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ANALYTICAL STUDY OF FICTITIOUS CRACK PROPAGATION IN CONCRETE BEAMS USING A BILINEAR σ - w RELATION

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Abstract
The fictitious crack model (FCM) by Hillerborg et al (1976) is a well known model in describing the fracture in concrete structures. Generally a numerical method like two-dimensional finite element method is used to obtain numerical results for a concrete beam. Recently Ulfkjaer et al (1995) have presented a one-dimensional model for the fictitious crack propagation in concrete beams. In their study they have used a linear softening relation (σ-w relation). It is known that a more realistic σ-w relation is the bilinear one. In the present paper the authors have presented a beam analysis by extending the model of Ulfkjaer et al by using a bilinear σ-w relation. The results agree closely with those by numerical method studied by Brincker and Dahl (1989).
Key words : Fictitious crack method, bilinear σ-w relation, crack propagation in concrete beams.

1 Introduction

It is realized today that linear elastic fracture mechanics (LEFM) is not applicable to quasibrittle materials like concrete. A number of models based on nonlinear fracture mechanics have been proposed to describe fracture in concrete. Of these, the fictitious crack model (FCM) by Hillerborg et al (1976) and the crack band model (CBM) by Bazant and Oh (1983) have been successful in describing completely the fracture of concrete considering its softening behaviour. In FCM, the softening curve, a material property of concrete is given by a stress versus crack opening displacement (σ-w) relation. Both these models employ finite element method (FEM) to obtain the fracture behaviour. In this paper, the FCM will be used to describe fracture in a three point loaded concrete beam.

Recently a one-dimensional model for the bending failure of concrete beams by development of a fictitious crack in an elastic layer with a thickness proportional to the beam depth has been presented by Ulfkjaer et al (1995). A linear softening relation has been used in their analysis. The model is validated by comparing the results with those from a more detailed numerical model.

2 Description of the model

A simply supported beam in three point bending is considered. The model assumes that a single fictitious crack develops in the mid-section of the beam. As the load is progressively increased, points on the crack extension path are assumed to be in one of the three possible states :- (i) a linear elastic state, (ii) a fracture state where the material is softened caused by cohesive forces in the fracture process zone, and (iii) a state of no stress transmission. In the fictitious crack zone the σ-w relationship is used, where σ is the stress and w is the crack opening displacement (distance between the cracked surfaces). In general, the relationship is expressed by the equation $\sigma = f(w)$. The material function f is to be determined by uniaxial tensile tests. The area under the curve f(w) is termed as the specific fracture energy G_f which is assumed to be a material parameter. The model developed by Ulfkjaer et al (1995) is based on two assumptions :- (i) the elastic response of the beam is approximated by two contributions i.e., a local flexibility due to the crack represented by a thin layer of springs and a global beam type flexibility, (ii) the softening relation is assumed linear. The thickness h of the equivalent elastic layer, representing local stiffness of the beam is taken as h=kb (independent of crack length) where b is the depth of the beam and k = 0.5. They have indicated that with this value of h, the results of their model are in good agreement with finite element analysis.

The results of Ulfkjaer et al (1995) can be summarized as follows :-

1. Many of the characteristic features associated with cracking of a concrete beam such as size dependency of maximum load etc. are captured by this simple model
2. The point on the load-deformation curve where the fictitious crack starts to develop and the point where the real crack starts to grow correspond to the same bending moment
3. Closed form solutions for the maximum size of the fracture zone and the minimum slope on the load-deformation curve are given.

As mentioned earlier, Ulfkjaer et al have used a linear softening relation. It is known that in general this relationship (σ-w) is nonlinear (Reinhardt 1984). A bilinear relation is used by Petersson (1981) in his numerical studies. Brincker and Dahl (1989) based on a numerical study of three point bending problem by approximating the σ-w relation by bilinear and trilinear segments have concluded that a bilinear relationship is sufficient. In this paper, the method of analysis given by Ulfkjaer et al (1995) is extended for a three point bending problem with a bilinear softening relationship (Fig. 1).

