Fracture Mechanics of Concrete Structures,
Proceedings FRAMCOS-2, edited by Folker H. Wittmann,
AEDIFICATIO Publishers, D-79104 Freiburg (1995)

AN ADAPTIVE MESH STRATEGY BASED ON THE ALE FORMULATION TO TRANSIENT FINITE ELEMENT ANALYSIS OF LOCALISATION

L. Bodé,
Laboratoire de Génie Civil, I.U.T. Egletons, France
G. Pijaudier-Cabot,
Laboratoire de Mécanique et Technologie, ENS Cachan / CNRS /
Université P. et M. Curie, Cachan Cedex, France
A. Huerta,
ETS Ingenieros de Caminos, Dpto de Matematica Aplicada III, Universitat
Politecnica de Catalunya, Barcelona, Spain

Abstract
Finite Element analysis of strain localisation in transient dynamics requires a mathematically consistent model for the description of localisation and a rich Finite Element discretisation in order to describe correctly the localised zones. In this paper, a mesh adaptive method based on the Arbitrary Lagrangian Eulerian formulation is coupled to a non local damage model. A bending beam example illustrate the application of this strategy in the context of beam analysis. A first attempt to develop the ALE strategy to two-dimensional problems is discussed.

1 Introduction

Numerical analysis of concrete and reinforced concrete in transient dynamics addresses theoretical and computational problems. First a mathematically consistent model for the description of localisation and of the post localisation behaviour of the material must be implemented.

Models that introduce an internal length scale, as non local models, guaranty that finite element computations on strain softening materials up to failure remain sound from a theoretical and computational viewpoint . But when localisation occurs, high localised gradient zones appear. An accurate description of the localised zones implies a rich Finite Element discretisation, for instance a very fine mesh (Huerta et al. (1994b)). As the size of the localised zones are generally small compared to the size of the entire structure, an uniform fine mesh is not necessary and would provided a prohibitive computer cost. An adaptive refinement strategy based in reducing the element size on the localisation bands, seems to be an interesting solution.

We have chosen to use such a technique based on a special cinematic formulation, the Arbitrary Lagrangian Eulerian (ALE) formulation. The efficiency of coupling such a method to transient Finite Element analysis of localisation in a non local continuum, has been already illustrated on one-dimensional examples (Huerta et al. (1992)). This paper deals firstly with an extension of the ALE for beam problems, and secondly with a first feasibility study for two-dimensional problems.

The non local constitutive relations used will be briefly presented. Then after having recalled the basis of the ALE formulation, the attention will be focused on the treatment of the update problem and on the choice of the remeshing strategy. Finally, an example of application of ALE in beam analysis will be presented.

2 Non Local Constitutive Relation

The localisation limiter used in this study is a non local damage model. The stress-strain relation is identical to that of a scalar continuous damage model

$$\sigma_{ij} = (1 - D)C_{ijkl}\varepsilon_{kl} \tag{1}$$

in which σ_{ij} and ε_{ij} are the components of the stress and strain tensors respectively. The scalar D is a damage variable and C_{ijkl} are the components of the elastic stiffness of the undamaged material. The growth of damage is defined by a loading function :

$$f(\overline{Y}(\mathbf{x}), D) = \int_0^{\overline{Y}(\mathbf{x})} F(z)\,dz - D \tag{2}$$

where F is a function which describes the growth of damage, and $\overline{Y}(\mathbf{x})$ is the average energy release rate due to damage at point $\mathbf{x}$. This quantity introduces the non local nature of the model :

$$\overline{Y}(\mathbf{x}) = \frac{1}{Vr(\mathbf{x})} \int_V \psi(\mathbf{s}-\mathbf{x}) Y(\mathbf{s}) d\mathbf{s} \quad \text{with} \quad Y(\mathbf{x}) = \frac{1}{2} \varepsilon_{kl} E_{ijkl} \varepsilon_{ij} \tag{3}$$

