Fracture Mechanics of Concrete Structures,
Proceedings FRAMCOS-2, edited by Folker H. Wittmann,
AEDIFICATIO Publishers, D-79104 Freiburg (1995)

MIXED-MODE FRACTURE IN CONCRETE: A NON-LOCAL DAMAGE APPROACH

M. di Prisco,
Department of Structural Engineering, Politecnico di Milano, Italy
J. Mazars,
Laboratoire de Mécanique et Technologie ENS de Cachan/CNRS/Univ. Paris 6 and GRECO Géomatériaux CACHAN, France.

Abstract

Experimental investigations, recently performed on mixed-mode fracture in concrete emphasize the path-dependency and, inducing localization, are strongly affected by specific phenomena linked to the heterogeneity of the material such as aggregate interlock.
In the framework of non-local damage mechanics a new approach to model cracking by means of a non-local anelastic strain tensor is presented. The model is able to take into account the induced anisotropy due to cracking, the position, the mean openings and the residual interface stresses transmitted along the cracks.
With reference to experimented paths, some comparisons are performed in order to check the reliability of the model, and highlight some problems related to the non-local nature of the model.

1 Introduction

Fracture in concrete is a difficult problem because it induces localization and discontinuity in the displacement field, not only at the micro-level, but also at the meso- and macro-level. While for a metal continuum it is possible to identify the constitutive behaviour on a meso-scale structure, assuming a sufficient homogeneity in the test, concrete material is strongly affected by its considerable heterogeneity (14 magnitude orders characterize its constituent aggregates) and only the initial elastic behaviour can be identified on a meso-scale structure without the rising of localization on a meso- and macro-level. In order to take into account this experimental evidence, different approaches were investigated by various authors. The first, from an historical point of view, Cauchy continuum formulation regards the meso-scale concrete structure used in the tests as a homogeneous equivalent material, in order to identify the constitutive behaviour analysing different stress paths characterized always (with the exception of large triaxial hydrostatic pressures) by softening and localization on a meso- and macro-level. Owing to localization, the test dimensions affect the constitutive behaviour, and the description of the post-peak evolution of yield-function becomes rather complex in order to take into account induced anisotropy. Because softening causes spurious size-effect, excessive localizations and spurious mesh sensitivity, a localization limiter was introduced with physical explanations which have been mainly phenomenologic and empirical and only recently Bazant (1994) has tried to adduce certain micromechanic arguments. Other approaches, such microplane, lattice-model or microstructural concrete, take into account the heterogeneity of mesoscale and tries to identify the mesoscale structure behaviour by means of simpler relations describing the interactions among the largest aggregates and the matrix in the contact regions (Bazant and Gambarova, 1984, Fokwa, 1992). This is a fascinating perspective, but from an experimental point of view only meso-scale kinematic and static quantities can be reasonably identified, and thus the constitutive equations must be deduced by interpolating integral results. The proposed model, following the non-local damage approach, tries to describe the meso-scale structure behaviour starting from the physical assumption that only 4 kinds of failure can occur: three related to cracking, pointed out by Van Mier (tensile fracture, short inclined shear planes in two directions and pronounced shear band - see di Prisco and Mazars, 1994), and one to crushing. This assumption permits us to describe the softening behaviour only with reference to those paths where kinematic quantities are sufficiently homogeneous and experimentally measurable. This is obviously true, the only exception being tensile fracture where localization is unavoidable. In this case two solutions can be adopted: the first one, performed by Bazant and Pijaudier-Cabot (1989) tries to "stabilize" the crack growth by means of a parallel elastic spring achieved by metallic bars

glued to the concrete specimen (see also Fokwa 1992). But another solution can be exploited adopting an approach based on the *theory of distributions* according to other recent formulations (Simo et al., 1993).
Let us assume to know the exact displacement field in a cracked concrete region which can be described by a vectorial function **u**(x,y,z) belonging to L^1_{loc} (the whole of the functions which can be integrated according to Lebesgue's definition). The lack of continuity prevents the evaluation of strains which require the conventional derivation along directions parallel to the coordinate axes, according to the small displacement assumption. Thus, to extend the continuum approach, let us introduce derivation in the sense of the Distributions, and imagine choosing a particular test-function $\varphi \in \mathbf{C}_\infty$ which has a support defined according to fracture mechanics considerations or physical observations. The functional $<\mathbf{u},\varphi>$ corresponds biunivocally to displacement function **u**, if it is known $\forall\varphi \in \mathbf{D}$ (where $D = D(\Re)$ is the vectorial space of complex functions $\in \mathbf{C}_\infty(\mathbf{R})$). The function **u** can be decomposed in two vectorial functions, one continuous, and so derivable in the traditional sense which reproduces local elastic strain ($\mathbf{u}_1(\mathbf{x}) \rightarrow \varepsilon^{el} = \mathbf{B}\mathbf{u}_1(\mathbf{x})$ where **B** is the usual differential operator), and the second one $\mathbf{u}_2(\mathbf{x})$, discontinuous, derivable only in the sense of distributions which describes all the discontinuities due to cracks:

