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SCALING THEORIES FOR QUASIBRITTLE FRACTURE: RECENT ADVANCES AND NEW DIRECTIONS

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Abstract
A review of the basic theories of scaling in solid mechanics is presented. The problem of scaling is approached through dimensional analysis, laws of thermodynamics and asymptotic matching. Definitive conclusions on the relative importance of various sources of size effect (energy release, Weibull statistics, and crack fractality) are drawn. The size effect laws for crack initiation from smooth surface and for both cracked and uncracked specimens are presented. A simpler, one-size version of the size effect method of fracture energy testing is proposed. Finally, promising research directions are pointed out.

1 Introduction

Scaling is the most fundamental property of every physical theory. In structural mechanics, however, little attention has been paid to the scaling of failure and until about a decade ago it has been generally assumed that the observed size effect on nominal strength of structures must always be explained by the randomness of strength. Detailed analysis shows, however, that this scaling theory does not capture the main cause of size effect for quasibrittle materials such as concrete, sea ice, rocks, tough ceramics and composites which exhibit a large fracture process zone and allow stable growth of large cracks prior to failure. Rather, the dominant source of size effect appears to be deterministic and consists in the release of stored energy and the associated stress redistribution.

A historical discussion of the size effect in concrete must begin with the work of Walsh (1972) who made the plot of logarithm of nominal strength versus logarithm of the size of similar fracture specimens that he tested and observed that the plot deviated significantly from the slope -1/2 required for linear elastic fracture mechanics (LEFM). He and others (e.g., Kesler et al., 1971) concluded from such deviations that fracture mechanics does not apply, but what they meant was LEFM, the only kind of fracture mechanics available at that time for nonductile materials. In 1983, a simple, approximate size effect law (Bažant 1983, 1984) was proposed and derived theoretically to describe the aforementioned size effect plot. This law subsequently received extensive and diverse justifications, including: (1) comparisons with tests of notched fracture specimens as well as unnotched reinforced concrete structures, (2) derivation based on energy release arguments and dimensional analysis, (3) comparison with discrete element (random particle) numerical model for fracture, (4) derivation as a deterministic limit of a nonlocal generalization of Weibull statistical theory of strength (Bažant and Xi, 1991), (5) comparison with finite element solutions based on nonlocal model of damage. The simple size effect law has been shown useful for incorporation into the design formulas for load capacity in various brittle modes of failure of reinforced concrete structures, as well as for evaluation of material fracture characteristics from tests. Significant contributions to the study of size effects have been made by Carpinteri (1986), Planas and Elices (1988 a, b) and others (e.g. van Mier, 1986).

The present lecture, after a brief review of the current status, will focus on presenting several recent advances made at Northwestern University, concerned with the asymptotic theory of the size effect, the possible role of the fractal nature of crack surfaces in the size effect (already discussed for concrete by Carpinteri et al., 1993, 1995; Carpinteri, 1994; Lange et al., 1993, and Saouma et al., 1990, 1994), and extension of the size effect law to failures at crack initiation from a smooth surface. Some implications for a new simplified size effect testing method for fracture characteristics will be also indicated, and the size effect predicted by the alternative Weibull-type statistical theory of strength will be put in perspective.

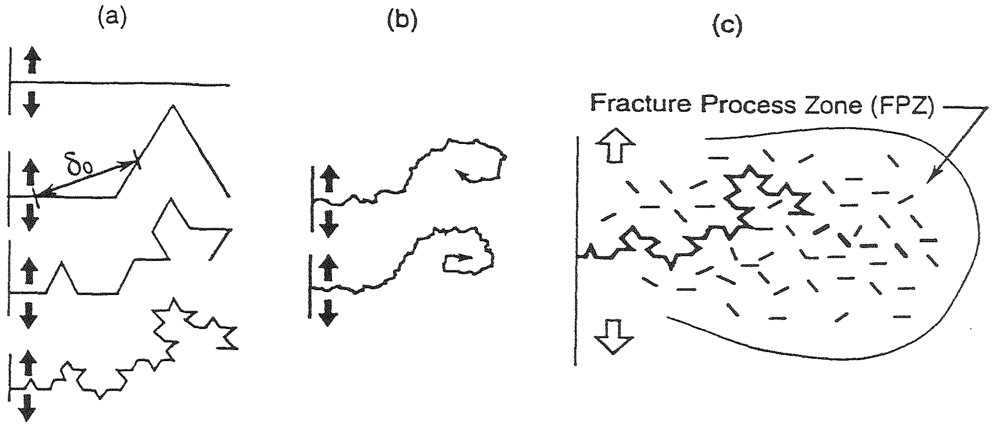

Figure 1: Von Koch curves as examples of fractal crack at progressive refinement; (b) recessive and spiraling segments which can be exhibited by fractal path; (c) fractal crack forming in a fracture process zone.

