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ADAPTIVE MESH REFINEMENT PROCEDURE OF FINITE ELEMENT METHOD FOR FRACTURE MECHANICS

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Abstract

This paper describes an adaptive mesh refinement procedure by the finite element method for a two-dimensional fracture mechanics problem. The finite element approximation error can be estimated by calculating a stress error norm on the basis of the discontinues analysis of stress components around each element boundary. The optimal discretization can be realized by combining a mesh generator of unstructured mesh with finite element computation, and adaptive mesh refinement is used to analysis stress filed at crack front tip on the fracture mechanics.

1 Introduction

Finite element method or FEM is one of the discretized numerical method, which is widely applied to solve various engineering problems and has shown it's advantages day by day. It is a key problem that continuous body is divided into many elements or discretized as finite element mesh in finite element method. Discretized method and quality of generated mesh as well level of optimal discretization play an important part in the reliability and accuracy of approximated solution. The problem of optimal discretization has hindered the development and wide application of finite element method sometimes. Discretized method and adaptive analysis based on the error estimation has been studying widely and obtained good results for approximation solution of the finite element method.

The adaptive analysis is classified into two types: one is *p*-method by applying higher-order polynomials to shape functions of element(Zienkiewicz and Taylor, 1990). The *p*-method has almost no relations with mesh generators, because it keeps the structure of initial mesh. The hierarchical idea to update shape functions is the key in this method. Sometime coarse meshes work well by *p*-method. With this reason, it is useful especially in 3-dimensional analysis. However, solution might be vibrated near the borders of the update element because of the higher-order polynomials(Babuska and Rheinboldt, 1988). Also post-processing results of finite element approximation are not satisfying by this method.

Another is *h*-method by refining generated meshes in part of the domain(Peraire et al., 1987). The idea is that every finite element possesses almost equal amounts of the approximated error. The number of nodes and elements are easily increased by this method because of relations with mesh generators. It can handle the complicated shaped domain and stress concentration problem as well transient dynamic analysis. Also it improves post-processing results of finite element computation markedly. The process of mesh adaptation can be classified into two steps: one is the estimation of error measure, the other is the control of meshes.

2 Error estimation of the finite element approximation

In finite element analysis, the process to decompose a whole domain into many elements and generally provide uniform meshes for arbitrary domain. However, uniform meshes are not get acceptable solutions in most cases, because loaded distributions are not uniform such as stress concentrations which properly biased meshes are required. A basic idea of the adaptive finite element method is that an approximation error is estimated without knowing the exact solution, but just knowing the finite element approximation and smooth solution based on mathematical methods. A general elliptic linear equation in finite element analysis is written as

$$\mathbf{L}\mathbf{u} = \mathbf{S}^T\mathbf{D}\,\mathbf{S}\,\mathbf{u} = \mathbf{f} \qquad \text{in } \Omega \tag{1}$$

Together with appropriate boundary conditions is typical elastic problems, where $\mathbf{u}$ is the displacement, $\mathbf{S}$ defines strains and stresses, $\mathbf{D}$ is the elastic matrix as

$$\varepsilon = \mathbf{S}\ \mathbf{u} \qquad \text{and} \qquad \sigma = \mathbf{D}\ \mathbf{S}\ \mathbf{u} \tag{2}$$

With a standard finite element approximation using shape function

$$\mathbf{u} \approx \hat{\mathbf{u}} = \mathbf{N}\ \bar{\mathbf{u}} \tag{3}$$

and

$$\hat{\varepsilon} = \mathbf{S}\ \hat{\mathbf{u}} \qquad \text{and} \qquad \hat{\sigma} = \mathbf{D}\ \mathbf{S}\ \mathbf{u} \tag{4}$$

A C_0 continuous approximation for $\hat{\mathbf{u}}$ has been assumed in finite element method and this resulted in discontinuous stresses $\hat{\sigma}$ among elements(Hinton and Owen, 1977). See Fig. 1, σ^* and $\hat{\sigma}$ are the theory solution and finite element approximation respectively, $\overline{\sigma}^*$ is the smooth solution.