$$\langle \mathbf{Bu_2(x)}, \varphi(\mathbf{s}-\mathbf{x}) \rangle = -\langle \mathbf{u_2}, \mathbf{B}\varphi \rangle = \psi_{ij}(\mathbf{x}) = \varepsilon_{ij}{}^{\mathbf{an}}(\mathbf{x}) \tag{1}$$

where **s**, **x** are position vectors in the Euclidean space and ψ is a tensorial function. Because of the heterogeneities connected to the macroscopical nature of concrete, only a good choice of the support function φ taking into account the microscopic mechanical description of the representative volume, could allow the evaluation of a significant ψ_{ij} which represents a reasonable average measure of the irreversible strains due to the discontinuities. Nevertheless, according to the theory of distributions, only the knowledge of a succession $\psi_{ij}{}^n$ related to a base φ^n permits the identification of $\mathbf{u}_2(\mathbf{x})$. The absence of this information prevents the knowledge of $\mathbf{u}_2(\mathbf{x})$, but the non-local tensor $\varepsilon_{ij}{}^{an}$ furnishes an average measure of the discontinuities located along the cracks. A sort of a macroscale kinematic compliance is assumed to maintain a traditional approach by means of F.E., with the displacement field described through polynomial functions $\in C_n$, and only repeated analyses with different φ^n could allow the determination of a discontinuous displacement field in the mesoscale. Anelastic strains, which represent the irreversible part of the global macro-strain, are expressed as function of non-local damage for their module and assume the expression of a non-local tensor being related to the gradient of a potential function of a non-local invariant of strains $\bar{\varepsilon}_+$. The result is a model with "rotating" cracks, but the crack orientation is related to the strain path according to Hassanzadeh (1991).

2 Analytical model

The model regards damage as an isotropic scalar variable like Mazar's previous model ('84): this choice is a simplifying assumption rather than an accurate description of the behaviour of concrete. Damage represents the reduction of area able to transfer stress which decreases with the growing of microcracking and decreases even more with macrocracks. It is associated to the stiffness controlled by Young's modulus E. The volume also changes when the cracking process develops. The irreversible increase of volume is described by means of anelastic strains strictly related to damage, while the reversible part is associated to the evolution of ν, the Poisson's coefficient. Its evolution, which is assumed to be monotonic in order to respect thermodynamic principles, is described by another internal variable δ, and can reach a maximum value fixed lower than 0.5 for numerical stability. Having introduced the anelastic strain tensor rate $\dot{\varepsilon}_{ij}^{an}$ and accepted the additive cumulation with elastic strains $\dot{\varepsilon}_{ij}^{el}$:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{el} + \dot{\varepsilon}_{ij}^{an} = \dot{\varepsilon}_{ij}^{el} + \dot{\varepsilon}_{ij+}^{an} + \dot{\varepsilon}_{ij-}^{an} = \dot{\varepsilon}_{ij}^{*} + \dot{\varepsilon}_{ij-}^{an} \qquad (2)$$

the stress-strain relationship becomes:

$$\sigma_{ij} = \frac{E_0(1-D)}{1+\nu_0(1+\delta)}\varepsilon_{ij}^{el} + \frac{E_0(1-D)\nu_0(1+\delta)}{[1+\nu_0(1+\delta)][1-2\nu_0(1+\delta)]}\varepsilon_{ij}^{el}\delta_{ij} \qquad (3)$$

According to this scalar description of damage, concrete is assumed to remain isotropic up to failure in its mechanical constitutive behavior, but it shows induced anisotropy due to anelastic strains ε_{ij}^{an}.

The internal variables D and δ range from 0 for the virgin material to 1 and $(\frac{\nu_{lim}}{\nu_0} - 1)$ respectively at asymptotic failure $(\varepsilon_{ij} \to \infty)$. A limit is necessary for the presence of positive anelastic non-local strain components: this limit can be established for rough cracks in terms of an average quantity which guaranteees the interlocking of the discontinuities and consequently the transmission of stresses with consequent elastic strains. It can be approximated by:

$$\max \varepsilon_{i}{}^{an}{}_{+} < d_a / 2l_c \qquad (4)$$

where d_a is the maximum size of aggregates and l_c is the characteristic length of material (Bazant and Pijaudier-Cabot 1987). Beyond this limit, D is imposed equal to 1 which means stiffness vanishes, showing that the present state of the model is not able to describe the unilateral character of concrete.