2 Asymptotic analysis of scaling for nonfractal and fractal cracks

We consider a crack representing a fractal curve (Fig. 1) whose length is defined as $a_\delta = \delta_0(a/\delta_0)^{d_f}$ where d_f = fractal dimension of the crack curve (≥ 1) and δ_0 = lower limit of fractality implied by material microstructure, which may be regarded as the length of a ruler by which the crack length is measured (Mandelbrot et al., 1984). The energy $\mathcal{W}_f$ dissipated by a fractal crack in a two-dimensional body of thickness b may defined as $\mathcal{W}_f/b = G_{fl}a^{d_f}$ where G_{fl} = fractal fracture energy (dimension Jm^{-d_f-1}). A nonfractal crack is the special case for $d_f = 1$, and in that case $G_f l$ reduces to G_f, representing the standard fracture energy (dimension Jm^{-2}).

We adopt three hypotheses: (1) Within a certain range of sufficiently small scales, the failure is caused by propagation of a single fractal crack. (2) The fractal fracture energy, G_{fl} is a material constant correctly defining energy dissipation. (3) The material may (although need not) possess a material length, c_f.

The rate of macroscopic energy dissipation $\mathcal{G}_{cr}$ with respect to the 'smooth' (projected) crack length a is:

$$\mathcal{G}_{cr} = \frac{1}{b}\frac{\partial \mathcal{W}_f}{\partial a} = G_{fl} d_f a^{d_f-1} \tag{1}$$

(Borodich, 1992; Molosov and Borodich, 1992). To characterize the size effect in geometrically similar structures of different sizes

D (characteristic dimensions), we introduce, as usual, the nominal stress $\sigma_N = P/bD$ where D = characteristic size (dimension) of the structure, P = dead load applied on the structure (or load parameter), and b = structure thickness in the third dimension (we restrict attention to two-dimensional similarity, although generalization to three-dimensional similarity would be easy). When $P = P_{max}$ = maximum load, σ_N = nominal strength.

The material length, c_f, may be regarded as the size (smooth, or projected) of the fractal fracture process zone in an infinitely large specimen (in which the structure geometry effects on the process zone disappear). The special case $c_f = 0$ represents fractal generalization of. Alternatively, if we imagine the fracture process zone to be described by smeared cracking or continuum damage mechanics, we may define $c_f = (G_{fl}/W_d)^{1/(2-d_f)}$ in which W_d = energy dissipated per unit volume of the continuum representing in a smeared way the fracture process zone (area under the complete stress-strain curve with strain softening). As still another alternative, with reference to nonlinear fracture mechanics such as the cohesive crack model, we may define $c_f = (EG_{fl}/f_t^2)^{1/(2-d_f)}$ in which f_t = material tensile strength, and E = Young's modulus.

We have two basic variables, a and c_f, both having the dimension of Euclidean length. The dimensionless variables may be chosen as

$$\alpha = a/D\,, \qquad \theta = c_f/D \tag{2}$$

According to Buckingham's theorem of dimensional analysis (e.g. Sedov, 1959; Barenblatt, 1979), the complementary energy Π^* of the structure may always be written as

$$\Pi^* = \frac{\sigma_N^2}{E}\, bD^2 f(\alpha, \theta) \tag{3}$$

in which f is a dimensionless continuous function of α and θ, characterizing the geometry of the structure and loading.

To express the first law of thermodynamics (energy balance), we note that the energy release from the structure as a whole must be calculated on the basis of a rather than a_δ. Indeed, the smooth length a is the length that matters for the overall energy of the elastic stress field on the macroscale. Therefore, $\partial\Pi^*/\partial a = \partial\mathcal{W}_f/\partial a$. Substituting (3) and differentiating, we get

$$\frac{\sigma_N^2}{E} D g(\alpha, \theta) + 2D^2 f(\alpha, \theta) \frac{\sigma_N}{E} \frac{\partial \sigma_N}{\partial a} = \mathcal{G}_{cr} \tag{4}$$

in which $g(\alpha, \theta) = \partial f(\alpha, \theta)(\partial \alpha)$ = dimensionless energy release rate.

The second law of thermodynamics yields the condition of stability of equilibrium state of a structure, which is equivalent (Bažant and Cedolin, 1991, chapter 10) to the condition $\partial P/\partial a > 0$. At the stability limit, $\partial P/\partial a = 0$ which coincides with the condition of maximum load. Therefore, if we want to know the size effect on the load at the limit of stability, that is the maximum load or nominal strength, we have $\partial \sigma_N/\partial a = 0$. So, Eq. (4) gives:

$$\sigma_N = \sqrt{\frac{E\mathcal{G}_{cr}}{Dg(\alpha_0, \theta)}} \tag{5}$$

where α_0 = relative crack length α at maximum load.

Because function $g(\alpha_0, \theta)$ ought to be smooth, we may expand it into Taylor series about the point $(\alpha, \theta) \equiv (\alpha_0, 0)$. Eq. (4) thus yields:

$$\sigma_N = \sqrt{\frac{E\mathcal{G}_{cr}}{D}} \left[g(\alpha_0, 0) + g_1(\alpha_0, 0)\frac{c_f}{D} + \frac{1}{2!}g_2(\alpha_0, 0)\left(\frac{c_f}{D}\right)^2 + \ldots \right]^{-1/2} \tag{6}$$

where $g_1(\alpha_0, 0) = \partial g(\alpha_0, \theta)/\partial\theta$, $g_2(\alpha_0, 0) = \partial^2 g(\alpha_0, \theta)/\partial\theta^2, \ldots$, all evaluated at $\theta = 0$. In the last equation we acquired the large-size asymptotic series expansion of size effect.

