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Minimum steel in concrete beams – An assessment based on discrete crack model

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ABSTRACT: Discrete crack model offers realistic modelling of physical discontinuities and thus plays an important role, in verifying the rationality of certain codal provisions as used in the design of concrete structures. The present work aims at examination of minimum reinforcement requirement in concrete flexural members using cohesive crack approach. The quasi-brittle failure of reinforced concrete beam with minimum steel is modelled by adopting fracture mechanics model for concrete and an empirical model for interface bond. The initial finite element model does not contain any predefined contact regions. Exponential softening law is used to approximate tension-softening behaviour of concrete. The size dependent tensile strength of concrete is computed using Carpinteri's *multifractal scaling law*. The effects of concrete strength and member size on the minimum steel requirement are studied and compared with the provisions specified in various international design codes.

1 INTRODUCTION

The provision of steel reinforcement bars in the tensile zone of flexural members is to off-set the inherent weakness of concrete in tension and to enable the concrete in compression zone to develop its full capacity. The failure mode of reinforced concrete members depend on structural geometry, material properties, load configuration and also on the amount, location and orientation of reinforcing steel. Debonding and slipping at the rebar-concrete interface influences the development of transverse cracks in concrete. Study of the behaviour of cracks in a structural member is of crucial importance to assess its reliable load carrying capacity. Current strength design of reinforced concrete beams is based on a strength-based failure criterion in conjunction with stress equilibrium, strain compatibility and constitutive laws of materials. In the conventional design methods, it is difficult to account for the debonding and slipping effects on the crack bridging action of the reinforcing steel. Hence empirical approaches are used to account for cracking. The complexity of the problem make it imperative to adopt robust computational procedures to accurately model the fracture mechanisms in concrete and at the bi-material interface. Discrete crack models simulate the development of discontinuities in reinforced concrete (RC) members in a realistic manner. The present work aims at examination of minimum reinforcement requirement in concrete flexural members by studying the crack bridging action of the rebars.

Minimum reinforcement requirements applies to flexural members, which for architectural or other reasons, are larger in cross-section than required for strength. This is required to prevent sudden failure of reinforced concrete member subsequent to concrete cracking. With minimum tensile reinforcement, the computed moment strength of a cracked RC section should be greater than that of the corresponding unreinforced concrete section computed from its modulus of rupture. The current codal provisions related to the minimum steel in concrete beams are essentially empirical in nature without regard to the fracture processes involved. Fracture mechanics approaches based on a single crack have been reported to study the minimum tensile steel. The minimum reinforcement requirement for concrete flexural members was examined using principles of linear elastic fracture mechanics (Baluch et al. 1990). Bosco & Carpinteri (1992) proposed a minimum reinforcement ratio for concrete beams. An empirical transitional brittleness number was suggested to represent ductile to brittle failure modes. Hawkins & Hjorteset (1992) proposed minimum reinforcement ratio based on finite element studies using the fictitious crack model. A new parameter called *fracture strength* was defined which is a function of material fracture parameters and the size of the beams. Gerstle et al. (1992) proposed a minimum reinforcement ratio which is independent of the yield strength of steel.

In the present study, the quasi-brittle failure of reinforced concrete beam with minimum steel is modelled by adopting Hillerborg's fictitious crack model for concrete and an empirical model to describe debonding and slipping at the bi-material interface. The effects of concrete strength and member size on the minimum steel requirement are studied and compared with the provisions specified in various international design codes. The codal provisions are examined from the fracture mechanics perspective by performing numerical experiments on a three point bend concrete beam using *CoMICS* program. First a plain concrete three point bend beam is analyzed by using exponential softening law for concrete fracture. Then the beam is modelled with the minimum reinforcement $A_{st,min}$ and the load capacity of the RC beam is compared with that of the unreinforced concrete beam.

Crack modelling is a key point in the present study. In RC members, one cannot predict the behaviour up to ultimate state only by modelling of cracks. For the predictions to be realistic, bond behaviour between concrete and reinforcement, and non-linearity in compression stress state of concrete must be considered and suitably be modelled. The stress paths for the materials are considered to be proportional and the scope of the analysis is therefore limited to study response of the member within elastic range.