In case of strain-softening, spurious localization and lack of mesh objectivity may occur. To avoid these unrealistic features, the key idea of treating the elastic part of the strain as local, as suggested by Pijaudier-Cabot and Bazant '87, is here adopted.

The non-local equivalent strain $\bar{\varepsilon}_+$, which is mainly related to opening mode 1 in microcracking and to mixed modes 1 and 2 when macrocracks appear, controls the growth of all the damage laws associated to the three kinds of failure indicated by Van Mier '86. This non-local invariant remains the key of the model, as in the previous one (Mazars '84), and it is defined thus (Saouridis and Mazars '92):

$$\bar{\varepsilon}_+(x) = \frac{1}{V_r(x)} \int_V \tilde{\varepsilon}_+(s)\varphi(s-x)dV \tag{5}$$

with

$$V_r(x) = \int_V \varphi(s-x)dV \ ; \ \tilde{\varepsilon}_+ = \sqrt{\sum_{i=1}^{3}\left(\langle\varepsilon_i^*\rangle_+\right)^2} \qquad \begin{array}{ll} \langle\varepsilon_i^*\rangle_+ = 0 & \text{if } \varepsilon_i^* < 0 \\ \langle\varepsilon_i^*\rangle_+ = \varepsilon_i^* & \text{if } \varepsilon_i^* \geq 0 \end{array} \tag{6}$$

ε_i^* are the principal strains; $\varphi(s-x)$ is the weighting function $\in C_\infty$;and $V_r(x)$ is the representative volume at point **x**:

$$\varphi(s-x) = \exp\left[\left(-\frac{k|s-x|}{l_c}\right)^2\right]; \quad V_r(x) = \int_V \varphi(s-x)dV \tag{7,8}$$

As specified by Bazant and Pijaudier-Cabot ('88), k can be chosen equal to $\sqrt{\pi}$, 2 or $(6\sqrt{\pi})^{1/3}$ respectively for 1,2 or 3D problems; l_c is the characteristic length of the non-local continuum and it is proportional to the smallest size of the damage localization zone.

An initial elastic domain characterizes the model and it is represented by all the points in the principal strain space inside the surface:

$$F(\bar{\varepsilon}_+) = \bar{\varepsilon}_+ - \varepsilon_{t0} \leq 0. \tag{9}$$

According to a formalism, which is similar to associate plasticity, when the surface is reached the first time (F=0), damage rises and the yield surface is described by $F_1 \cup F_2$:

$$F_1\left[\varepsilon_{ij}^*, D, \delta, \alpha_c(\sigma_{ij}), \alpha_t(\sigma_{ij})\right] = D - \alpha_c D_c(\bar{\varepsilon}_+) - \alpha_t D_t(\bar{\varepsilon}_+) \leq 0 \tag{10}$$

$$F_2\left[\varepsilon_{ij}^*, D, \delta, \alpha_c(\sigma_{ij})\right] = \delta - \alpha_c \delta_c(\bar{\varepsilon}_+) \leq 0 \tag{11}$$

The evolution is described by:

$$\left.\begin{array}{l|l} F_1 = 0:\ \dot{D} = 0 \ \text{ with } \ \dot{D} \geq 0; \dot{F} \leq 0 & \dot{F}_2 = 0:\ \dot{F}_2\,\dot{\delta} = 0 \ \text{ with } \ \dot{\delta} \geq 0; \dot{F}_2 \leq 0 \\ \quad \dot{\varepsilon}^{an}_{ij\,+} = \varepsilon_{t0} \dfrac{\partial f(D,c)}{\partial D} \dot{D} \dfrac{\partial F(\bar{\varepsilon}_+)}{\partial \varepsilon_{ij}^*} & F_2 < 0 :\ \dot{\delta} = 0 \\ F_1 < 0:\ \dot{D} = 0 & \end{array}\right. \tag{12}$$

$$f(D,c)=\frac{D^{1+c}}{(1-D^{x})^{y}};\ x,y>0.;\ c=c(\varepsilon_i^{el})=\frac{\langle \min\left[\varepsilon_i^{el}(1-2\nu)+\nu\varepsilon_{kk}\right]\rangle_-}{(1-2\upsilon)\sum_{i=1}^{2}\langle\varepsilon_3^{el}-\varepsilon_i^{el}\rangle_-^{2}} \quad (13)$$

where c takes into account the confinement for multiaxial compression stress states.