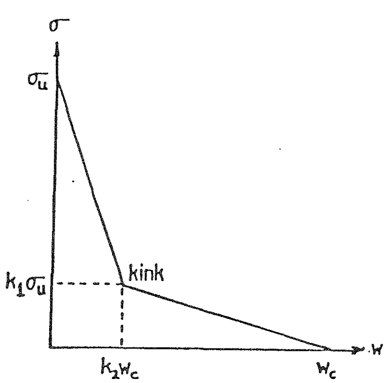

Fig. 1 Bilinear softening relation showing the kink at $k_1\sigma_u$ and $k_2 w_c$

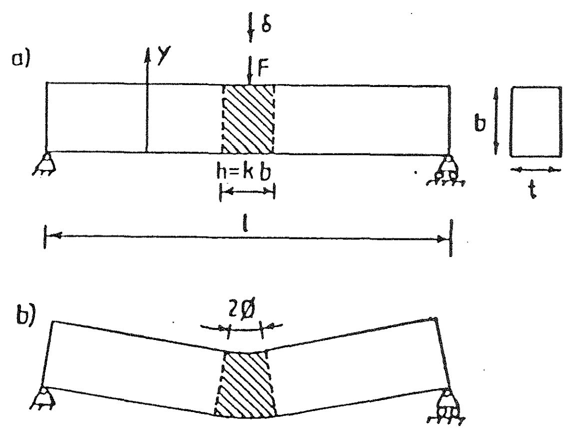

Fig. 2 a) Beam where hatched area is elastic layer and b) Deformed beam where only rigid body displacements are considered

3 Solution for moment-rotation relationship

Basic assumptions :-

1. The elastic response of the beam is linear.
2. A crack is initiated at a point when the maximum normal stress (σ) reaches the tensile stress (σ_u). The crack forms normal to the direction of σ.
3. After the crack is initiated, the fictitious crack progresses and in the fracture process zone σ-w relationship is bilinear as shown in Fig. 1.
4. When the crack opening displacement (w) reaches a critical value (w_c), the stress transfer becomes zero and a real crack starts to grow.

The specific fracture energy which is the area under the softening curve is given by

$$G_f = \frac{1}{2} \sigma_u w_c (k_1 + k_2) \tag{1}$$

The flexibility of the beam is divided into two contributions :-

1. A local layer of bilinear spring, and
2. A global linear beam flexibility.

First the deformation of the spring is considered seperately (Fig. 2).
The fracture process is divided into three phases :-

1. Phase I i.e., before the tensile strength (σ_u) is reached in the tensile side of the beam.
2. Phase II i.e., development of a fictitious crack in the elastic layer of thickness h = kb. This is again divided into two stages :-

    - Stage 1 - development of a fictitious crack before kink
    - Stage 2 - progress of the crack after kink.

3. Phase III i.e., real crack propagation.

The stress distribution or the load deformation curve in the three phases are shown in Fig. 3.

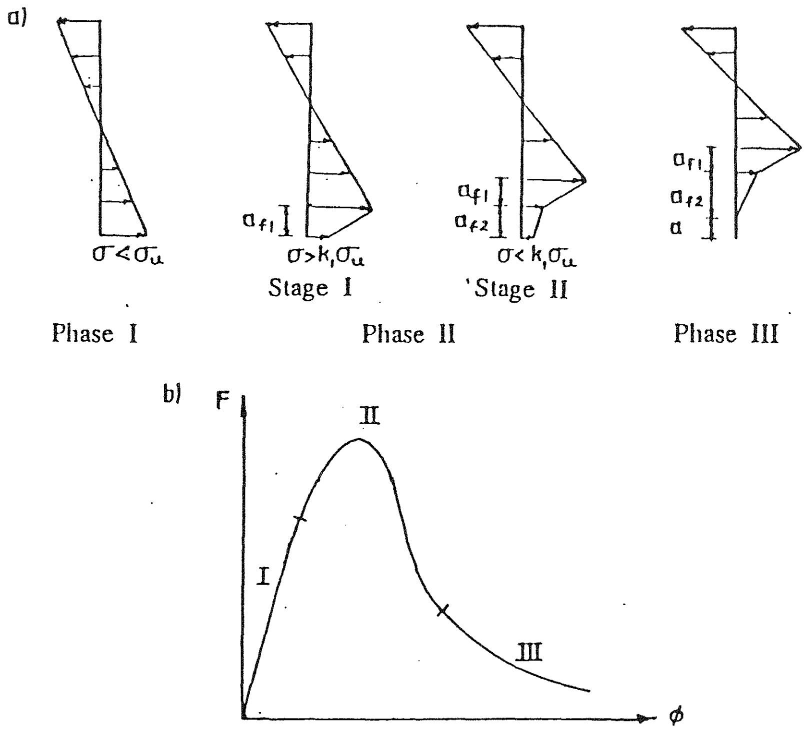

Fig. 3 a) Stress distribution at each phase and b) Load deformation curve

3.1 Phase I

The elongation $v_e = \sigma h/E$. By simple geometric consideration, $v_e = \phi(b-2y)$, where ϕ is the rotation, b = beam depth and y = vertical coordinate (Fig 2). Following Ulfkjaer et al bending moment M and rotation ϕ are normalized as

$$\mu = M\frac{6}{\sigma_u b^2 t} \quad ; \quad \theta = \phi\frac{bE}{h\sigma_u} = \phi\frac{E}{k\sigma_u} \tag{2}$$

giving the simple moment rotation relation corresponding to the elastic spring layer

$$\mu(\theta) = \theta \qquad \text{for } 0 < \theta < 1 \tag{3}$$

At the end of phase I, for y =0, $\sigma = \sigma u$ and $\mu=1$. Thus in phase I, the (μ,θ) curve is a straight line between the origin and $(\mu,\theta) = (1,1)$.