This paper is organized in the following way: Section 2 outlines the fictitious crack model for analyzing concrete fracture. Attention is focussed on identification of material parameters associated with this non-linear fracture model. Section 3 gives a brief description of the finite element program *CoMICS* used for the numerical simulations. Section 4 outlines the minimum steel provisions of various codes that are studied. Section 5 presents the details of the specimens studied and the results of FE simulations. A summary of the conclusions closes the paper.

2 FICTITIOUS CRACK MODEL

Hillerborg et al. (1976) proposed fictitious crack model (FCM) for progressive fracture in concrete. The essence of this model is the description of nonlinearity by means of a relationship between cohesive stresses and crack openings. Figure 1 shows the terminology and concepts associated with the fictitious crack model. The stress state in the fracture process zone is presumed to depend upon a post-peak tensile stress-separation relationship in tension (Figure 2). Exponential tension-softening approximation (Equation 1) is used in the present study.

$$\sigma = f_t \, exp\left(-\frac{f_t}{G_F} w\right) \tag{1}$$

Although this non-linear fracture mechanics model is physically very appealing, there are however numerous difficulties associated with identification of material parameters. Proper description of the material parameters is essential for obtaining valid predictions with cohesive crack models.

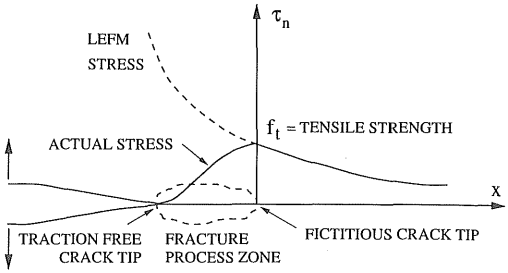


Figure 1. Fracture process zone

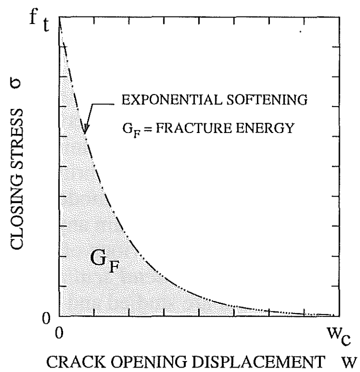


Figure 2. Tension-softening model

2.1 *Material properties*

The material parameters required for numerical analysis with cohesive crack model are the modulus of elasticity E, the tensile strength of material f_t, the fracture energy G_F, and the shape of the tension-softening diagram. The direct tensile test is the only test which gives these parameters. In view of the difficulties involved in conducting stable tensile tests in laboratories, these are usually obtained by conducting stable bending tests.

2.1.1 *Fracture energy*

According to the cohesive crack model, the fracture energy is the energy required to produce a unit area of traction free crack. RILEM Technical Committee 50-FMC had recommended the *work-of-fracture* method proposed for concrete by Hillerborg (1985a) to determine the fracture energy G_F. In the absence of test data, G_F may be computed from the following empirical relation given by the CEB-FIP Code (1990).

$$G_F = K_d \, f_c^{0.7} \tag{2}$$

where f_c is the mean compressive strength in MPa and K_d is an empirical coefficient that depends on the maximum size of the aggregate.

Hillerborg (1985b) had reported the results of an extensive round-robin series of tests on nearly 700 three-point bend beams. The fracture parameter G_F was observed to be structure size dependent. The round-robin series test data are useful in providing the G_F required for the numerical computation.

2.1.2 *Tensile strength*

Tensile strength of concrete is quoted according to the method used for determination. It can be determined either from direct tensile test or from Brazilian cylinder splitting test. It is generally agreed that material tensile strength f_t is geometry dependent. For flexural applications the *modulus of rupture* measured for beams is an experimentally convenient measure of strength owing to the relative simplicity of the test procedure. The strong size dependence of the rupture modulus has been detected earlier in several experimental investigations. Scaling laws were proposed by Bažant (Bažant 1984) and Carpinteri et al. (Carpinteri et al. 1997) to arrive at the nominal strength for practical range of specimen sizes. In the present study, the following approximations are made in arriving at the nominal strength for the particular specimen geometry under study.