To obtain a simplified approximation, we now truncate the asymptotic series after the linear term. Then, introducing the notations:

$$D_0 = c_f \frac{g_1(\alpha_0, 0)}{g(\alpha_0, 0)}, \qquad Bf_t' = \sqrt{\frac{EG_f}{c_f g_1(\alpha_0, 0)}} \tag{7}$$

we finally obtain the following size effect of fractal fracture:

$$\sigma_N = Bf_t' D^{(d_f - 1)/2} \left(1 + \frac{D}{D_0}\right)^{-1/2} \tag{8}$$

For $d_f \to 1$, this law reduces to the nonfractal size effect law deduced by Bažant (1983, 1984), i.e.

$$\sigma_N = \frac{Bf_t'}{\sqrt{1+\beta}}, \qquad \beta = \frac{D}{D_0} \tag{9}$$

where β is called the brittleness number (Bažant 1987; Bažant and Pfeiffer, 1987).

For geometrically similar fracture test specimens, the value of α_0 is constant (independent of D). For brittle failures of geometrically similar reinforced concrete structures without notches, such

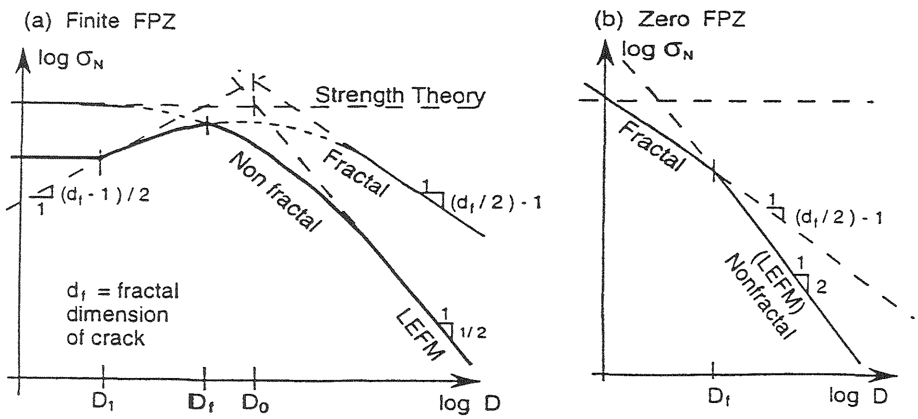

Figure 2: Size effect of geometrically similar fractal and nonfractal cracks for (a) finite and (b) zero size of the fracture process zone (cohesive zone).

as diagonal shear, punching of slab, torsion, anchor pullout or bar pullout, and splice failure, extensive laboratory evidence as well as finite element solutions (e.g. FraMCoS1, 1992; Bažant et al. 1994) show that the failure modes are in most cases approximately similar and $\alpha_0 \approx$ constant for a broad enough (albeit not unlimited) range of D. Then k, c_0, D_0, σ_N^0 and Bf_t' are also constant. In these typical cases, (8) describes the dependence of σ_N on size D only, that is, the size effect. Fig. 2 shows the size effect plot of $\log \sigma_N$ versus $\log D$ at constant α_0. Two size effect curves are shown: (1) the fractal curve and (2) the nonfractal curve (for which the possibility of a cut-off of fractality at both left and right ends is also shown). For the special case when $c_f \to 0$ (LEFM), the plot is shown in Fig. 2b. The transition from the first to the second power law corresponds to what is called the renormalization group transformation.

Among the specialists gathered at the FraMCoS conference, it is needless to elaborate on the fact that the fractal scaling seen in Fig. 2 does not agree with the bulk of experimental evidence. Just to give some examples, see the Fig. 3 showing the data for diagonal shear failure of reinforced concrete beams (Bažant and Kazemi, 1991) or double-punch compression failure of concrete cylinders (Marti, 1989). So we must conclude that the size effect is not affected nor explained by crack surface fractality.

There exists another objection to the fractal hypothesis. The fracture front does not consist of a single crack, but a wide band of microcracks and plastic-frictional slip planes, which all must form first and must dissipate energy if the fracture should grow. Only very few of the microcracks and slip planes eventually coalesce into a single continuous crack. Thus, even though the final crack surface

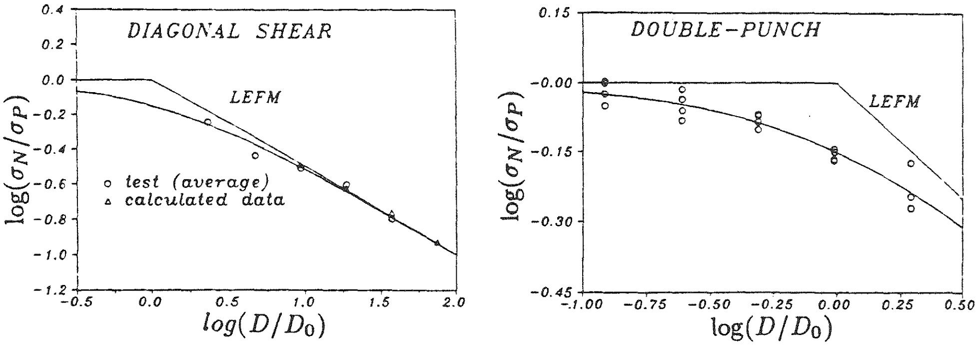

Figure 3: Size effect test results for diagonal shear failure of longitudinally reinforced concrete beams without stirrups (Bažant and Kazemi, 1990) (left), and for double punch compression failure of concrete cylinders (Marti, 1989) (right), both compared to the nonfractal size effect law; after Bažant et al. (1994).