The average energy release rate is defined as a weighted average of the local energy rate given in Eq. (3) over the entire material domain denoted as V. $V_r(\mathbf{x})$ is the representative volume defined as :

$$V_r(\mathbf{x}) = \int_V \psi(\mathbf{s}-\mathbf{x}) d\mathbf{s} \tag{4}$$

The weighting function ψ is a normalised bell-shaped function :

$$\psi(s-x) = \exp\left[-\frac{\|\mathbf{s}-\mathbf{x}\|^2}{2l_c^2} \right] \tag{5}$$

where lc is the internal length of the non local continuum. The evolution law of damage is :

$$D = \frac{\delta g}{\delta \overline{Y}} \tag{6}$$

with the Kühn Tucker conditions $\delta \geq 0, f \leq 0$ and $\delta f = 0$. g is the evolution potential which is simply g = $\overline{Y}(x)$ in the present model. In the applications, function F is of the form :

$$F(\overline{Y}) = \frac{B}{\left[1 + B(\overline{Y} - Y_0)\right]^2} \tag{7}$$

where B and Y_0 are material parameters.

3 ALE Formulation

The ALE approach is based on an arbitrary motion of the mesh, independently of the material particles motion and independently of the spatial frame. In others words, the adaptivity of the discretisation size is made moving the nodes during the computation.

This formulation has been used in the past for large boundary motions problems in fluids (Donea et al. (1982, Huges et al. (1981) and more recently in solids (Liu et al. (1986), Huerta et al. (1994a)). The same formulation is used here not to account for boundary motions but rather to refine the spatial interpolation in order to capture localisation areas. Compared to others adaptive techniques its application to strain localisation problems in dynamics is essentially motived by the fact that it allows a precise description of the moving interface while the number of degrees of freedom and the mesh connectivity remain unchanged from one time step to other.

The ALE cinematic formulation introduces a third configuration (reference configuration attached to the mesh) that can move independently of the two classical configurations : the Lagrangian one (attached to material particles) and the Eulerian one (attached to spatial domain). The material points are noted X and their coordinates are noted, x in the space at time t and χ in the ALE domain.

For a general physical property denoted as f, the following fundamental relation can be obtained (Donea et al. (1982)) :

$$\left.\frac{\partial f}{\partial t}\right|_X = \left.\frac{\partial f}{\partial t}\right|_\chi + c_i.\frac{\partial f}{\partial x_i} = \left.\frac{\partial f}{\partial t}\right|_\chi + \vec{c}.\vec{grad}\, f \tag{8}$$

with the notation $\frac{\partial f(x,\cdot)}{\partial t} = \left.\frac{\partial f}{\partial t}\right|_x$ that means "partial time derivative at x fixed".

$\vec{c}$ is the convective velocity defined as $\vec{c} = \vec{v} - \hat{\vec{v}}$ (9a)

$$\text{where : } \hat{\vec{v}} = \left.\frac{\partial \vec{x}}{\partial t}\right|_\chi \quad \text{and} \quad \vec{v} = \left.\frac{\partial \vec{x}}{\partial t}\right|_X \tag{9b}$$

The conservation of momentum that governs the motion of the continuum may also be written in the ALE description (Liu et al. (1986)) :

$$\rho\left.\frac{\partial \vec{v}}{\partial t}\right|_\chi + \rho\vec{c}.grad\,\vec{v} = \vec{div}\,\sigma + \vec{b} \tag{10}$$

where is the mass density and $\vec{b}$ are the body forces.

The weak formulation from this equation leads to a standard form :

$$\int_V \delta\dot{\varepsilon}:\sigma\, dV + \int_V \rho\delta\vec{v}.\left.\frac{\partial \vec{v}}{\partial t}\right|_\chi dV + \int_V \rho\delta\vec{v}.(\vec{c}.grad\,\vec{v})dV - \int_V \delta\vec{v}\vec{b}\, dV - \int_\Gamma \delta\vec{v}.\vec{f}\, d\Gamma = 0 \tag{11}$$

with $\vec{f}$ = external forces applied on frontier Γ.

In the following, no body forces have been considered. Upon discretisation, this equation reads :

$$Ma = f_{ext} - f_{conv} + f_{int} \tag{12}$$

where M is the lumped matrix, f_{ext} are the external forces, f_{int} the internal forces and a is the nodal acceleration vector.

The only difference with the equations obtain in the classical Lagrangian formulation is that an additional convective term appears: f_{conv} are the convective forces.

A central difference scheme is used for the integration of the momentum equation. This integration scheme has the advantage of being explicit. Nevertheless, the stability of this scheme is only assured for a time step lower than a critical time step.