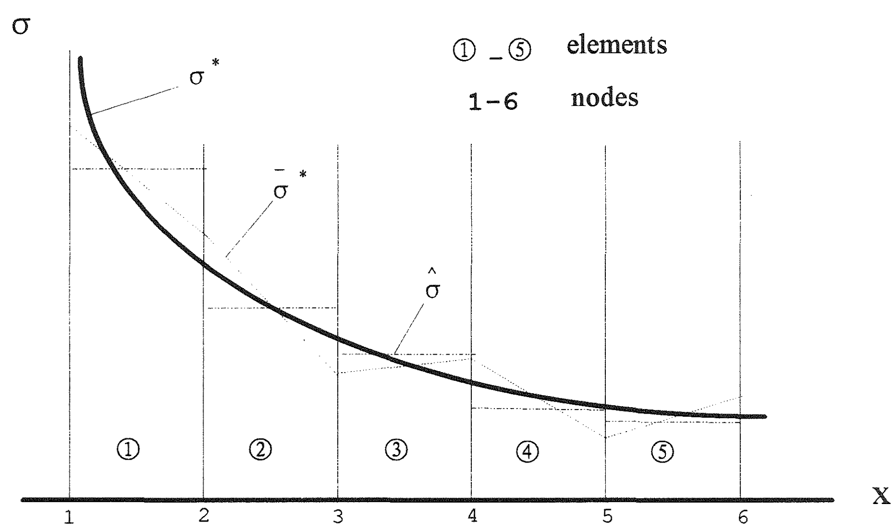


Fig. 1. Stress distribution among elements and smooth solutions(linear u)

This discontinuity just shows the errors from mesh discretization and stress concentration. A continuous set of stresses σ^* can be obtained by 'projection' and averaging(Hinton and Campbell, 1974) or superconvergence recovery technique(Zhu and Zienkiewicz, 1990) and same shape functions are used for representing the σ^* field,

$$\overline{\sigma} = \mathbf{N} \ \overline{\sigma}^* \tag{5}$$

the approximation equation is achieved by weighted residual requirement for equality between σ^* and $\overline{\sigma}^*$,

$$\int_{\Omega} \mathbf{N}^{\mathrm{T}} \left(\sigma^* - \hat{\sigma} \right) \mathrm{d}\,\Omega = 0 \tag{6}$$

by substituting (5) into (6),

$$\left(\int_{\Omega} \mathbf{N}^{\mathrm{T}}\mathbf{N} \ \mathrm{d}\,\Omega \right) \overline{\sigma}^* = \int_{\Omega} \mathbf{N}^{\mathrm{T}} \hat{\sigma} \ \mathrm{d}\,\Omega \tag{7}$$

giving

$$\overline{\sigma}^* = \mathbf{M}^{-1} \int_{\Omega} \mathbf{N}^{\mathrm{T}} \hat{\sigma} \ \mathrm{d}\,\Omega \tag{8}$$

and

$$\mathbf{M} = \int_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{N} \ \ \mathrm{d}\,\Omega \tag{9}$$

The above computation is made particularly simple if lumped or diagonal is used to M. Also smooth solution $\overline{\sigma}^*$ is approximations of higher order than finite element solution $\hat{\sigma}$ and two sets of values can be used to estimate the spatial discretization error.

$$\mathbf{e} = \sigma^* - \hat{\sigma} \tag{10}$$

For an integral measure of a pointwise defined variables, various norms may be adopted. One of the commonly used norm is the L_2 norm written as

$$\| \mathbf{e} \| = \left(\int_{\Omega} \mathbf{e}^{\mathrm{T}} \mathbf{e} \ \ \mathrm{d}\,\Omega \right)^{1/2} \tag{11}$$

for vector cover domain Ω.