may be to a large extent fractal, the fractality is irrelevant for the fracture process zone advance. Most of the energy is dissipated in the fracture process zone by microcracks and plastic-frictional slip planes that do not become part of the final crack surface and thus can have nothing to do with the fractality of the final crack surface. (The difficulty of correlating energy dissipation or G_f to γ is also supported by the observations of Cahn, 1989.)

So it transpires we should distinguish two types of fractality: (1) Fractality of the final crack surface, which is an undisputed morphological feature (although only for a limited range of scales); and (2) fractality of the fracture process controlling energy dissipation. The latter is not a significant property of concrete.

There are still further problems with the fractal hypothesis. The crack morphology must be kinematically admissible, such that the zones of material adjacent to the crack could move apart as two rigid bodies. But a fractal curve can have recessive segments and even spiraling segments (Fig. 1b) which preclude such movement.

Material length c_f can, in particular, be rigorously and unambiguously defined as the LEFM-effective length (measured in the direction of propagation) of the fracture process zone in a specimen of infinite size. In that case, $\theta = c_f/D = (a - a_0)/D = \alpha - \alpha_0$, and so $g(\alpha, \theta)$ reduces to the LEFM function of one variable, $g(\alpha)$. Also, $g(\alpha_0, 0)$ reduces to $g(\alpha_0)$, $\partial/\partial\theta = d/d\alpha$, and $g_1(\alpha, 0)$ takes the meaning of $g'(\alpha) = dg(\alpha)/d\alpha$. Eq. (7) thus yields:

$$D_0 = c_f \frac{g'(\alpha_0)}{g(\alpha_0)}, \quad Bf_t' = \sqrt{\frac{EG_f}{c_f g'(\alpha_0)}}, \quad \sigma_N^0 = \sqrt{\frac{EG_{fl} d_f \alpha_0^{d_f - 1}}{c_f g'(\alpha_0)}} \tag{10}$$

and so Eq. (9) takes the form:

$$\sigma_N = \sqrt{\frac{EG_{fl} d_f \alpha_0^{d_f - 1}}{g'(\alpha_0) c_f + g(\alpha_0) D}} \tag{11}$$

which involves the material fracture parameters. For $d_f = 1$, this reduces to the form of size effect law derived in a different manner by Bažant and Kazemi (1990, 1991) (also Eq. 12.2.11 in Bažant and Cedolin, 1991). The present derivation is simpler and more general. Same as for the nonfractal case, fitting this equation to size effect data could be used for determining G_{fl} if fracture of some material were a fractal process.

One may alternatively introduce more general dimensionless variables $\xi = \theta^r = (c_f/D)^r, h(\alpha_0, \xi) = [g(\alpha_0, \theta]^r$, with any $r > 0$. Then, expanding in Taylor series function $h(\alpha_0, \xi)$ with respect to ξ, one obtains by a similar procedure as before a more general large-size asymptotic series expansion (whose nonfractal special case was derived in Bažant, 1985, 1987):

$$\sigma_N = \sigma_P \left[\beta^r + 1 + \kappa_1 \beta^{-r} + \kappa_2 \beta^{-2r} + \kappa_3 \beta^{-3r} + \ldots \right]^{-1/2r} \tag{12}$$

where $\beta = D/D_0$ and $\kappa_1, \kappa_2, \ldots$ are certain constants. However, based on experiments as well some limit properties, it seems that $r = 1$ is the appropriate value for most cases.

Retaining more terms of the large-size asymptotic expansion (12), we improve accuracy for large D. But the expansion diverges for $D \to 0$. To get a better description of the size effect for small D, we need the small-size asymptotic expansion.

The previous energy release rate equation $(\sigma_N^2/E) D g(\alpha, \vartheta) = \mathcal{G}_{cr}$ (Eq. 4) is not meaningful for constitutive models such as the smeared cracking or the continuum damage mechanics. For such models, the material failure must be characterized by W_f rather than G_f. Therefore, instead of Eq. (4), the energy balance equation (first law) for $\partial \sigma_N / \partial a = 0$ (second law) must now be written in a dimensionally correct form as follows:

$$\frac{\sigma_N^2}{E} \left[\psi(\alpha, \eta) \right]^r = W_f \tag{13}$$

$\psi(\alpha, \eta)$ = dimensionless function of dimensionless variables $\alpha = a/D$ and $\eta = (D/c_f)^r = \vartheta^{-r}$ (variable ϑ is now unsuitable because $\vartheta \to \infty$ for $D \to 0$), and exponent $r > 0$ is introduced for the sake of generality, same as before. Because, for very small D, there is a diffuse failure zone, a must now be interpreted as the characteristic