4 Update problem

In ALE description, convective terms appears in all time derivative quantities, including in the constitutive relations. In the case of non linear solids, the stresses at a given time depend of the total history of state variables. For such materials, the ALE expression of incremental constitutive relations leads to systems of differential equations. For each scalar component of a tensorial variable, we have :

$$\left.\frac{\partial \tau}{\partial t}\right|_{\chi} + \vec{c}.\vec{grad}\tau = q \qquad \text{(q non linear term)} \tag{13}$$

The numerical integration of theses differential equations introduces an additional difficulty compared to the Lagrangian case : Due to the mesh motion, the integration points correspond at different material particles from one time step to other. Simultaneous, the state variables have to be transferred from the old mesh to the new mesh.

This is performed in two step : First, a pseudo-Lagrangian integration is performed (we assume that the mesh has not move). Secondly, the variables are then updated from the old mesh to the new mesh. We use here integrated equations of evolution. Hence the first step is immediate : Starting with τ(t), we obtain τ*(t+t).

In beam analysis, the second step is then performed using a full upwind method (Huerta (1994a. This update algorithm is explicit.

$$\tau_e(t+\Delta t) = \tau_e{}^*(t+\Delta t) - \Delta t.\left\{ \langle c_i \rangle_+ \frac{\tau_e - \tau_{e-1}}{h_e} + \langle c_{i+1} \rangle_- \frac{\tau_{e+1} - \tau_e}{h_e} \right\} \tag{14}$$

with τ_e is the value of in element e, h_e is the length of element e, i and i+1 are the two nodes of the element e and c_i, c_{i+1} are the values of the convective velocity at nodes i, i+1 respectively. $\langle\ \rangle_+$ and $\langle\ \rangle_-$ are respectively the positive part value and negative part value operators.

In two-dimensional analysis, we transferred state variables values making, for sake of simplicity, a projection from the old mesh to the new mesh.

5 Remeshing strategy

The remeshing consists in expressing the field of the convective velocity at each node. Then the motion of the mesh will be perfectly defined. As it is currently done, the remeshing strategy that we used is defined, first by a mesh indicator and second by a mesh optimality criterion.

5.1 Mesh indicator

In the present problem, the interest is focused on obtaining the best description of the damage localisation zone. Therefore, the mesh indicator will be defined in order to concentrate elements in the neighbourhood of sharp variations of damage.

Our mesh indicator belongs to the family of remeshing indicators related to the quality of the interpolation of the unknown variables. It is simply expressed as a function of the variation of a dependant variable, damage or strain invariant. It provides only a control of the quality of the mesh, that is it simply detects when the mesh is too coarse. As the number of elements in the mesh is unchanged, if an area of the mesh needs more elements, the number of element will decrease in the rest of the mesh.

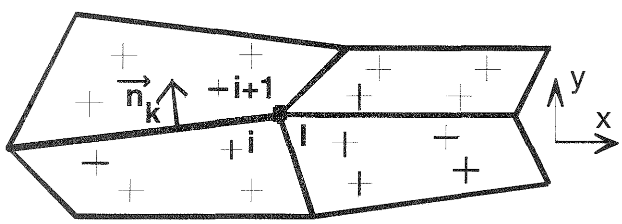

Fig. 1. Computation of the mesh indicator (node I)

The mesh indicator, denoted as κ_I, computed at the nodal point I is a function of the state variables denoted generally as τ_i defined at the gauss points i of adjacent elements. In two dimensions (Fig 1), the mesh indicator is given by the formula :

$$\kappa_I = a\left\|\kappa_{Ix}\vec{x} + \kappa_{Iy}\vec{y}\right\| + b \tag{15}$$

$$\text{with} \quad \kappa_{Ix} = \frac{\left|\sum_{k=1}^{N} F_k(\tau)n_k^x\right|}{\left|\sum_{k=1}^{N} n_k^x\right|} \quad , \quad \kappa_{Iy} = \frac{\left|\sum_{k=1}^{N} F_k(\tau)n_k^y\right|}{\left|\sum_{k=1}^{N} n_k^y\right|} \tag{16}$$

where N is the number of element boundaries arriving to node and $F_k(\tau)$ is a function of the state variables at Gauss points which are apart from the considered boundary. This function may be absolute value of the jump of state variables across the boundary (Eq. 17a) or their average (Eq. 17b) computed over the Gauss points which are separated by the boundary :

$$F_k(\tau) = \sum\left|\tau_{i+1} - \tau_i\right| \tag{17a}$$

$$F_k(\tau) = \sum\left|\frac{\tau_{i+1} + \tau_i}{2}\right| \tag{17b}$$

The constants a and b define the sensitivity of the mesh indicator.