Damage is split into three parts according to the three above mentioned failure modes (Van Mier '86):

$$D=\alpha_c D_c+\alpha_t D_t=\alpha_c(\sigma_{ij})\left\{\left[2-\eta\left(\varepsilon_{ij}^{*}\right)\right]D_c^{I}(\bar{\varepsilon}_+)+\left[\eta\left(\varepsilon_{ij}^{*}\right)-1\right]D_c^{II}(\bar{\varepsilon}_+)\right\}+\alpha_t\left(\sigma_{ij}\right)D_t\left(\bar{\varepsilon}_+\right) \quad (14)$$

where $D_c^{I}(\bar{\varepsilon}_+)$, $D_c^{II}(\bar{\varepsilon}_+)$ (I or II indicates one or two directions with positive strains) and $D_t(\bar{\varepsilon}_+)$ are spline functions identified only once, at the beginning, taking anelastic strains ε_{ij}^{an} into account and imposing consistency along each failure mode. The α_c,α_t and η factors are expressed as non-dimensional functions of the principal strains.

Adopting the notation :

$$\langle\varepsilon_i^{el}\rangle_+=\varepsilon_{ci}^{el}+\varepsilon_{ti}^{el} \quad (15)$$

in which ε_{ti}^{el} are the positive elastic principal strains owing to positive stresses and ε_{ci}^{el} are the positive elastic principal strains owing to negative stresses (Poisson's effect), α_c and α_t are defined as (Mazars '84):

$$\alpha_t=\alpha_t(\sigma_{ij})=\sum_{i=1}^{3}\left[\frac{\varepsilon_{ti}^{el}(\varepsilon_{ti}^{el}+\varepsilon_{ci}^{el})}{\tilde{\varepsilon}_+^{el2}}\right];\quad \alpha_c=\alpha_c(\sigma_{ij})=\sum_{i=1}^{3}\left[\frac{\varepsilon_{ci}^{el}(\varepsilon_{ti}^{el}+\varepsilon_{ci}^{el})}{\tilde{\varepsilon}_+^{el2}}\right] \quad (16)$$

$$\text{with : } \tilde{\varepsilon}_+^{el}=\sqrt{\sum_{i=1}^{3}\langle\varepsilon_i^{el}\rangle_+^{2}} \quad\text{and}\quad \eta=\eta\left(\varepsilon_{ij}^{*}\right)=\frac{\sum_{i,j=1}^{3}\left|\langle\varepsilon_i^{*}\rangle_- -\langle\varepsilon_j^{*}\rangle_-\right|^{2}}{2\tilde{\varepsilon}_-^{*2}} \quad (17)$$

assuming the notation $\langle\varepsilon_i\rangle_-=0 \quad if \quad \varepsilon_i\geq 0;\ \langle\varepsilon_i\rangle_-=\varepsilon_i \quad if \quad \varepsilon_i\leq 0$.

The function $\delta_c(\bar{\varepsilon}_+)$, is related only to damage in compression $D_c(\bar{\varepsilon}_+)$ according to a trigonometric function:

$$\delta_c=\frac{M-\nu_0}{\nu_0}\left[\sin^2\left(\frac{\pi}{2}\right)D_c^{\,n}\left(\bar{\varepsilon}_+\right)\right],\qquad 0\leq \mathrm{M}<0.5;\quad \mathrm{n}\cong 1.1 \quad (18)$$

As mentioned above, crushing in compression is taken into account too, but following the assumptions of the model, concrete behaviour is modelled by means of anelastic strains without any influence on damage. The uncoupling between crushing and cracking permits us to take into account the former as a local process because it does not cause softening; thus, in order to simplify the F.E. implementation a local description was preferred in spite of a previous choice (di Prisco and Mazars, '94). To "symmetrize" the model a function $G(\tilde{\varepsilon}_-)$ is introduced:

$$G(\tilde{\varepsilon}_-)=\tilde{\varepsilon}_- -\tilde{\varepsilon}_-^{M}\leq 0 \quad (19)$$

where $\tilde{\varepsilon}_- = \sqrt{\sum_{i=1}^{3}(\langle\varepsilon_i\rangle_-)^2}$ $\qquad \tilde{\varepsilon}_-^M = Sup\{\tilde{\varepsilon}_-\} \geq \varepsilon_{c0}$ (20)

The evolution is controlled by the internal variable s, defining $s = \tilde{\varepsilon}_-^M - \varepsilon_{c0}$:

$$G=0:\ \dot{G}\dot{s}=0 \ \text{ with } \ \dot{s}\geq 0, \quad \dot{G}\leq 0$$

$$\dot{\varepsilon}_{ij-}^{an} = \varepsilon_{c0}\frac{dg(\tilde{\varepsilon}_-)}{d\tilde{\varepsilon}_-}\dot{s}\frac{\partial G(\tilde{\varepsilon}_-)}{\partial\varepsilon_{ij}}; \qquad g(\tilde{\varepsilon}_-) = g_\infty\left\{1-\frac{1}{\exp\left[(\tilde{\varepsilon}_- - \varepsilon_{c0})^Z\right]}\right\} \tag{21}$$

$$G<0:\ \dot{s}=0$$

g_∞, Z are constants (g_∞=130., Z =1.2 are chosen in the following examples). The identification of $g(\tilde{\varepsilon}_-)$ is related to the fourth kind of failure, typical of hydrostatic process.

