of the contact problem without friction in elasticity and small deformation. These phenomena are characterised by locally non-linearity at the contact surface. These non-linearity implies the inequality variational problems.

2 Formulation of the contact problems

We consider here the Signorini problem of contact of a linearly elastic body with a rigid foundation along which non linear contact condition exists (Zhong and Mackerle,1992). We begin our analysis by considering a linearly elastic body, occupying a domain Ω of R^3 (of usual reference (o,x,y,z)). The boundary Γ of the body is assumed to consist of three disjoint parts, Γ_u, Γ_σ and Γ_c, where Γ_u and Γ_σ are the parts of the boundary on which the displacements and tractions are prescribed, respectively, and Γ_c is the eventual contact area ; i.e. Γ_c is a portion of the boundary that contains the material surface which comes in unilateral contact with a rigid foundation during the application of loads (see fig 1). The external forces on the body consist of a prescribed body force field intensity **F** per unit volume and of surface tractions of intensity **t** per unit surface area. We shall assume that Γ_u is perfectly fixed, so that

$$u = 0 \quad \text{on } \Gamma_u \tag{1}$$

On Γ_σ, we shall have

$$\sigma_{ij}(u).n_j = t_i \quad \text{on } \Gamma_\sigma \tag{2}$$

where $\sigma_{ij}(u)$ is the stress produced by u and n_i are the components of the unit outward normal n to Γ.

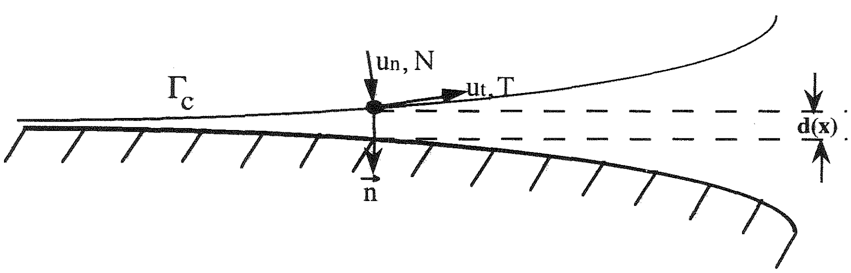

Figure 1 Contact between elastic body and rigid foundation.

The components of the displacements and traction are

$$\begin{aligned} u_N &= u_i.n_i \\ u_T &= u - u_N.n \\ \sigma_N &= \sigma_{ij}.n_j.n_i \\ \sigma_T &= \sigma_{T_i} = \sigma_{ij}.n_j - \sigma_N.n_i \end{aligned} \qquad \textbf{(3)}$$

Here and throughout our presentation, cartesian index notation and the summation convention are employed. Since the body is assumed to be linearly elastic, Hooke's law holds so that

$$\sigma_{ij}(u) = E_{ijkl}.u_{k,l} \qquad \textbf{(4)}$$

where E_{ijkl} are the usual elastic constants of the materials,

$$u_{k,l} = \frac{\partial u_k}{\partial u_l} \qquad \textbf{(5)}$$

$$\sigma_{ij}(u)_{,j} + f_i = 0 \quad \text{in}\,\Omega \qquad \textbf{(6)}$$

The unilateral motion of particles of the body on the material surface initially defined by $\sigma_N(u) = 0$ is constrained by the presence of a rigid foundation which is at a given distance **d** from the body prior to the application of loads. Mathematically, this constraint is represented by the requirement that the normal displacement of boundary points cannot exceed **d**. Let us examine the different cases:

a-Condition of non-penetration of the solid onto this rigid foundation

$$(u_N - d(x)) \leq 0 \quad \text{on} \;\; \Gamma_c \qquad \textbf{(7)}$$

If $(u_N - d(x))$=0, then contact is established, while $(u_N - d(x))$**<0** indicates the existence of a gap between the support and the body. Thus (7) represents a non-penetration condition. If contact has not ocurred $(u_n - d < 0)$, then the normal contact pressure $\sigma_N(u) = 0$.

b-The unilateral contact condition

$$u_n - d < 0 \qquad \textbf{(8)}$$

c-The contact condition on the free side

$$\sigma_N(u).[u_N - d(x)] = 0 \qquad \textbf{(10)}$$

alternatively, $u_n - d = 0$ at a point on Γ_c, then σ_N must be nonpositive. $(u_N - d(x))$=0 implies $\sigma_N(u) \leq 0$.