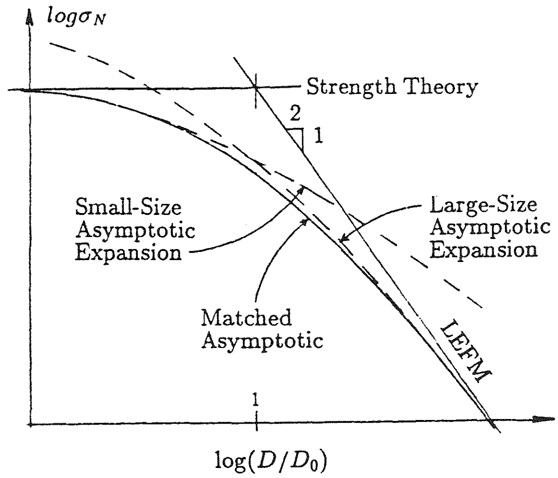

Figure 4: Asymptotic expansions of size effect and approximate size effect law obtained as matched asymptotics.

size of the failure zone, e.g., the length of cracking band. The same procedure as before now yields:

$$\sigma_N = \sigma_P \left[1 + \beta^r + b_2\beta^{2r} + b_3\beta^{3r} + \ldots\right]^{-1/2r} \tag{14}$$

in which $b_2, b_3, \ldots$ are certain constants and

$$\sigma_P = \sqrt{\frac{EG_{fl}}{c_f^{2-d_f}[\psi(\alpha_0, 0)]^r}}, \quad D_0 = c_f \left[\frac{1}{\psi(\alpha_0, 0)} \frac{\partial\psi(\alpha_0, 0)}{\partial\eta}\right]^{-1/r} \tag{15}$$

Eq. (14) represents the small-size asymptotic series expansion (Fig. 4). This expansion of course cannot yield the asymptotic limit for $D \rightarrow \infty$.

There is one important common feature of the large-size and small-size asymptotic series expansions in Eqs. (12) and (14) (Fig. 4). They have in common the first two terms. If either series is truncated after the first two terms, it reduces to the same generalized size effect law (Bažant, 1985):

$$\sigma_N = \sigma_P(1 + \beta^r)^{-1/2r} \qquad (\beta = D/D_0) \tag{16}$$

Because this law, including its special case for $r = 1$, is anchored to the asymptotic cases on both sides and shares with both expansions the first two terms, it represents an intermediate approximation of uniform applicability for any size, called the matched asymptotic (e.g. Bender and Orszag, 1978; Barenblatt, 1979). The value $r = 1$ appears, for various reasons, most appropriate for practical use.

In some problems, e.g. compression tests or, probably, the brazilian test, a plastic mechanism can operate simultaneoulsy with fracture. In that case, one of the following two generalizations of Eq. (16) with nonzero residual nominal strength σ_r may be appropriate (Bažant, 1987):

$$\sigma_N = \sigma_P \left(1 + \beta^r\right)^{-1/2r} + \sigma_r \tag{17}$$

$$\sigma_N = \text{Max}\left[\sigma_P \left(1 + \beta^r\right)^{-1/2r}, \sigma_r\right] \tag{18}$$

3 Universal size effect law for cracked and uncracked structures

Consider now unnotched quasibrittle structures that reach the maximum load when the crack initiates from a smooth surface, as in the test of modulus of rupture f_r of a plain concrete beam. Applying the size effect law in (1) for $\alpha_0 \to 0$ is impossible because $g(\alpha_0, 0)$ vanishes as $\alpha_0 \to 0$. To tackle this case, one must truncate the large-size asymptotic series expansion only after the third term. Then, considering that $r = 1$ (and $g(\alpha_0, 0) = 0$), we get, for the nonfractal case, instead of (11),

$$\sigma_N = \sqrt{\frac{EG_f}{g'(0)c_f + \frac{1}{2}g''(0)c_f^2 D^{-1}}} = \sigma_N^\infty \left(1 - \frac{2D_b}{D}\right)^{-1/2} \tag{19}$$

where $\sigma_N^\infty = \sqrt{EG_f/g'(0)c_f}$ and $D_b = -[g''(0)/4g'(0)]c_f$ (with subscript b referring to the boundary layer, in which the crack tip is located at crack initiation). Then it is convenient to apply the approximation $(1 - 2\xi)^{-1/2} \approx 1 + \xi$ with $\xi = D_b/D$, which does not change the size effect for large D. The resulting size effect law for failures at crack initiation from a smooth surface is

$$\sigma_N = Bf_r^\infty \left(1 + \frac{D_b}{D}\right) = f_r^\infty \left[1 - 0.0634 g''(0)\frac{c_f}{D}\right] \tag{20}$$

which coincides with the formula derived by Bažant, Li and Li (1995) in a different manner; f_r^∞ is the modulus of rupture for infinitely large beam (but not so large that Weibull statistical size effect would become significant), and B is a dimensionless parameter. It is important to note that the limiting value $g'(0)$ is shape independent, and so is Bf_r^∞, provided that the crack does not initiate from a sharp corner tip; always $g'(0) = 1.12^2\pi$ which leads to