The uncoupling guaranteed by the introduction of $\varepsilon_{ij}^* = \varepsilon_{ij} - \varepsilon_{ij-}^{an}$ implies a mixed hardening of $F_1 \cup F_2 \cup G$ domain in principal strain space, and it is easy to notice that the model results unassociated for a general path.

3 Mode I and mixed mode cracking

The previous analytical presentation of the proposed model highlights the key role of the strain invariant $\tilde{\varepsilon}_+$ in the description of cracking processes. When a mode one is induced, two different solutions can be followed in order to consider the softening branch: the discrete approach as Fictitious Crack Model (FCM) proposed by Hilleborg (see Nooru Mohamed, 1992) and the smeared approach as the Crack Band Model (CBM) proposed by Bazant and Oh (1983). The former is able to take into account the discontinuity of the displacement field and considers the maximum tensile strength as crack initiation criterion and a process zone where normal stresses are related to normal displacements by means of a crack evolution law (Carpinteri et al., 1993). The latter (CBM) permits a description of crack in terms of stress and strain, and consequently, the numerical implementation is easier, since the approach only requires a change of the stiffness matrix of the element after cracking, while the topology of the original F.E. mesh remains preserved. Three broad categories accept the smeared crack concept: fixed, fixed multidirectional and rotating crack approaches. The problems related to these approaches were well analysed and discussed by de Borst and Nauta (1985) and de Borst (1991 - see in Nooru Mohamed 1992).

With reference to mode I, the proposed model shows a localization band which is similar to the previous Mazar's model as regards its depth. Fig.1a shows damage risen in a continuum strip, where the central zone is characterized by a smaller threshold in tension ε^*_{t0} (ε^*_{t0}= $0.8\varepsilon_{t0}$, A_t=0.8, B_t=20000; 4-nodes F.E.). With these parameters, Mazars' model shows a central band which has a dimension close to $3l_c$ (l_c=12mm), and two other bands close to the bounds. Enlarging the defect depth, a single band is obtained. The proposed model does not present three bands, but only one which becomes wider when damage increases. The anelastic strains which are related to damage by the function *f(D,c)* (see eq.13), tend to concentrate more and more when damage increases. When a local approach is followed, both the models show localization in a very narrow zone close to the defect. Fig.1b shows the load versus the vertical displacement of the strip, in the local and non-local approaches for both the models. When there is a stress concentration due to a notch the evolution of crack is different. Fig.2 presents the results obtained analysing the test geometry proposed by Hassanzadeh's specimens (1991) in plane strain condition. The tensile parameters are the previous one. The strain invariant $\bar{\varepsilon}_+$ represents the fracture criterion and it is shown as different characteristic lengths can influence the location and the evolution of the process zone, when a constant vertical displacement on the two horizontal opposite sides is