The direct tensile strength is the most appropriate measure from the point of crack initiation and growth. The asymptotic uniaxial tensile strength for a large size specimen, f_t' is approximated to be ten percent of the uniaxial compressive strength f_c of the material. Thus:

$$f_t' = 0.10\, f_c \tag{3}$$

Then the multifractal scaling law (MFSL) proposed by Carpinteri et al. (1997) is used to arrive at the nominal strength for the particular specimen geometry under study.

In the FCM, the characteristic length l_{ch} is a measure of the length of the fracture process zone. It is defined as

$$l_{ch} = \frac{EG_F}{{f_t}^2} \tag{4}$$

According to the MFSL, the nominal tensile strength f_t is related to the member size D:

$$f_t = f_t'\left[1 + \frac{l_{ch}}{D}\right]^{1/2} \tag{5}$$

The characteristic length l_{ch} makes the transition between a range of sizes (usually coincident with the laboratory-testing sizes) where the scaling effect is pronounced and the larger sizes, where the scaling can be neglected and a constant value of strength f_t' can be defined. Since l_{ch} is a function of f_t, Equation 5 is to be solved iteratively.

3 COMPUTER PROGRAM - *CoMICS*

CoMICS (Computational Model for Investigation of Cracks in Structural concrete) is a special purpose program with interactive graphics facilities (Prasad 2000) developed duly addressing the following main issues related to discrete crack models in the finite element analysis of concrete members:

1. An efficient mesh discretization technique,
2. an accurate criterion for nodal release,
3. realistic models for contact between uniform media and non-uniform media,
4. automatic remeshing scheme to model advancing crack and bond-front, and
5. a reliable solution algorithm.

An iterative procedure based on verification of equilibrium condition and congruence condition has been formulated to solve the non-linear problem. Concrete is modelled with four-nodded bilinear quadrilateral elements with selective-reduced integration technique to improve bending behavior. Singular elements are not used at the fictitious crack tip following the assumption of singularity cancellation inherent in the cohesive crack model. In the present study the convergence is verified by means of the congruence condition that compares the calculated reaction at the crack tip to the closing force in the crack line. The non-linear problem is solved by incremental-iterative procedure, based on Newton-Raphson scheme. The crack propagates along the inter-element boundaries. The analysis determines a load factor associated with the incremental length of the crack.

Interface elements are used for implementation of constitutive equations for concrete in tension-softening and also to model bond-slip relations at the steel-concrete interface. Theoretically, bond cracking is fracture and so the energy required to form the interface cracks should also be taken into account. For this the energy characterizing the complete stress-separation curve at the bi-material interface must be available. However, due to scarcity of test data, no energy dissipation mechanism is considered at the steel concrete interface. A simpler and less general empirical formulation is adopted in this study.

3.1 *Mesh generation scheme*

In solving complex engineering analysis problems by finite element method, considerable effort is required in preparing data for a problem. This is particularly true in RC problems. To enhance the usefulness and efficiency of the FEM for RC applications, there is a need to develop exclusive mesh generation schemes that guarantee sufficiently refined meshes. A mesh generation module is developed by providing the option for two materials in the existing quadrilateral meshing algorithm(MSD), proposed by Krishnamoorthy et al. (1995). The adaptive strategy is based on the node patch based superconvergent error-estimation technique proposed by Zienkiewicz and Zhu (1992) and integrated into an environment for automated mesh design for RC planar members. The mesh density distribution in concrete regions is governed by the node spacing requirements as obtained from the estimated error in the elastic strain energy in the uncracked member. For steel elements the element aspect ratio considerations provide convenient bounds on the size of the elements.

In complex analysis of cohesive crack growth in concrete members, the uncertainty in the fracture mechanisms and the resulting approximations in the contact boundary conditions applied to the FE model can significantly outweigh the possible accuracy due to mesh refinement. The meshing strategy implemented in *CoMICS* is a first step to provide a viable platform to study crack growth in RC members.