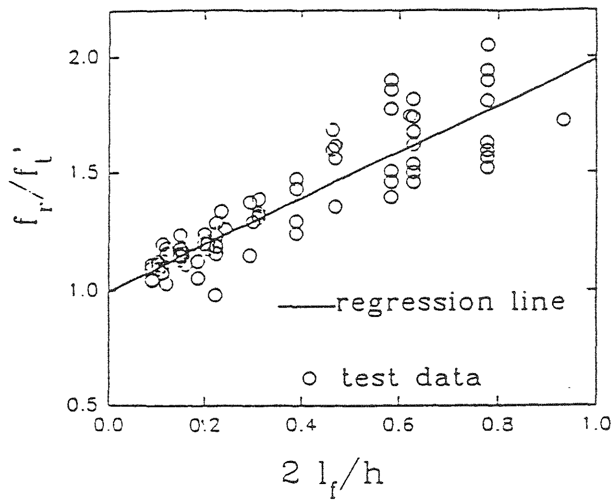

Figure 5: Test data for the dependence of modulus of rupture f_r of unreinforced concrete beams on beam depth D, and their optimum fit by the nonfractal formula (g); $f_t' =$ direct tensile strength, $h = D =$ beam depth, $l_f =$ constant (after Bažant and Li, 1993).

the last expression in (17). Note that Eq. (20) can be arranged as a linear regression equation σ_N versus $1/D$, which is helpful for identifying the constants from tests (Fig. 5).

The universal size effect law valid for failures at both large cracks and crack initiation from surface may now be proposed:

$$\sigma_N = \sigma_0 \left(1 + \frac{D}{D_0}\right)^{-1/2} \left\{1 + \frac{1}{s}\left[\left(\eta + \frac{D}{D_b}\right)\left(1 + \frac{D}{D_0}\right)\right]^{-1}\right\}^{s} \tag{21}$$

where, denoting $g = g(\alpha_0)$, $g' = g'(\alpha_0)$, $g_0' = g'(0)$, $g'' = g''(\alpha_0)$,

$$\sigma_0 = c_N\sqrt{\frac{EG_f}{c_f g'}}, \; D_0 = \frac{g'}{g}c_f, \; D_b = \frac{\langle -g''\rangle}{4g'}\bar{c}_f, \; \bar{c}_f = \kappa c_f \tag{22}$$

and $\eta =$ empirical constant close to 1; $\kappa = 1$ for $\alpha_0 \geq c_f$, $\kappa =$ constant > 1 for $\alpha_0 = 0$. Approximately, $s \simeq 1$.

Eq. (21) can be proven by expressing σ_N^{-2} in terms of ϑ and expanding it into Taylor series in ϑ about point $\vartheta = 0$. This yields (6) if $\alpha_0 > 0$, and $\sigma_N/f_r^\infty = 1 + D_b/(D + \eta D_b)$ if $\alpha_0 = 0$. The latter differs from (20) by constant η, but this does not affect the first two terms of the expansion in D^{-1} in the denominator in Eq. (19). Introducing constant η achieves that σ_N be finite for $D \to 0$, for both $\alpha_0 > 0$ and $\alpha_0 = 0$. Eq. (21) represents the matched asymptotic satisfying the first three (rather than just two) terms of the large-size expansion in ϑ and the first two terms of the small-size expansion in D/D_b.

4 Proposal of one-size version of size-effect method for measuring fracture characteristics

The size effect law serves as the basis of the size effect method for measuring G_f, c_f and other fracture characteristics (FraMCoS 1992). This method is simple to use because one needs to measure only the maximum loads of notched fracture specimens of sufficiently different brittleness numbers β (or sufficiently different sizes). However, the need to produce specimens of different sizes may sometimes be inconvenient. Two new kinds of the size effect method in which notched fracture specimens of only one size suffice will now be proposed.

4.1 Method using zero-size limit

Consider that the maximum loads or nominal strength σ_N of fracture specimens of only one size and one geometry are measured, and that the nominal strength σ_P for the zero-size (plastic) limit for specimens of this geometry can be calculated according to plasticity from a known value of tensile strength f_t'. Then, using Eq. (9), one finds that the fracture characteristics can be calculated from the formulas:

$$G_f = \frac{\sigma_P^2 D_0}{c_n^2 E} g(\alpha_0), \qquad c_f = \frac{g(\alpha_0)}{g'(\alpha_0)} D_0 \tag{23}$$

in which

$$D_0 = \frac{D}{(\sigma_P/\sigma_N)^2 - 1} \tag{24}$$

and the nominal strength is defined as $\sigma_N = c_n P_{max}/bD$ where c_n is a factor chosen for convenience. The value of σ_P can be easily calculated according to Mohr-Coulomb yield criterion, assuming a bi-rectangular stress distribution along the ligament of the specimen, with stresses on one and the other side equal to the tensile and compressive strength. An ongoing research at Northwestern University by Zhengzhi Li (private communication, 1995) has already shown that, for tensile strength equal to f_r, this method works very well for notched three-point-bend concrete fracture specimens of span-to-depth ratio 2.5, and gives results in good agreement with the original size effect method proposed by Bažant and Pfeiffer (1987), provided that the notched specimens are large enough (a depth of 6 in. appears to suffice, but the larger the better).