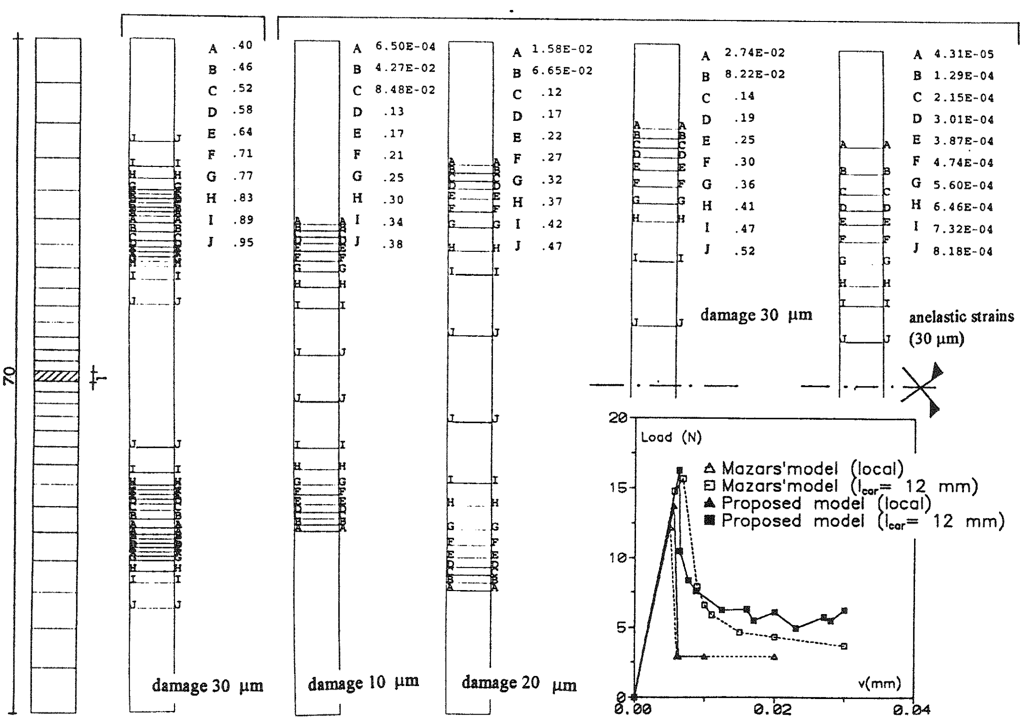

Fig.1. Localization in a concrete strip subject to tension: (a) damage patterns; (b) load-versus displacement curve (E=32900. MPa, ν= 0.2)

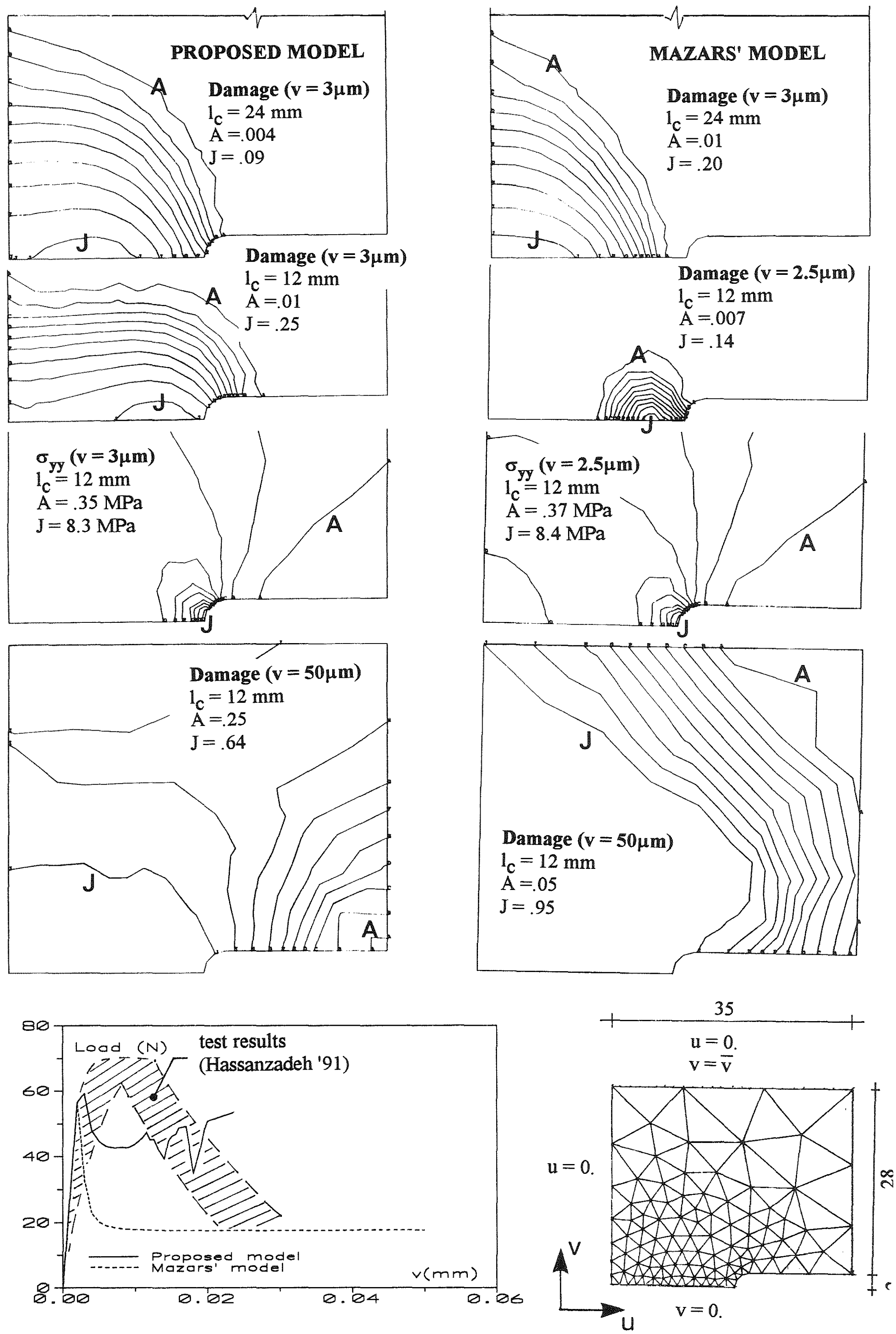

Fig.2. Hassanzadeh's tensile test analysed by Mazars' and proposed models.