5.2 Remeshing strategy

When the distribution of the mesh indicator is computed, the following step consists in expressing the new position of the nodes, which verifies the chosen mesh optimality criterion. We use a classical mesh optimality criterion based on the equidistribution of the variation of over the entire domain. This implies to defining a mapping between the referential coordinates and the material coordinates x.

It has been already shown (Pijaudier-Cabot (1995)) that the one-dimensional expression of a remeshing equation that translates this application, could be written :

$$\frac{\partial}{\partial \chi}\left(\kappa(x)\frac{\partial x}{\partial \chi}\right)=0 \tag{18}$$

Eq. (18) with the known position of the boundary defines a Dirichlet problem governed by an elliptic equation. It forms a non linear system which must be solved iteratively at each time step of the dynamic problem. Such an implicit resolution is a drawback in the context of transient dynamic problems where equations of motion are generally solved with explicit integration schemes which induce a large number of time steps. Hence, we have modified the remeshing equation as follows :

$$\frac{\partial}{\partial \chi}\left(\kappa(x)\frac{\partial x}{\partial \chi}\right)=\gamma\frac{\partial x}{\partial t} \tag{19}$$

This is a parabolic equation which can be integrated explicitly at each time step. In fact solving this equation is equivalent to solving eq. (18) with a relaxation method where only one iteration is performed.

For two-dimensional problems, as the mesh indicator defines in eq. (15) is a scalar (isotropic remeshing), eq. (19) becomes :

$$\begin{cases} \dfrac{\partial}{\partial \chi_1}\left(\kappa(x_1,x_2)\dfrac{\partial x_1}{\partial \chi_1}\right)+\dfrac{\partial}{\partial \chi_2}\left(\kappa(x_1,x_2)\dfrac{\partial x_1}{\partial \chi_2}\right)=\gamma\dfrac{\partial x_1}{\partial t} \\ \dfrac{\partial}{\partial \chi_1}\left(\kappa(x_1,x_2)\dfrac{\partial x_2}{\partial \chi_1}\right)+\dfrac{\partial}{\partial \chi_2}\left(\kappa(x_1,x_2)\dfrac{\partial x_2}{\partial \chi_2}\right)=\gamma\dfrac{\partial x_2}{\partial t} \end{cases} \tag{20}$$

This is system of two equations which are similar to transient non linear heat equations with an isotropic conductivity in two dimensions.

6 Beam implementation

The ALE method has been implemented in a layered beam Finite Element program (EFICOS) with a non local damage model (§2). Each beam element is subdivided into layers whose behaviour is one-dimensional. The computation of the additional convective terms f_{conv} that appears in the momentum equations is performed using a multi-layered integration scheme

$$\int_{V^e} f(x,y)\,dV^e = \sum_{m=1}^{N} b_m \int_0^1 \int_{y_m-\frac{h_m}{2}}^{y_m+\frac{h_m}{2}} f(\xi,y)\,dy\,L^e\,d\xi \tag{21}$$

with V^e is the spatial domain of element e, N is the number of layers in the element, h_m and b_m are respectively the height and the width of the layer m, and y_m is the distance between the neutral axis of the layer and the neutral axis of the beam.

L = 2.8 m , l =1.65 m , x = 0.2 m , Width b = 0.2m , h = 0.25 m , 10 layers

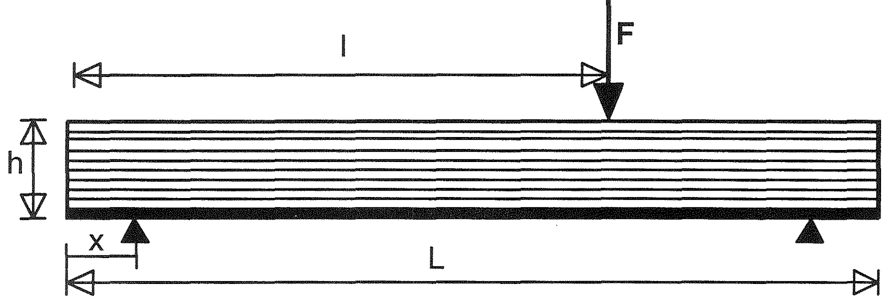

Fig. 2. Impact on beam (Geometry)