There is, however, one aspect which must be handled empirically. The value of tensile yield strength f_t', which is needed for calculating σ_P according to plasticity, is not predicted by fracture mechanics, and fracture mechanics does not even guarrantee the proper value of f_t' to be the same for various specimen geometries. In fact, upon equating the zero size limit $\sigma_P = Bf_t'$ to $c_n[EG_f/c_f g'(\alpha_0)]^{1/2}$ where $B = B(\alpha_0)$ = parameter to be calculated according to plasticity, we conclude that the tensile yield strength value to be used must satisfy the relation:

$$f_t' = c_n[EG_f/c_f g'(\alpha_0)]^{1/2}/B(\alpha_0) \tag{25}$$

This value is not constant. It varies with specimen geometry. Therefore, it is by chance that good results for the aforementioned specimens are obtained with $f_t' = f_r$ = modulus of rupture. For different geometries, different values of f_t' have to be used (e.g. $1.5f_r$ or $0.8f_r$). They would have to be calibrated empirically for each geometry to be specified as a standard testing method. Although this is not a practical problem, it does mean that this method does not have a complete theoretical foundation but contains an empirical ingredient. This feature is not surprising because the size effect law (9) is valid only within the approximate range $0.22 \leq D/D_0 \leq 4.5$, which excludes zero size.

4.2 Method using strength of unnotched specimen

The aforementioned empirical ingredient can be avoided at the cost of slightly more complicated calculations based on universal law (21). Instead of zero-size limit, one can experimentally determine the strength of unnotched specimens, preferably (but not necessarily) of the same size and shape. This is similar to the standardized test of modulus of rupture. The size effect on the modulus of rupture (Eq. 20) must of course be taken into account simultaneously with the size effect for notched specimens (Eq. 11). This can be accomplished by fitting the maximum load data with the extended size effect law in (21), which is valid for both cases. With (21), the optimum fit of the data for notched specimens of different sizes (Bažant and Pfeiffer, 1987) is in general not exactly the same as with (9). However, it is the same for typical notched beams, because $g'' > 0$ or $\langle -g'' \rangle = 0$, $D_b = \infty$ (and $\chi = 1$ in Eq. (26) below).

The fitting of Eq. (21) to the measured nominal strength of notched and unnotched specimens cannot be accomplished by linear regression. However, a computer library subroutine such as Levenberg-Marquardt nonlinear optimization algorithm readily yields the values of G_f and c_f that provide the best fit. Alternatively, iteration of linear regressions can also be used. Eq. (21) can be rearranged to the form $Y = AX + C$ in which (for $s = 1$):

$$X = \frac{g}{g'}D, \; Y = \frac{Ec_n^2}{g'\sigma_N^2}\chi, \; \chi = \left\{1 + \left[\left(\eta + \frac{4g'D}{\langle -g''\rangle \bar{c}_f}\right)\left(1 + \frac{gD}{g'c_f}\right)\right]^{-1}\right\}^2 \quad (26)$$

and $A = 1/G_f, C = \bar{c}_f/G_f$ (note that for $\alpha_0 = 0$ we have $g = 0$ and $X = 0$). For typical notched beams $\chi = 1$ because $g'' < 0$. But $\chi \neq 1$ for notchless beams. Parameter χ is assumed 1 for the first iteration and its value is then updated after each iteration. Z. Li (priv. comm., 1995) has already shown that the iterations converge very well and that this method, which has a consistent theoretical foundation, gives excellent results, very close to those obtained by Bažant and Pfeiffer (1987) with the original size effect method. Again, the higher the brittleness number of the notched specimens, the better the results. The specimen geometry and notch length should be chosen to as to minimize D_0, that is, the ratio g'/g.

5 Is Weibull-type size effect theory relevant to concrete?

Until about a decade ago, the size effect observed in concrete structures has been universally explained by randomness of strength and calculated according to Weibull theory. Recently, however, it has been shown (Bažant and Xi, 1991) that this theory cannot apply when large stable fractures can grow in a stable manner prior to maximum load. The main reason is the redistribution of stresses caused by stable fracture growth prior to maximum load and localization of damage into a fracture process zone. If the Weibull probability integral is applied to the redistributed stress field, the dominant contribution comes from the fracture process zone whose size is nearly independent of structure size D. The contribution from the rest of the structure in nearly vanishing, which means the fracture cannot occur outside the process zone. Because this zone has about the same size for specimens of very different sizes, the Weibull-type size effect must, therefore, disappear. A generalized version of Weibull-type theory, in which the material failure probability depends not on the local stress but on the average strain of a characteristic volume of the material, has been shown to yield realistic size effect and also to approach the size effect law in Eq. (9) as its deterministic limit (Bažant and Xi, 1991).

For concrete, Weibull-type size effect might be taking place only in very large structures that fail right at crack initiation, for example, in very deep unnotched plain concrete beams. Because for beam depths such as $D = 10D_b$ the stress redistribution in the boundary layer, underlying Eq. (18), is still significant, the beam depth beyond which the Weibull-type size effect could begin to dominate must be at least $D = 100D_b$. Hardly any case satisfying this condition exists in concrete practice. Besides, good practice requires designing structures so as not to fail at crack initiation.