3.2 *Bond-slip model*

Displacements are initially continuous at the material interface. At the onset of cracking a localized slip occurs at the bar-to-concrete interface. This is accounted in a direct way by gradual loss of bond through the provision of contact elements for bond. The bond stress Vs. slip relationship is one of the basic constitutive property required in the non-linear finite element analysis of reinforced concrete structures. The bond-slip relationship in *CoMICS* is based on the material constitutive law adopted by the CEB-FIP model code (1990). The average bond strength tangential to the bi-material interface is assumed to be

$$\tau_{btu} = 1.75\sqrt{f_{ck}} \qquad (6)$$

where f_{ck} is the cube compressive strength of concrete. Bond stress-slip relationship for monotonic loading for average bond conditions is shown in Figure 3. It may be of interest to compare the bond-slip model with the tension-softening model for concrete (Figure 2) dissipating a finite quantity of energy G_F during the progressive failure process. Though these two types of constitutive relations are quite similar as far as the failure process is concerned, no account has been made for the energy dissipation in the bond failure.

3.3 *Bond-split model*

CEB-FIP model code (1990) relates the bond splitting strength to the characteristic cube compressive strength of the concrete f_{ck}. The splitting strength at the bi-material interface is assumed to be

$$\tau_{bnu} = 1.0\sqrt{f_{ck}} \qquad (7)$$

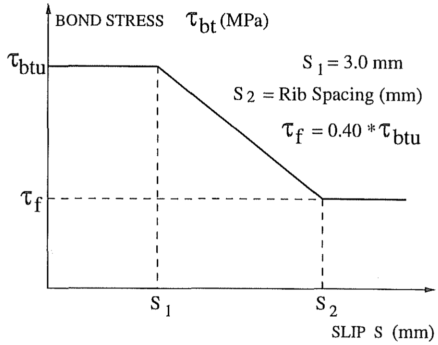


Figure 3. Bond-slip model

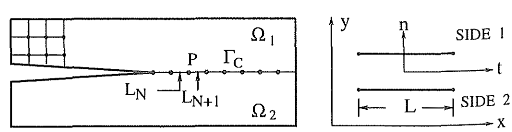
The bond-split model is included in *CoMICS* to provide generality in the finite element program. Splitting bond failure is not critical for the problem under study.

3.4 *Crack modelling*

Following difficulties arise in the use of the estimated nodal stress to model discrete crack propagation by nodal release technique:

1. In the displacement based finite element method equilibrium is only satisfied in a "weak" sense by equivalent nodal forces and evaluation of stress fields is prone to error.
2. Strength criterion based on maximum tensile principal stress is valid to initiate fracture at a point. But in finite element discretization, the point contact is represented by line contact with finite dimension.
3. In case of contact between dissimilar media (eg. steel-concrete interface) the stress field is discontinuous as a result of the abrupt change of material property.
4. Inadequate nodal release criterion leads to severe convergence problem in the non-linear analysis of contact problems and thus seriously limits the applicability of discrete crack model in analyzing multiple cracks in reinforced concrete applications.

In *CoMICS* the element internal forces are used to derive physically meaningful criterion for nodal re-



(a) Line contact (b) Contact element

Figure 4. Line contact in FE discretization

lease in conjunction with the cohesive crack models used to model fracture. The criterion followed is based on FE nodal force $\mathbb{F}$ and the admissible tractions on the element boundaries. The closing force $\mathbb{C}$ at the cohesive crack tip, P (Figure 4) is,

$$C_n = \int_0^1 \tau_n(\triangle u_n)\, J_N dr + \int_{-1}^0 \tau_n(\triangle u_n)\, J_{N+1} dr \quad (8)$$

$$C_t = \int_0^1 \tau_t(\triangle u_t)\, J_N dr + \int_{-1}^0 \tau_t(\triangle u_t)\, J_{N+1} dr \quad (9)$$

For no slip ($\triangle u_n = \triangle u_t = 0$) condition τ represents the material strength parameters. The force criterion to describe cohesive mode I fracture in concrete is $F_n = C_n$. The crack propagation angle is assumed to be perpendicular to the direction of maximum principal stress. To implement this criterion, an inter-element boundary at the crack tip node should be available in the crack propagation direction.