3 Computation of element sizes for the optimal mesh

The adaptive mesh refinement scheme is based on the idea that after remeshing each element will have the same error and the overall estimated percentage errors

$$\eta_{est} = \frac{\| \mathbf{e} \|_{est}}{\| \sigma^* \|} \times 100\% \tag{12}$$

will be equal to specified percentage errors

$$\eta_{spe} = \frac{\| \mathbf{e} \|_{spe}}{\| \sigma^* \|} \times 100\% \tag{13}$$

From the definition of L_2, it is noted that the square of the overall error norm over the whole domain is equal to the sum of squares of local error norm of each element, i.e.

$$\| \mathbf{e} \|^2 = \sum_{i=1}^{m} \| \mathrm{e} \|_i^2 \tag{14}$$

It has been assumed that after remeshing each element will have the same local error, for element i,

$$\| \mathrm{e} \|_{i\,spe}^2 = \| \mathbf{e} \|_{spe}^2 / \mathrm{m} \tag{15}$$

and

$$\| \mathrm{e} \|_{i\,spe} = \| \mathbf{e} \|_{spe} / \mathrm{m}^{1/2} = \| \sigma^* \| \, \eta_{spe} / \mathrm{m}^{1/2} \tag{16}$$

where i and m are the number of element and the total number of elements respectively. For elements without singularity, it is assumed that the rate of convergence of the local error is O (h^{λ}), where h is the size of the element and λ is the polynomial order of the interpolation function(Zienkiewicz and Zhu, 1987). Thus the new element size can be predicted by the following equations

$$(h_i)_{new} / (h_i)_{old} = \left[\left(\| \mathrm{e} \|_i \right)_{spe} / \left(\| \mathrm{e} \|_i \right)_{est} \right]^{1/\lambda} \tag{17}$$

In order to avoid too large and too small elements, both the upper and the lower limits for the element size are often specified as

$$h_{min} \le (h_i)_{new} \le h_{max} \tag{18}$$

For problems where singularity exists, the mesh size predicted by equation(17) needs to modification(Zienkiewicz et al., 1988)

$$h' = q^*(h_i)_{new} \tag{19}$$

where the grading factor q is less than one and the value of q=0.5 is used for the singular points.

4 Numerical example

Fig. 2 shows a central cracked panel under uniform tension. The stress intensity factor K_I is defined in fracture mechanics along cracked line(x axis, $\theta = 0$ and $\sin\theta = 0$), i.e.

$$\sigma_y = \sigma_x = K_I / \sqrt{2\pi r} \quad (r \ll a) \tag{20}$$

For thin panel of the finite size, stress intensity factor K_I can be expressed as(Chu, 1979)

$$K_I = F\left(\frac{a}{2w}\right) \sigma \sqrt{\pi a} \tag{21}$$

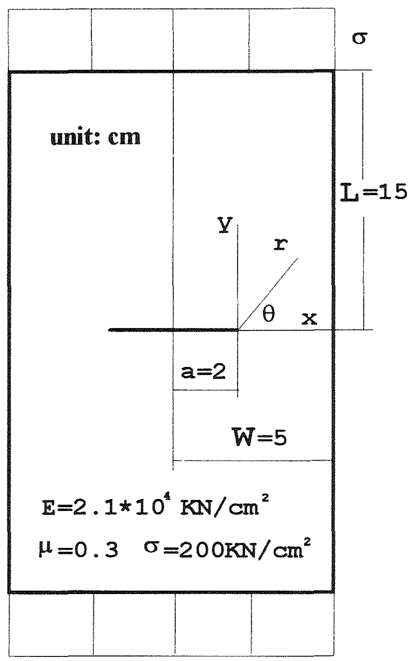


Fig. 2. a centre cracked panel

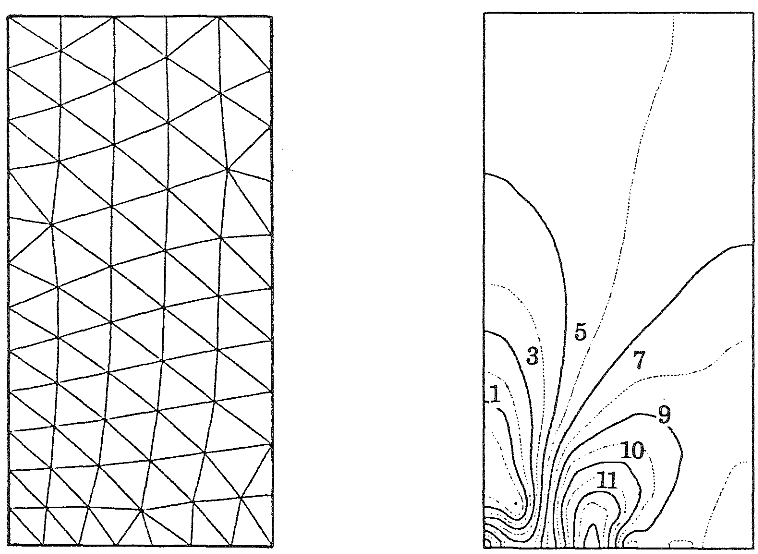
where $F\left(\frac{a}{2w}\right) = \sqrt{\sec\left(\frac{\pi a}{2w}\right)}$ and $F\left(\frac{a}{2w}\right)$ is modified factor.