6 Promising research directions

Although much has been learned about the size effect and scaling aspects of concrete structures during the last dozen years, large gaps of knowledge still persist. As a nonexhaustive list of promising research directions, the following can be offered:

1. *Compression fracture of concrete:* This is actually the most important type of fracture, and it is known to exhibit size effect, in some cases very strong and others mild or nonexistent. Standard test cylinders in compression seem to exhibit almost no size effect, but very long ones do (e.g., van Mier, 1986). Reduced scale tied columns have been experimentally shown to exhibit a very strong size effect, the stronger the higher the slenderness (Bažant and Kwon, 1993). Compression fracture is more difficult than tensile fracture, because it is inherently a triaxial phenomenon, strongly sensitive to lateral confining stresses, while tensile fracture is essentially a uniaxial phenomenon. Compression fracture is not a primary mode of failure, but a secondary mode appearing as a result of microscopic splitting tensile fractures and plastic-frictional shear slips. It seems that a good model for compression fracture is the propagation of a band of axial splitting cracks in which the slabs of the material between parallel cracks undergo post-critical buckling (Bažant, 1993). The band propagates laterally to the direction of splitting cracks, depending on boundary conditions, and if it runs obliquely it terminates with what looks as a shear failure. This mechanism has been formulated analytically (Bažant, 1993) and has provided reasonable predictions for the size effect. It appeared that the difference of nominal strengths from a certain residual strength diminishes as size$^{(-2/5)}$. The same type of size effect has been theoretically derived for compression breakout of boreholes in rock, by using energy release analysis based on Eschelby theorem (Bažant et al., 1993). Much is also known about microscopic mechanisms of compression fracture, including various wedging configurations causing axial splitting microcracks, as well as wing-tip cracks emanating from shear loaded cracks and axial splitting cracks emanating from pores. However, these mechanisms do not ex-

plain the compression fracture globally, only its initiation. One needs to analyze interaction of such microfractures in a smeared, continuum manner. In a general and fundamental sense, compression fracture is still not really understood.

2. *Effect of loading rate and duration on scaling:* Concrete fracture is time-dependent, for two reasons: (a) the crack growth is rate-dependent, and (b) creep in the bulk of the structure plays a significant role. The former phenomenon is important for ceramics and rocks, the latter for polymers, but for concrete both phenomena are important. Experimental results have already shown that the size effect is rather different for loading at different rates, and particularly that the brittleness of response increases with a decreasing loading rate, which is due to creep and is not observed in ceramics and rocks (Bažant et al., 1995). However, much deeper understanding of the rate and time effects is needed.
3. *Micromechanical aspects:* The aggregrate size, gradation, shape, and mix ratios of concrete, strengths of the interfaces between mortar and large aggregates, etc., affect the quasibrittle fracture characteristics, including G_f, c_f and size effect parameter D_0. Understanding of this problem is particularly weak at present.
4. *Scaling aspects of cohesive crack and nonlocal damage models:* It is generally agreed that these models, and especially the latter, are capable of providing a more general and fundamental continuum description of the fracture process. However, the types of size effect inherent to these models have been little explored, and effective numerical methods to obtain the size effect for these models (whose study was attempted by Li and Bažant, 1994) are also developed insufficiently.
5. *Interference of other types of size effect:* In practice, the size effect due to energy release in fracture is often combined with other size effects. Aside from the statistical effect, these include the size effect from diffusion phenomena such as water migration through concrete (drying) and heat conduction (variable environmental temperature, hydration heat). Drying and other diffusion phenomena cause significant size effects, generally strongly time-dependent and very different from the size effect discussed here. Significant studies have already been made (e.g., Planas and Elices, 1993), however, much more is needed. Another type of size effect arises in the boundary layer of concrete, due to its different composition as well as lack of lateral restraint.

7 Concluding remarks

As explained in the present lecture, the size effect in quasibrittle structures can be analyzed on the basis of asymptotic series expansions and asymptotic matching. This approach, widely used in fluid mechanics, is very powerful because, for normal sizes, the problem at hand is extremely difficult, but becomes much simpler for very large sizes (LEFM) and for very small sizes (plasticity). Asymptotic matching is an effective way to obtain a simplified description in the normal, intermediate range of sizes. Correlation between the size effects after large stable crack growth and at crack initiation from a smooth surface is also possible on the basis of the asymptotic energy release analysis, and a universal size effect law comprising both types of size effect can be formulated.

The fractal aspect of the morphology of crack surfaces observed in concrete does not appear to play a significant role in fracture propagation and the size effect.

Knowledge of the size effect law is useful for identifying material fracture characteristics from tests. In contrast to the original size effect method of testing the fracture energy of concrete, in which specimens of very different sizes need to be used, it is possible to formulate a one-size version of the size effect method, for which only the maximum loads of notched specimens of one size need to be measured and the maximum load of either a specimen extrapolated to zero size or a specimen of the same size but without a notch is determined by either plastic analysis or experiment.

The statistical size effect as described by Weibull's theory of random strength does not play a significant role in concrete structures, except for very large structures failing at crack initiation — a behavior which is neither typical nor desirable.

The theory of scaling for concrete structures is a rapidly moving field in which significant advances still remain to take place.

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