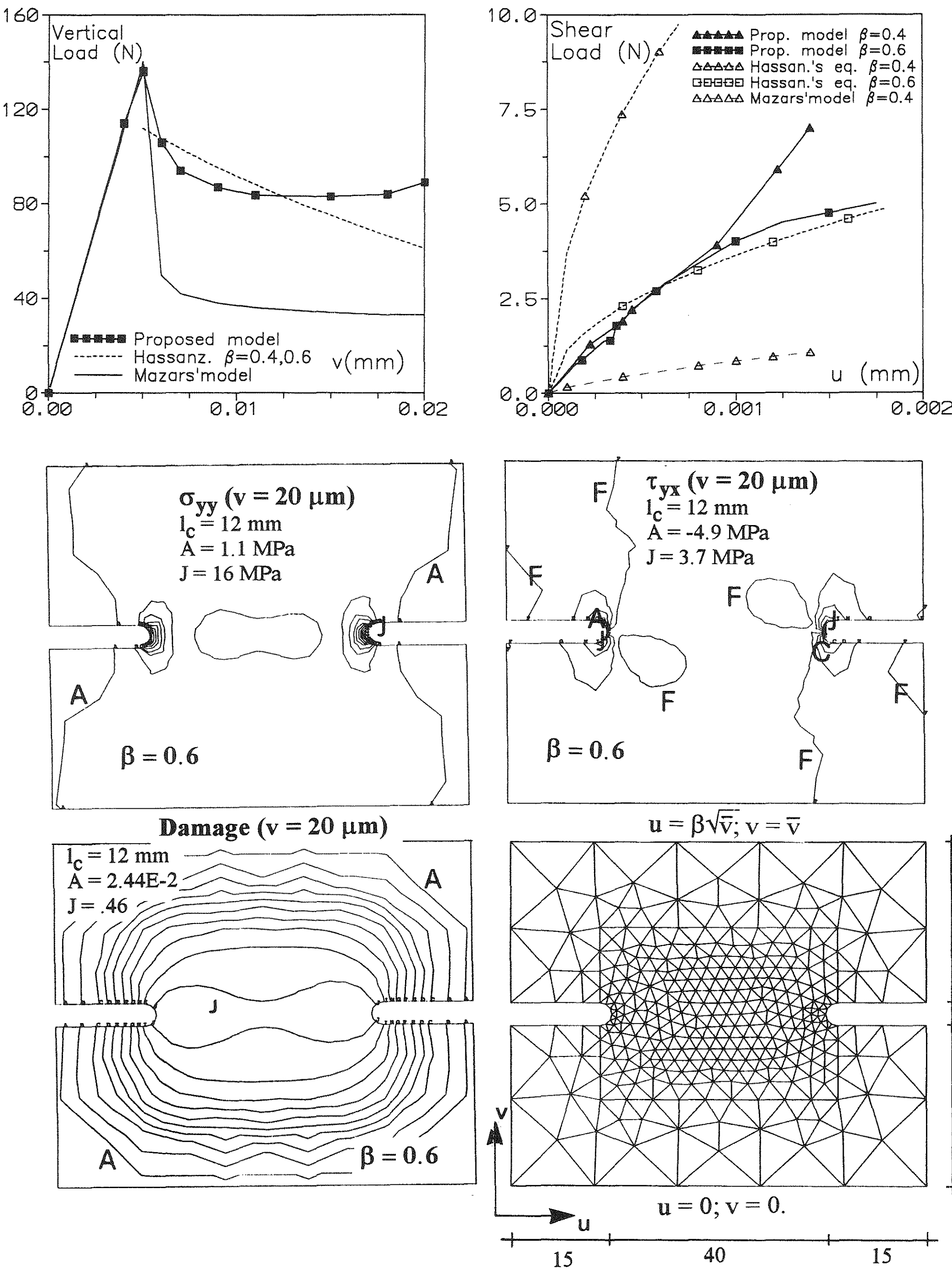

Fig.3. Hassanzadeh's mixed mode tests (parabolic paths).

imposed. It is interesting to note that the same does not occur when a Compact Tension Specimen test is performed. The reason of this anomalous behaviour must be looked for in the definition of the non-local strain invariant $\bar{\varepsilon}_+$. When the *local action principle* is not respected as in the non-local continuum theory, the constitutive law becomes topologically dependent. Thus, if the region dimensions are related only to the maximum aggregate size, it is necessary to introduce the effect of the boundary conditions when they affect the influence coefficients of the points inside the same region. It is relevant to highlight as the consistency imposed "a priori" on the three radial paths related to damage is imposed in a local homogeneous identification process. Thus, when a non-local approach is followed, the lack of a local constraint permits stresses normal to the crack in the process zone which are much larger than homogeneous peak strength accepted in tension. Moreover, the shape of the tension curve determined by the above specified parameters induces an unrealistic damage pattern for high values of the vertical displacements. Fig.3 shows the first phases of two parabolic paths ($v = \beta\sqrt{u}$, β= 0.4, 0.6) tested by Hassanzadeh (1991). Conserving the same tension parameters, reasonable results were obtained. It is interesting to note the improvement in relation to Mazars'model (di Prisco and Mazars, 1992) with regard to shear capacity, owing to the presence of anelastic strains in tension.