In many experiments and finite element methods, stress intensity factors are calculated by mathematical recurrence method near the crack tip generally. This often increases computational error because they have two approximated calculations and produce the accumulative error. However, adaptive mesh refinement procedure is easy to process stress concentration problems and can improve reliability and accuracy of the finite element approximation. The computation begins with an initial uniform mesh as given in Fig. 3(a) and Fig. 3(b) shows stress contours. When choosing different η_{spe}, new meshes are generated, which are shown in Fig. 4 – 6 together with the refined meshes and stress contours. From mesh and computational results in Fig. 6, stress $\sigma\ y$ can be obtained as $\sigma\ y$ =1335.7KN/cm2 (x=0.0254cm from cracked tip). Also this result could be used to calculate K_I by equation (20), because of x=0.0254 <<a and r/a=0.01275. Theoretical solution K_I can be calculated by equation (21). $(K_I)_1$=557.36KN/cm$^{-3/2}$ and $(K_I)_2$= 533.6KN/cm$^{-3/2}$ are theoretical solution and finite element proximation respectively. The error is calculated as

$$\delta = \frac{(K_I)_1 - (K_I)_2}{(K_I)_1} \times 100\% = \frac{557.36 - 533.6}{557.36} \times 100\% = 4.26\% \qquad (22)$$

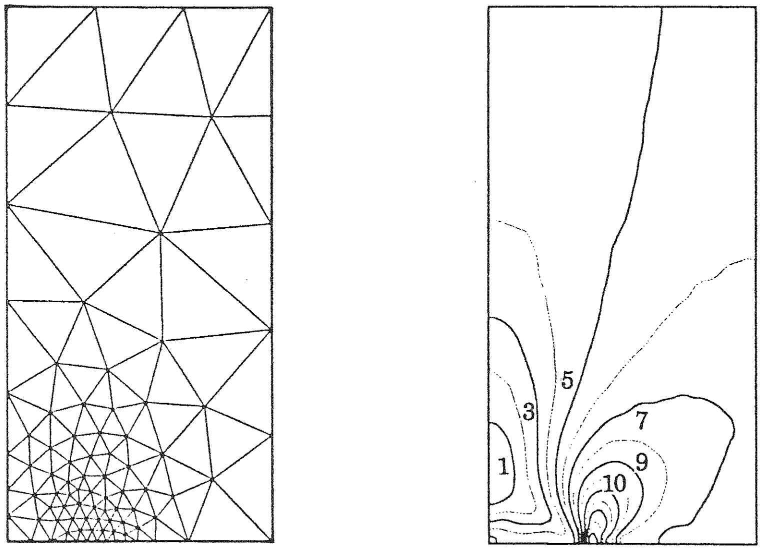
5 Conclusions

Adaptive mesh refinement plays important part to analysis stress field at crack front tip for the fracture mechanics. This method increases the reliability and accuracy of the finite element approximations. Also it improved post-processing results of the finite element computation markedly. All of these works has a great significance for further research work and practical application in engineering.