3.2 Phase II

In phase II there will be two stages i.e., one before the kink and the other after the kink.

3.2.1 Stage 1 - before the kink

The stress distribution in the elastic layer in this stage is shown in Fig. 4.

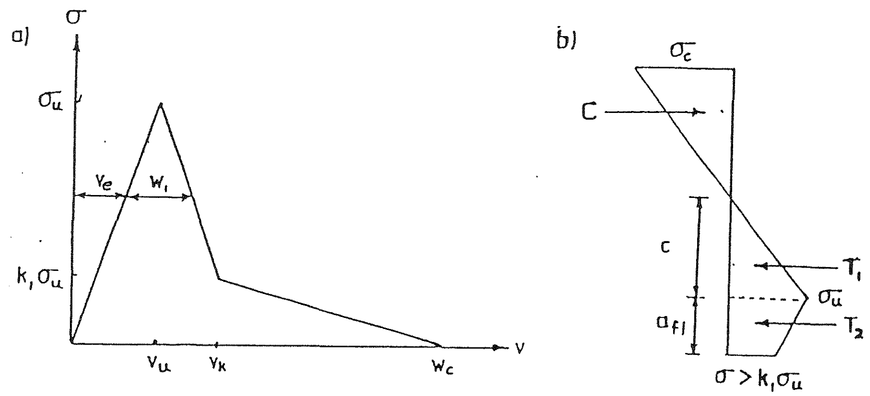

Fig. 4 a) Stress displacement relation and b) Stress diagram - Stage I, Phase II

Considering the deformation as well as stress equilibrium ($C = T_1 + T_2$), the value for the fracture process zone a_{f1} (Fig. 4(b)) is obtained by the equation

$$\alpha_{f1} = \left\{B\frac{k_1}{k_2} - \frac{B}{k_2} + 1\right\} - \sqrt{\left\{B\frac{k_1}{k_2} - \frac{B}{k_2} + 1\right\}^2 - \left\{1 - \frac{1}{\theta}\right\}\left\{B\frac{k_1}{k_2} - \frac{B}{k_2} + 1\right\}} \qquad (4)$$

$$\text{where } \alpha_{f1} = \frac{a_{f1}}{b} \;;\quad B = \frac{v_u}{w_c} = \frac{\sigma_u h}{w_c E} = \frac{(k_1 + k_2)\sigma_u^2 h}{2G_f E} \qquad (5)$$

The brittleness number B varies from zero corresponding to ideal ductile behavior to one corresponding to ideal brittle behavior. The moment can be found by taking moment about the compressive force. In its non-dimensional form it is given by

$$\mu(\theta) = \left\{\frac{1}{\theta} + 2\alpha_{f1} - \frac{\alpha_{f1}}{2\theta} - \alpha_{f1}^2\right\} + \frac{\sigma}{\sigma_u}\left\{2\alpha_{f1} + \frac{\alpha_{f1}}{2\theta}\right\} \tag{6}$$

In order to stay in stage I, $\theta < \theta_k$. At kink $\theta = \theta_k$, $\alpha_{f1} = \alpha_{fk}$ and $\sigma = k_1 \sigma_u$. Similar equation can be derived for stage II of phase II. Details are given by Ravikumar (1997). In order to stay in phase II, w at the bottom of the beam must be smaller than w_c. The rotation at the end of phase II (θ_c) is given by equating the stress σ and displacement w at the bottom to 0 and w_c respectively.

3.3 Phase III

In phase III the real crack starts to propagate. The length of the real crack 'a' (Fig. 5) can be obtained from geometrical and equilibrium considerations (Ravikumar 1997).

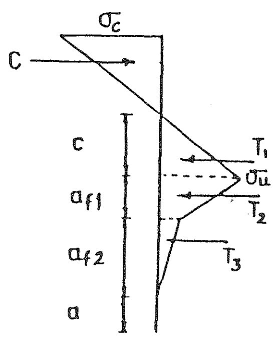

Fig. 5 Stress diagram - Phase III

Till now only the deformation of the spring has been considered. Elastic deformation in the beam parts outside the elastic layer are taken into account by a procedure similar to the one suggested by Ulfkjaer et al (1995). The load-deflection curve of the beam is obtained by using a simple approximate procedure (Ravikumar 1997).