4 Conclusions

The implementation of the proposed model in the F.E. Code *Castem 2000* and the first calculations have permitted to highlight the following concluding remarks.

The model shows localization confined to a band which has a depth related to the characteristic length: its evolution is strongly affected by the shape of the tension curve which must be identified with special care. The lack of a local consistency in the post-cracking behaviour causes tensile stresses normal to the crack much larger than peak homogeneous tensile strength in the process zone. In order to avoid this unrealistic behaviour, a sort of fixed multi-directional constraint should be introduced.

The influence of boundary conditions must be introduced in the non-local approach, in order to prevent damage evolutions which are unaccettable from a physical point of view.

Hassanzadeh's tests with parabolic displacement paths show reasonable results in the initial phases. The model does not introduce any special shear parameter, as retenction factor β, and can be considered as a rotating crack model. These features make this model suitable in order to analyse path dependency behaviour. The work is in progress and further numerical tests have to be performed.

Acknowledgements

The financial support to **A**lliance of **L**aboratories in **E**urope for **R**esearch and **T**echnology - Geomaterials of **C.E.C.** in the framework of program *Human Capital and Mobility* is gratefully acknowledged.

5 References

Bazant, Z.P. (1994) Nonlocal damage theory based on micromechanics of crack interactions. **J. of Engrg. Mech., ASCE**, 120 (3), 593-617.
Bazant, Z.P., and Gambarova, P.G. (1984) Crack shear in concrete: crack band microplane model. **J. of Struct. Engrg., ASCE**, 110 (9), 2015-2035.
Bazant, Z.P., and Pijaudier-Cabot, G. (1988). Nonlocal Continuum damage, localization instability and convergence. **ASME J. of Applied Mechanics**, 55, 287-293.
Bazant, Z.P., and Pijaudier-Cabot, G. (1989). Measurement of characteristic length of Nonlocal Continuuum. **ASCE J. of Engng. Mech.**, 115(4), 755-767.
Carpinteri, A., Valente, S., Ferrara, G., and Melchiorri, G. (1993) Is mode II fracture energy a real material property? **Computers & Structures**, 48 (3), 397-413.
de Borst, R., Nauta, P. (1985) Non-orthogonal cracks in a smeared finite element model. **Engineering Computations**, 2, 35-46.
di Prisco, M., and Mazars, J. (1992) On damage modelling of reinforced concrete subject to high-strain gradients. **Studi e Ricerche**, School for the Design of R/C Struct., Politecnico di Milano, 13, 33-81.
di Prisco, and M., Mazars, J. (1994) A Non-Local Damage Model to investigate Mixed Mode Cracking in Concrete. Proc. Int. Conf. on **Comput. Modelling of Concrete Struct. EURO-C**, Innsbruck, Austria, 22-25th March, (eds. H.Mang, N. Bicanic, R. de Borst), Cromwell Press, 145-154.
Fokwa, D. (1992) Materiaux Heterogenes: analyse experimentale et modelisation numerique par une approache hierarchique. **Ph. D. Thesis, L.M.T.** - University Paris 6, 1-128.
Hassanzadeh, M. (1991) Behaviour of fracture process zones in concrete influenced by simultaneously applied normal and shear displacements. **Technical Report** Div. of Building Mat., **Lund** Institute of Technology.
Mazars, J. (1984). Application de la mécanique de l'endommagement au comportement non lineaire et à la rupture du béton de structure. **Thesis de Doctorate d'Etat**, L.M.T. - University Paris 6, France.
Nooru-Mohamed, M.B. (1992) Mixed-mode fracture of concrete: an experimental approach, **Ph.D. Thesis**, **Delft** University of Technology.
Pijaudier-Cabot, G., and Bazant, Z.P. (1987) Nonlocal damage theory. **J. of Engrg. Mech**., **ASCE**, 113(10), 1512-1533.
Saouridis, C., and Mazars, J. (1992) Prediction of the failure and size-effect in concrete via a bi-scale damage approach. **Engng. Computations**, 9(3), ISSN 0264-4401, Swansea, UK, 329-344.
Simo, J.C., Oliver J., and Armero, F. (1993) An analysis of strong discontinuities induced by softening solutions in rate independent solids. **J. Comp. Mech.**, 12, 277-296.