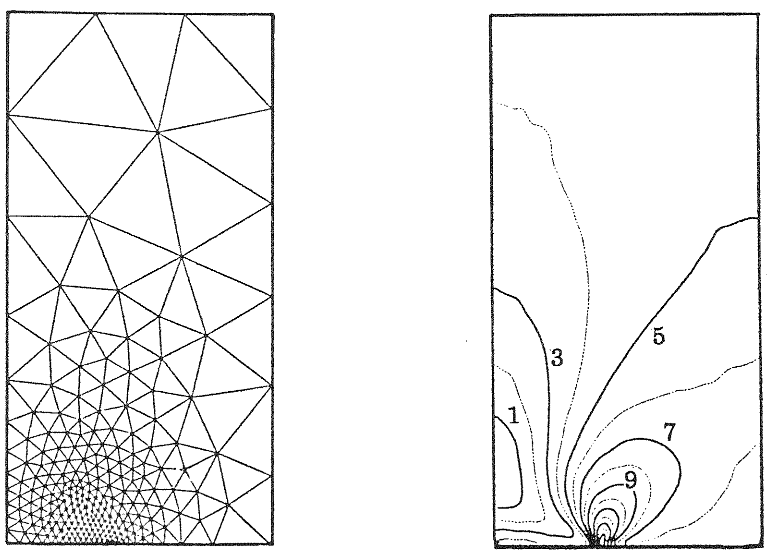
(a) initial mesh(N=71, E=106) (b) stress contours($\eta_{spe} \approx 25\%$)

Fig. 3. Initial uniform mesh and stress contours



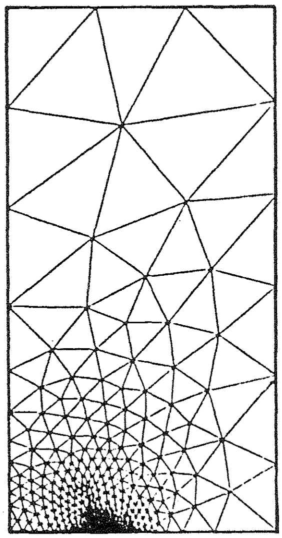
(a) refined mesh(N=78, E=145) (b) stress contours($\eta_{spe} \approx 10\%$)

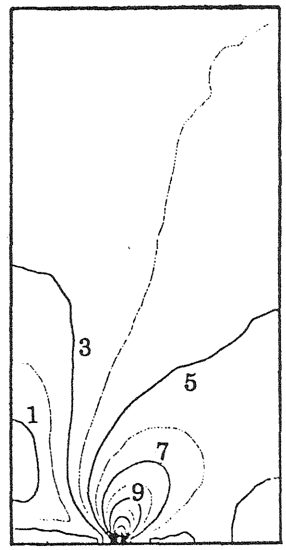
Fig. 4. Refined mesh and stress contours



(a) refined mesh(N=254, E=460) (b) stress contours($\eta_{spe} \approx 5\%$)

Fig. 5. Refined mesh and stress contours





(a) refined mesh(N=398, E=722) (b) stress contours($\eta_{spe} \approx 1\%$)

Fig. 6. Refined mesh and stress contours

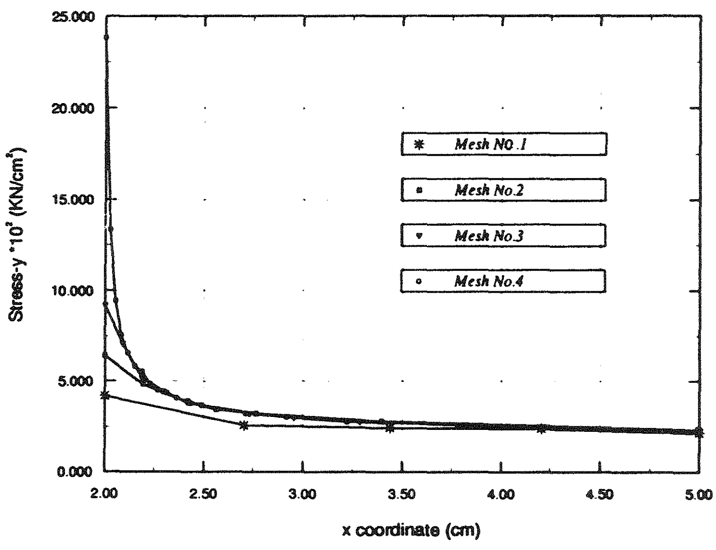


Fig. 7. Stress σ_y curves along cracked line(x axis) for different meshes

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