On similar lines, the force criterion to describe bond-slip at rebar interface can be written as $F_t = C_t$. At dissimilar contact the debonding front is constrained to develop along the interface. In such case, the coexisting orthogonal stress is reduced linearly in the subsequent analyses. The contact models are implemented in *CoMICS* together with efficient finite element meshing/remeshing strategies.

4 MINIMUM STEEL - CODAL PROVISIONS

The comparison is provided for the following concrete frame design codes:

- U.S. (ACI318-1999)
- British (BS8110-1997)
- Indian (IS456-2000)
- New Zealand (NZS3101-1995)

ACI building code (ACI318 1999) specification for the minimum flexural reinforcement requirement is a function of concrete strength f_c' (psi), yield stress of steel f_y (psi) and member dimensions (in). The minimum steel ($A_{st,min}$, in^2) to prevent sudden failure of flexural members, shall not be less than that given by

$$A_{st,min} = \left\{ \frac{3\sqrt{f_c'}}{f_y}\, b\, d \quad and \quad 200\, b\, d/f_y \right\} \quad (10)$$

The 1997 British standard code provision (BS8110 1997) for minimum reinforcement ($A_{st,min}$, mm^2) in flexural members is

$$A_{st,min} = 0.0013 A_c \quad (11)$$

where A_c is the total concrete area in mm^2. Yield stress of steel $f_y = 460MPa$.

Bureau of Indian standard code (IS456 2000) provision for minimum reinforcement in flexural members ($A_{st,min}$, mm^2) is independent of concrete material properties:

$$A_{st,min} = \frac{0.85}{f_y}\, b\, d \quad (12)$$

where f_y is the yield stress of steel in MPa, b and d (mm) are member breadth and effective depth respectively.

New Zealand code (NZS3101 1995) provision for minimum reinforcement in flexural members ($A_{st,min}$, mm^2) is dependent on concrete material properties:

$$A_{st,min} = \frac{\sqrt{f_c'}}{4\, f_y}\, b\, d \quad (13)$$

where f_y is the yield stress of steel in MPa, b and d (mm) are member breadth and effective depth respectively.

5 ASSESSMENT BASED ON DISCRETE CRACK MODEL

The finite element mesh for the three point bend beam specimen considered for this numerical experiment is shown in Figure 5. The details of the geometry and the material properties are given in the Table 1. The yield stress f_y adopted for the analysis of specimen with BS code specified steel is 460 MPa. The size dependent tensile strength of concrete is computed using Carpinteri's multifractal scaling law. G_F is computed using the empirical relation given in Equation 2 with $K_d = 10$. The minimum reinforcement according to the various codes for the beam specimen under study are given in Table 2. The plain concrete beam (PCB) has a load carrying capacity of 15.64 kN with exponential softening model. The corresponding moment capacity of this three point bend beam is 4.184 kN-m. Self weight of the beam is not considered in the analysis.

A beam with reinforcement less than the minimum fails due to the propagation of a single crack that develops at the section of maximum bending moment. Initially there are no contact elements at rebar-concrete interface in the FE mesh. They are inserted into the model when the criterion for initiation or

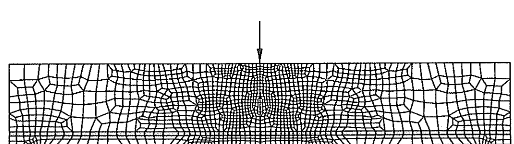


Figure 5. FE mesh to verify minimum steel

Table 1. Details of the RC beam

Length of beam	L	1220	mm
Clear span	l	1070	mm
Depth of the beam	D	200	mm
Effective beam depth	d	175	mm
Span to depth ratio	l/d	6.11	
Beam thickness	b	125	mm
Concrete			
Young's modulus	E_c	29270	MPa
Poisson's ratio	ν_c	0.195	
Tensile strength	f_t	4.10	MPa
Compressive strength	f_c	30.0	MPa
Cube strength	f_{ck}	37.5	MPa
Mode I fracture energy	G_F	100	N/m
Steel			
Young's modulus	E_s	200 000	MPa
Poisson ratio	ν_s	0.30	
Yield strength	f_y	415	MPa
Tangential bond strength	τ_{btu}	10.72	MPa
Splitting bond strength	τ_{bnu}	6.12	MPa
Rib spacing (assumed)	S_2	7.5	mm