Moreover we have to make sure that the updated generalised displacements still verify the Bernoulli hypothesis. This leads to several restrictions : First, the neutral axis of the beam must remain continuous so that strain update preserves the linearity of the strain distribution over the beam depth. Hence the nodal points corresponding to structural joints must remain fixed. Second, the thickness of each layer was assumed to remain constant from one time step to another and consequently the convective velocity remains collinear to the neutral axis of the beam. Under this two restrictions, the update algorithm (eq. (14)) is linear and Bernoulli assumption is conserved with the update. Finally, for sake of simplicity, we have introduce another restriction : The number of layers is kept constant from one element to other.

We now show a test example which is an impact on a plain concrete beam (Fig. 2). ALE computations with a mesh containing 12 elements have been compared with a reference solution containing 72 elements of constant length, and with the fixed mesh solution (ALE initial mesh remained fixed). Figure 3 shows the comparison of the strain distribution in the lower layer and figure 4 shows the evolution of the mesh.

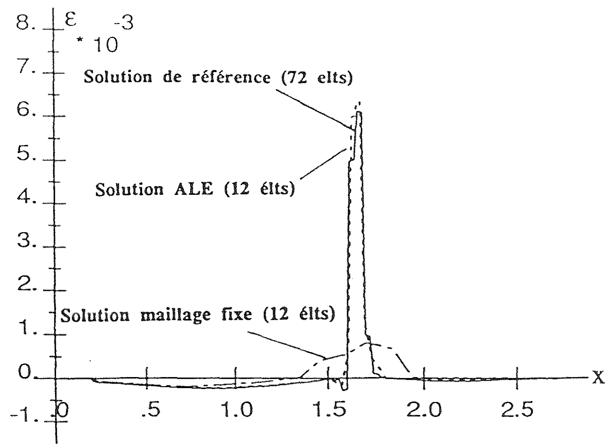

Fig. 3. Distribution of strain in the lower layer at t=1.6e-3 s

Here the mesh indicator has been expressed as a function of the average normalised strain ($\tau=\varepsilon(x)/\varepsilon_{max}$ in eq 17b) in order to points out the location of the localisation zone, and the remeshing starts at the beginning of the computation.

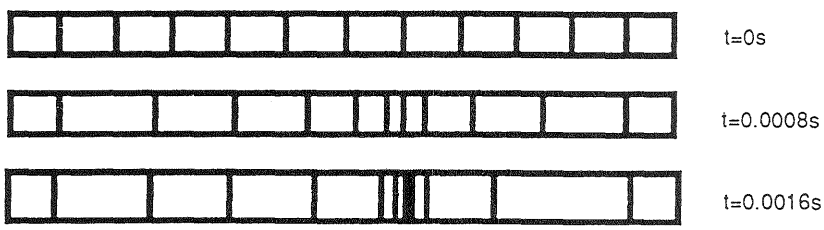

Fig. 4. Evolution of the mesh

7 Conclusions - ALE in two-dimensional problems

A first feasibility study of the ALE implementation to localisation analysis in two-dimensional continuum has been carried. Its goal was to evaluate the generalisation in two dimensions of the ALE explicit transient dynamic algorithm and of the remeshing strategy, that were used in one-dimensional and beam analysis.

We have decided to use the Finite Element Object Oriented Code CASTEM_2000. This code stores the information in objects that are manipulated by operators of a macro-language (Gibiane). It allows a rapid programmation of the mechanical problem equations. Hence the explicit integration of the momentum equation using the central differences scheme has been implemented. The writing of a Gibiane procedure has been needed in order to take account of the additional connective forces which appear in the ALE formulation of the momentum equations. Moreover, the mesh indicator (eq 15 to 17) have been implemented for two-dimensional massive element (Quadrangle with four nodes, Triangle with three nodes). The resolution of the remeshing equation (eq 20) has been easily implemented using a procedure of CASTEM_2000 which makes the resolution of a transient non linear heat equation. Finally, for sake of simplicity the state variables update has been performed with a projection from the old mesh to the new mesh.

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