4 Results

Numerical computations have been done for two beams which are analyzed by Ulfkjaer et al (1995) and Brincker and Dahl (1989). The geometrical and material parameters are given in Table 1.

Table 1. Geometry and material properties of the beams

Property	Symbol	Values	
		Ulfkjaer et al beam	Brincker and Dahl beam
Beam depth	b	100 mm	80 mm
Beam width	t	100 mm	40 mm
Beam length	l	800 mm	400 mm
Sp. Fr. energy	G_f	0.1 N/mm	0.1096 N/mm
Tensile strength	σ_u	3.0 N/mm^2	2.86 N/mm^2
Mod of elasticity	E	20,000 N/mm^2	32550 N/mm^2

The values of k_1 and k_2 are taken as 0.308 and 0.161 for the bilinear case. These values are taken from Brincker and Dahl (1989). Table 2 gives values for P_{max}, θ_c and μ_c for the two beams both for linear and bilinear σ-w relationships.

Table 2. Values for P_{max}, θ_c and μ_c

Quantity		Ulfkjaer et al beam	Brincker and Dahl Beam
P_{max}	Linear case	3972 N	2235 N
	Bilinear case	3491 N	2007 N
θ_c	Linear case	5.94	13.24
	Bilinear case	10.97	25.58
μ_c	Linear case	1.00	1.00
	Bilinear case	0.41	0.40

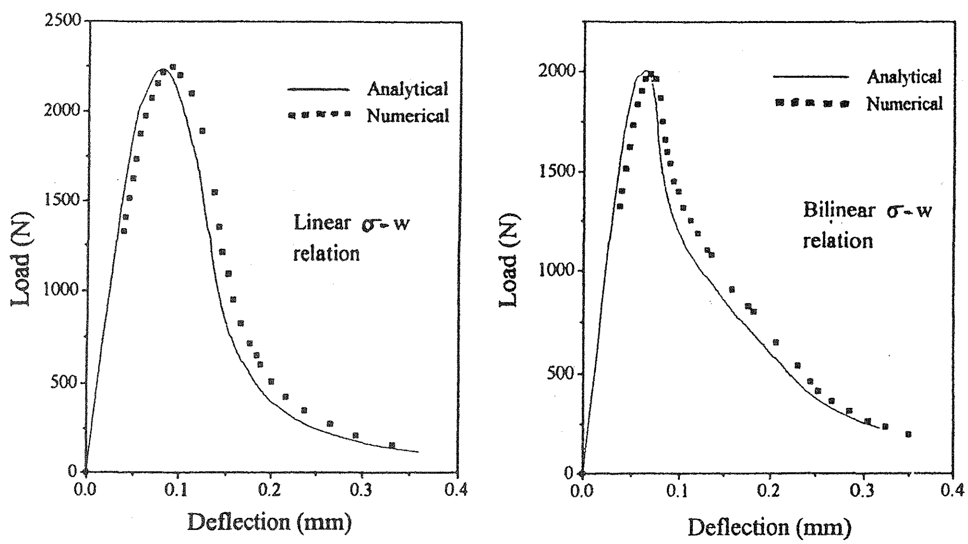

Fig. 6 Load deflection curve for Brincker and Dahl beam

Fig. 6 shows a comparison of load-deflection curves for linear and bi-linear softening relations for Brincker and Dahl beam. For comparison the values from a numerical study of Brincker and Dahl are also shown. In all the calculations, k is taken as 0.5

5 Conclusions

1. Taking k=0.5, the results (Fig. 6) for the bilinear model agree closely with those of the numerical study by Brincker and Dahl (1989). For the linear model Ulfkjaer et al have arrived at the same conclusion.
2. The maximum values of the load given in table 2 agree closely with those obtained by Ulfkjaer et al for the linear model and Brincker and Dahl for both linear and bilinear models.
3. P_{max} is slightly less in a bilinear model when compared to the value from a linear model.
4. The value of θ_c (θ at the end of phase II) for linear and bilinear models are very different.
5. In a linear model, the point on the load-deformation curve where the fictitious crack starts to develop and the point where the real crack starts to grow correspond to the same bending moment i.e., $\mu = 1$ for both the situations. The same conclusion has been arrived at by Ananthan et al (1990) with a different model in their studies using a linear strain softening relationship. However, as can be seen from the values given in Table 2, these two values in the case of a bilinear model are different and also the values of μ_c at the end of phase II for these beams are less than half the values given by a linear model.

6 References

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