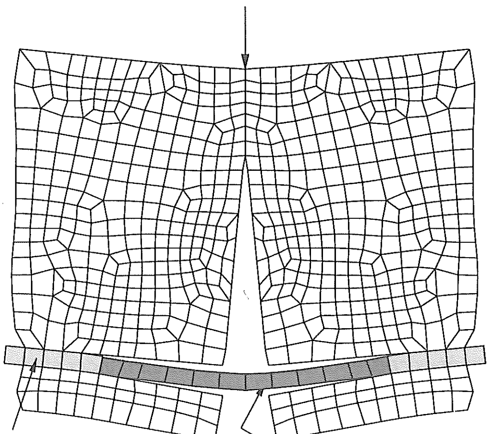


Figure 6. Discrete crack growth and debonding

propagation is exceeded. The reinforcement will deform when it is stressed. The geometry of the specimen is updated at the end of each analysis which means that the last configuration of the structure is assumed to be the reference configuration for the new mesh obtained following remeshing.

In the analysis four debonding fronts and one crack are initiated and propagated as shown in Figure 6. The reinforcement for this specimen correspond to the minimum requirement specified by the ACI code. For the sake of clarity only part of the beam is shown and the displacements are magnified 100 times. The analysis is terminated when the stress in the rebar at the notch location reached eighty percent of the yield

Table 2. Comparision of moment capacities

Code	$A_{st,min}$	$\left(\frac{100A_{st}}{bd}\right)$	Load	Moment
	mm^2	%	kN	kN-m
–	0.00	0.000	15.64	4.184
ACI	72.9	0.332	16.03	4.288
BS	32.5	0.149	9.39	2.512
IS	44.8	0.205	11.75	3.143
NZS	72.2	0.330	15.80	4.227

stress. At this stage, the steel stresses are subjected to a rapid increase with a small increment in load. Converged equilibrium states are obtained till the termination of the analysis.

It is of interest to compare the predicted moment capacity of the specimen with service moment obtained from the strength of materials approach:

$$M_{el} = f_t \left(\frac{1}{6}\, b\, D^2\right) \tag{14}$$

M_{el} computed with the size dependent tensile strength is 3.417 kN-m. The corresponding load on the member is 12.77 kN.

For the specimen studied, the minimum steel specified by the ACI & NZS codes is found to be reasonable whereas the BS & IS code requirements for the same is on lower side. The moment capacities obtained with the BS & IS code minimum steels are less than the ultimate moment capacity of the member obtained with exponential tension-softening model. They are also less than the service moment capacity M_{el} computed from Equation 14. The predictions are sensitive to the tensile strength parameter and a rational value for f_t based on Carpinteri's multifractal scaling law, is used in the present analysis. Modulus of rupture specified by the IS code for this concrete specimen is 4.287 MPa.

5.1 *Parametric study*

The intent of the study is to predict the effects of concrete strength and the member size on minimum steel requirement so as to provide an understanding whereby additional research could provide needed refinements and improvements to the design provisions.

Table 3. Details of parameters for varying f_c'

		Compressive strength f_c (MPa)			
		25	30	35	40
E	MPa	29240	31072	32711	34200
G_F	N/m	95	100	120	132
f_t	MPa	3.60	4.10	4.70	5.31
l/d		6.11	6.11	6.11	6.11
ACI	mm^2	72.9	72.9	77.9	83.3
BS	mm^2	32.5	32.5	32.5	32.5
IS	mm^2	44.8	44.8	44.8	44.8
NZS	mm^2	65.9	72.2	78.0	83.3