Finally, the unilateral contact conditions on the contact surface Γ_c are:

$$\left.\begin{aligned} &[u_N - d(x)] \leq 0, \;\; \sigma_N(u) \leq 0 \\ \text{or} \quad &\sigma_N(u).[u_N - d(x)] = 0 \end{aligned}\right\} \text{on} \;\; \Gamma_c \qquad \textbf{(11)}$$

condition $\sigma_N(u).[u_N - d(x)] = 0$ means that the pressure can only be nonzero when there is contact.

3 Variational principles for contact problem: relationship with the classical problem

We now introduce variational principles for the non linear Signorini problem and establish the relationship between the variational and classical formulations (Fichera,1964 and Oden and Pires,1983). Using the notation previously introduced , we define:
V=the space of admissible displacements. A displacement vector **V** will belong to **V** if and only if
(1) **V**=0 on Γ_c
(2) **V** produces finite (normalized) strain energy in the sense of the norm

$$\|V\|_V = \left\{ \int_\Omega v_{ij} \cdot v_{ij}\, dx \right\}^{1/2} \qquad \textbf{(12)}$$

K is a subset of *V* consisting of all admissible displacements **v** in **V** for which $(v_N - d(x)) \leq 0$ for points on the contact surface Γ_c
a(u,v)=the virtual work produced by the action of stresses $\sigma_{ij}(u)$ on strains caused by the displacement **v**

$$a(u,v) = \int_\Omega E_{ijkl} \cdot u_{k,l} \cdot v_{i,j}\, dx \qquad \textbf{(13)}$$

f(v)=the virtual work done by the external forces on the displacement **v**

$$f(v) = \int_\Omega f.v\, dx \; + \; \int_{\Gamma_\sigma} t.v\, ds \qquad \textbf{(14)}$$

With the foregoing definitions and notations now established, we consider the following variational boundary value problem:
find an admissible displacement vector **u** in the set **K** such that for all admissible displacements **v** in **K**:

$$a(u, v - u) \geq f(v - u) \qquad \textbf{(15)}$$

Inequality (16) is a statement of the principle of virtual work for an elastic body. Note that this characterizes equilibrium configurations by an inequality rather than by an equality because of the existence of the unilateral contact constraint (7). We also notice that the actual contact surface depends on the solution **u** and is, therefore, not known in advance.
The existence and the uniqueness of such a problem are given by the Lax-Milgram theorem generalisation. This theorem is based on the Korn inequality.
*The Lax-Milgram theorem: Let **a** be a bilinear and continuous form, **K** be a chosen convex. Then, it exists only one solution for the following problem:
find an admissible displacement vector **u** in the set **K** such that for all admissible displacements **v** in **K**, we have relation (16).

5 Approximation of the variational inequality

A displacement vector has to be found such that:

$$a(u, v-u) - (L, v-u) \geq 0 \quad \forall\, v \in K \tag{16}$$

Therefore, among all the displacements $\mathbf{v_n}$, only those which are negative or zero on the contact surface will be kept. So the domain of research of a solution would have to be relaxed and it would have to be expressed through the equations as following

find $u \in K_1$ such that $\forall\, v \in K_1$

$K_1 = \{v \in V \,/\, \text{ whatever } v \text{ on } \Gamma_c\}$

$$\begin{Bmatrix} a(u,v) = L(v) + b(v_n, \sigma_n) \\ b(u_n, \sigma_n) \geq 0 \end{Bmatrix} \tag{17}$$

since:

$$\int_{\Gamma_c} \sigma . v \, d\Gamma = \int_{\Gamma_c} (\sigma_n . v_n + \sigma_T . v_T) \, d\Gamma \tag{18}$$

and as the friction is neglected $\Rightarrow \sigma_T = 0$, the previously equation is simplified to:

$$\int_{\Gamma_c} \sigma . v \, d\Gamma = \int_{\Gamma_c} (\sigma_n . v_n) \, d\Gamma \tag{19}$$

finally:

$$b(v_n, \sigma_n) = \int_{\Gamma_c} \sigma_n . v . n \quad d\Gamma \tag{20}$$

we have also:

$$[b(u_n, \sigma_n) \geq 0] \cong \quad \sigma_n = \sigma_n + \rho . u_n \tag{21}$$

and after, variational inequality approximation can be expressed as follows:

$$\begin{Bmatrix} a(u,v) = L(v) + \int_{\Gamma_c} \sigma_n . v . n \quad d\Gamma \\ \\ \sigma_n = \sigma_n + \rho . u_n \, , \quad \rho > 0 \end{Bmatrix} \tag{22}$$

ρ: must be physically identified as a stiffness per unit area.

6 Discretisation of the variational inequality

In order to resolve the variational inequation problems already set, we use a finite element method. For that we replace the space of displacement U by a sub-space of finite size Uf. The parameter f caracterizes the mesh refinement. We are going to use P1 finite elements. There are three nodes triangles with two degrees of freedom (u,v) per nodes (see fig 2).
For this problem, it is more judicious to use a P1 triangle element. Indeed if we consider one element along its **AB** side for example, in the contact

surface, the relations $\mathbf{u_N(A) \leq 0}$ and $\mathbf{u_N(B) \leq 0}$ imply, because of the linear shape functions, that all the points of this side do not penetrate the obstacle

$$\left| \begin{array}{l} \forall \quad M \quad \in [A,B] \\ u_N(M) \leq 0 \end{array} \right. \qquad \textbf{(23)}$$

otherwise if the shape functions were quadratic, and if: $\mathbf{u_N(A) \leq 0}$ and $\mathbf{u_N(B) \leq 0}$,then:

$$\left| \begin{array}{l} \exists \quad M' \quad \notin [A,B] \\ \text{such as} \quad u_N(M') \leq 0 \end{array} \right. \qquad \textbf{(24)}$$

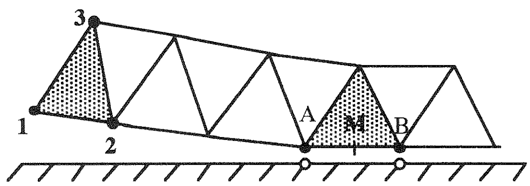

Figure 2 Finite elements discretisation.

Some details of the construction of the elementary stiffness matrix are now given (Batoz and Dhatt,1990):

$$\{U\} = [[N_1] \ [N_2] \ [N_3]].\{\{u_1\} \ \{u_2\} \ \{u_3\}\}^T$$

$$\{\varepsilon\} = [D].\{U\} = \frac{1}{2.\Delta}.[B]\{q_e\} \qquad \textbf{(25)}$$

$$[K]^e = \int_\Omega [B]^t.[H].[B] \ d\Omega$$

N_i: shape functions
u_i: displacement components
D,B: derivation matrix
q_e: nodal displacements
H: Hooke 's law
Now, we consider our variational equation:

$$a(u,v) = (L,v) + \int_{\Gamma_c} \sigma_n . v . n \quad d\Gamma \qquad \textbf{(26)}$$

The discretisation of our equation leads to:

$$\begin{array}{l} \langle v \rangle.[K].\{u\} = \langle v \rangle.\{F\} + \langle v \rangle.\{p\}.\{\sigma_n\} \\ \Rightarrow [K].\{u\} = \{F\} + \{p\}.\{\sigma_n\} \end{array} \qquad \textbf{(27)}$$

Final form of the equation system to resolve is written as:

$$\begin{cases} [K].\{u\} = \{F\} + \{p\}.\{\sigma_n\} \\ \sigma_n = \sigma_n + \rho.u_n \\ \rho > 0 \end{cases} \qquad \textbf{(28)}$$

7 Classical Lagrangian methods

These methods consist to solve the dual problem above. Obtaining a solution for the dual problem makes it possible to obtain a solution for primal problem (Minoux,1983). The main methods utilizing the Lagrangian duality have been given by:
*Dantzig (1959),
*UZAWA (1958) and Arrow-Hurwicz (1958).