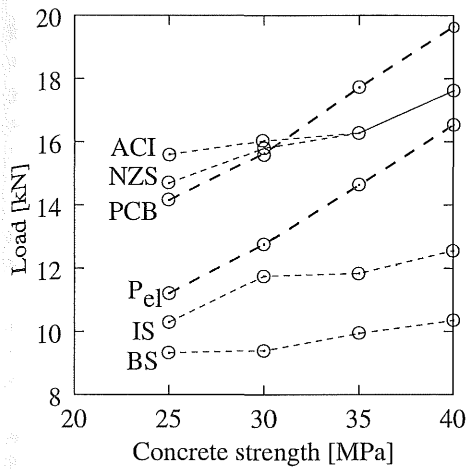


Figure 7. Effect of concrete strength (D = 200 mm)

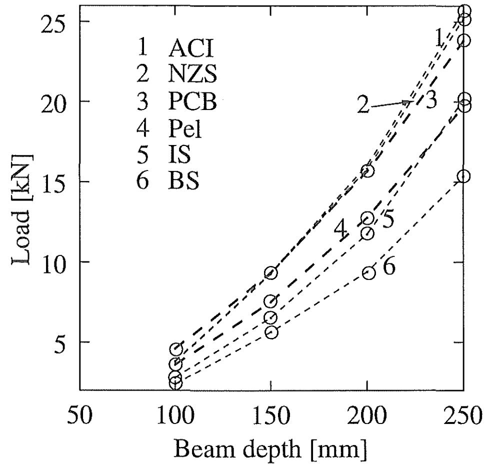


Figure 8. Effect of member size ($f_c{}' = 30$ MPa)

5.1.1 *Concrete strength*

The material properties associated with the four nominal concrete strengths used in this study are given in Table 3. Beam depth adopted for this specimen is 200 mm. Figure 7 shows a comparison of predictions of R-C beam load capacity for different concrete strengths with that of plain concrete beam (PCB) specimen. The service load for the specimen P_{el} is calculated based on Equation 14. The predicted loads for the beams with the BS & IS specified steels are less than the service load P_{el} computed for the plain concrete beam specimen in the range of concrete strengths studied. The design provisions in these two codes need to consider the concrete compressive strength in determining the minimum steel in flexural members.

5.1.2 *Member size*

Four beam specimens with different geometrical ratios are studied here to predict the structure size-effect on minimum steel requirement. Compressive strength of concrete $f_c = 30MPa$. The size dependent tensile strength f_t, computed based on Carpinteri's multifractal scaling law for the different beam depths is given in the Table 4. The deformed mesh at the time of analysis termination for all the specimens is typical to the one shown in Figure 6. Load versus member size plots for the four specimens studied are compared with the corresponding plain concrete members in Figure 8.

Table 4. Details of parameters for varying D

		Member depth D (mm)			
		100	150	200	250
E	MPa	29270	29270	29270	29270
G_F	N/m	100	100	100	100
f_t	MPa	4.62	4.30	4.10	4.05
l/d		12.6	7.9	6.1	4.8
ACI	mm^2	35.4	56.2	72.9	93.7
BS	mm^2	16.3	24.4	32.5	40.6
IS	mm^2	21.8	34.6	44.8	57.6
NZS	mm^2	35.1	55.7	72.5	92.8

6 CONCLUSIONS

Minimum reinforcement in RC beams is studied from the fracture mechanics perspective by performing numerical experiments on a three point bend concrete beam. The effects of concrete strength and member size on the minimum steel requirement are studied and compared with the provisions specified in various international design codes. The following observations can be made from the present study.

1. The quasi-brittle failure of reinforced concrete beam with minimum steel is realistically modelled using the efficient meshing/remeshing strategies available in the *CoMICS* program.
2. For the specimen studied, the minimum steel specified by the ACI & NZS codes is found to be reasonable whereas the BS & IS code requirements for the same are on the lower side. Load capacity of beams with minimum steel specified by BS & IS codes is less than the service load capacity P_{el} of the plain concrete member.
3. The design provisions in BS & IS codes need to consider the concrete compressive strength in determining the minimum steel in flexural members.

Presently work is in progress to extend the scope of the analysis to the range of non-proportional stress paths. Further studies using *CoMICS* are directed to achieve rationality with respect to the codal provisions for improved design of RC members.

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