7.1 Uzawa's algorithm

Uzawa's algorithm uses the classical gradient method for solving the dual problem. At each step the Lagrangian function is minimised. The method is described by the following phases:

(a) Start with a point $\lambda^0 \geq 0$

(b) At iteration k we have λ^k

we calculate $w(\lambda^k) = \mathrm{Min}\{f(u) + \lambda^k.g(u)\}$

$$w(\lambda^k) = f(u^k) + \lambda^k.g(u^k)$$

(c) we define λ^{k+1} by:

$$\lambda^{k+1} = \mathrm{Proj}\{\lambda^k + \rho_k.g(u^k)\}$$

where ρ_k is the displacement step in k iterations

(d) If the end test is satisfied we stop, else we choose $k = k+1$ and we return to (b)

In this algorithm, the displacement parameter ρ_k must be chosen adequately. In addition, the method consists to alternate the computations between the **u** primal variable and the λ dual variable spaces.
We are going to adapt the dual method to our case, and we have chosen to use the algorithm of Uzawa for the resolution of a contact problem without friction.

7.2 Algorithm

Uzawa's algorithm is adapted for solving the previously equation system.

(1) solve $[K].\{u\} = \{F\}$

obtaining of $\{u\} \Rightarrow \{u_n\}$

(2) calculate $\{\sigma_n\}^{i+1} = \{\sigma_n\}^{i} + \rho.\{u_n\}^{i}$

(3) solve $[K].\{u\} = \{F\} + \{p\}.\{\sigma_n\}$

(4) if $\left|\sigma_n^{i+1} - \sigma_n^{i}\right| <$ critical value $\Rightarrow$ end of calculations : END

ELSE: $i = i + 1$ and return in (2)

8 Application example

Contact between an elastic cylinder and rigid foundation (Hertz's problem). The contact problem between an elastic cylinder and rigid foundation is illustrated schematically in figure 3.

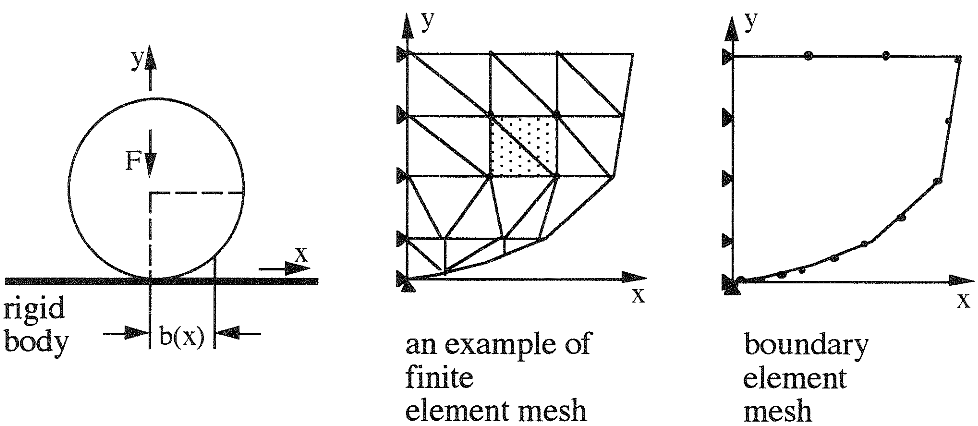

Figure 3 Hertz's nroblem analysis
a- problem description
b- finite element discretisation (the mesch refinement is not represented)
c- boundary element method discretisation

Figure 3.b shows the finite element idealization of the same problem. Plane strain conditions are assumed and the cylinder is considered to be made of isotropic linear elastic material. In the contact region, 14 nodes are concerned.

Analytical solution for Hertz's problem is briefly presented (Jhonsson,1985). The analytical solution for the contact between an elastic cylinder and rigid foundation is written as:

$$b(x) = 2.\sqrt{\frac{1-\mu^2}{E.P}.p.R}$$

$$P(x) = \frac{2.p}{b}.\sqrt{1-\frac{x^2}{b^2}} \qquad (29)$$

With:
R:radius of the considered cylinder
P:load applied on the cylinder
b:width of the contact surface
E:Young modulus
v:Poisson coefficient.

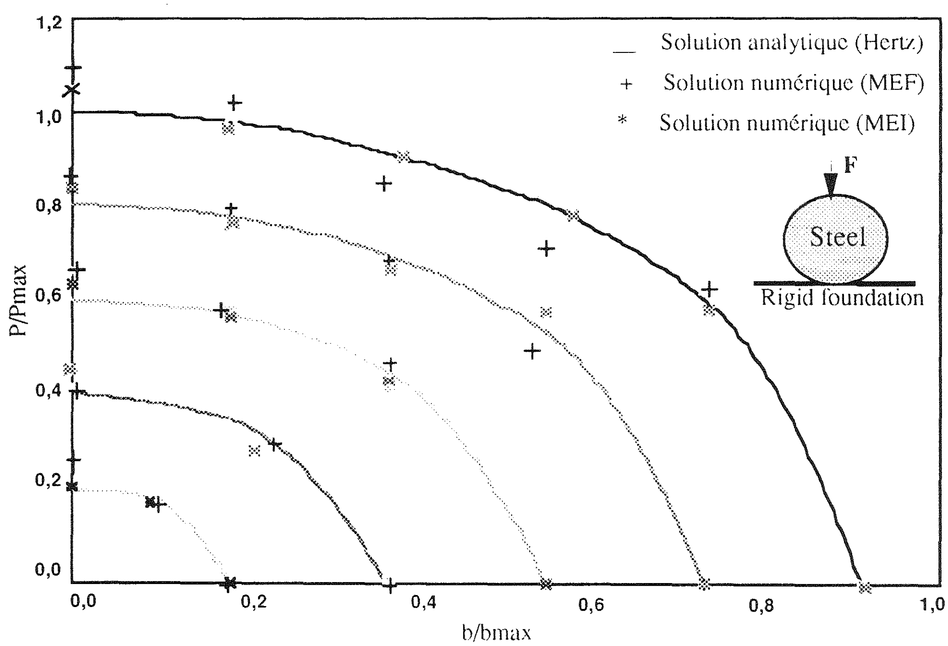

Figure 4 Comparative curves for the normal stress to the contact area:
a- Analytical solution (Hertz)
b- Numerical solution (MEF)
c- Numerical solution (MEI).

Figure 4 shows a comparative illustration between Hertz analytical solution, a solution given by boundary element method (Noune,1994) and the values issued from the proposed analysis using finite element method. This approach validates the developed algorithm.

9 Conclusion

A finite element solution procedure for two-dimensional frictionless contact problems has been presented which allows one to predict contact surface tractions and the area of contact. Although the new method is not theoretically superior to existing methods, it is simple to use and computationally efficient. The method is found to yield good results in a few iterations. Illustrative example shows good agreement with the published results

References

Batoz, J.L. and Dhatt, G. (1990) **Modélisation des Structures par les Eléments finis.** Volume 2, Edition Hermès, France.

Fichera, G. (1964) Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al condorno. **Mem. Accad.Naz.Lincei** VIII, 91-140.

Johnsson, M. (1985) **Contact Mechanics.** Cambridge University Press, London.

Minoux, M. (1983) **Programmation Mathématique: théorie et algorithmie.** Bordas, France.

Noune, A. (1994) Modélisation du contact avec frottement dans les géomatériaux: approche par la méthode des éléments de frontière. Rapport d'avancement de travaux de recherche, L2M-UCB-Lyon, France.

Oden, J.T and Pires, E.B. (1983) Nonlocal and nonlinear friction laws and variational principles for contact problems in elasticity. **J. Appl. Mech.**, 50,67-76.

Zhong, Z.H. and Mackerle, J. (1992) Static contact problem - a review. **Engineering Computations**, 9, 3-37.