Probabilistic assessment of fatigue crack propagation in concrete

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ABSTRACT: It is well known to engineers that fatigue accounts for a majority of structural failures. In plain and reinforced concrete structures, fatigue may lead to excessive deformations, excessive crack widths, debonding of reinforcement and rupture of the reinforcement or matrix leading to structural collapse. It is learnt from the available literature on fatigue behavior of concrete that the rate of fatigue crack growth depends on a number of parameters, such as, the tensile strength, stress history, stress intensity factor range and fracture properties. Furthermore, these parameters are random in nature. These factors together with the wide variations in material properties of concrete suggests that a statistical/probabilistic framework is required for fatigue life prediction in concrete. In this work, the probability of failure of concrete beams under fatigue is computed through Monte Carlo simulation and by considering the different parameters responsible for fatigue failure as randomly distributed. The sensitivity of different parameters involved in fatigue process pertaining to failure is also studied using stochastic sensitivity analysis.

1 INTRODUCTION

Most of civil engineering infrastructures are subjected to fatigue loading and it is important to consider the fatigue degradation during residual life evaluation. Numerous models are available to describe fatigue behavior in metals but very few of these can be directly applied to concrete. Based on linear elastic fracture mechanics theory, fatigue crack propagation in concrete can be described through suitable modification of the well known Paris Law as discussed in (Bazant and Kangming 1991), (Carpinteri 1991) (Slowik et al. 1996) and (Subramaniam et al. 2000). The modified laws include several parameters such as frequency of applied loading, the presence of fracture process zone, the size effect and the overload effects, which complicates the fracture process in concrete. Since the quantities entering into the evaluation process are not deterministic in nature, an application of the principles of probability theory becomes mandatory. In the present work, a statistical framework is discussed for assessing the fatigue life of plain concrete beams. As a first step, a modified LEFM based fatigue law for predicting crack propagation in concrete is discussed briefly. Subsequently, the reliability with respect to fatigue failure of the member under consideration is computed by considering different input parameters as randomly distributed. Monte Carlo simulation with Latin Hypercube Sampling technique is used for determining the failure probability. The Latin Hypercube sampling technique reduces the computational time considerably by a judicious choice of the random samples, as discussed later. Further, using stochastic sensitivity analysis, it is also determined in what degree the randomness of an input quantity influences the variability of the output.

2 LEFM BASED FATIGUE LAW FOR CONCRETE

As mentioned earlier, in metals fatigue is a well understood phenomenon, causing irreversible material damage (Paris and Erdogan 1963). Unlike in metals, the fatigue mechanism in concrete is different due to its quasi-brittle nature. Based on linear elastic fracture mechanics concepts, the fatigue crack propagation law originally proposed by Slowik et al. (1996) includes parameters such as fracture toughness, loading history, specimen size etc, except the frequency of externally applied load and is described by,

$$\frac{da}{dN} = C\frac{{K_{Imax}}^m \Delta {K_I}^n}{(K_{IC} - K_{Isup})^p} + F(a, \Delta\sigma) \tag{1}$$

where C is a parameter which gives a measure of crack growth per load cycle, K_{Isup} is the maximum stress intensity factor ever reached by the structure in its past loading history, K_{IC} the fracture toughness, K_{Imax} is the maximum stress intensity factor in a cycle, N is the number of load cycles, a is the crack

Table 1: Details of specimen geometry

Specimen	Depth (mm)	Width (mm)	Span (mm)	Initial notch(mm)
Large	152.4	38.1	381	25.4
Medium	76.2	38.1	190.5	12.7
Small	38.1	38.1	95.3	6.35

length, ΔK is the stress intensity factor range, and m, n, p, are constants. F is a function that accounts for the overload. These constant co-efficients are determined by Slowik et al through an optimization process using the experimental data and are 2.0, 1.1, 0.7 respectively.
Although, the fatigue crack propagation law given in Equation 1 is based on LEFM, the effect of the quasi-brittle nature of concrete and the presence of FPZ is accounted for by the parameter C. This parameter basically gives a measure of crack growth per load cycle. In concrete members this parameter indicates the crack growth rate for a particular grade of concrete and is also size dependent. Slowik et. al. (1996) proposed an equation for computing C values for a particular specimen depending on its fracture parameters such as the characteristic length l_{ch}, fracture energy G_f etc. Since, frequency of external loading also influences the crack propagation rate in a large amount, in an earlier work authors (Sain and Chandra Kishen 2003) have proposed a modified empirical equation for evaluating C incorporating the frequency of loading f as

$$Cf = -0.0193\left(\frac{L}{l_{ch}}\right)^2 + 0.0809\left(\frac{L}{l_{ch}}\right) + 0.0209 \quad (2)$$

The function $F(a, \Delta\sigma)$ in Equation 1 describes the sudden increase in equivalent crack length due to an overload (Slowik et al. 1996). A much detailed explanation on effect of overload can be found in (Sain and Chandra Kishen 2003). The proposed fatigue law given by Equation 1 together with Equation 2 is validated using Bazant and Xu's experimental results for small, medium and large beam specimens. The details of the specimen geometry is given in Table 1.

The stress intensity factor for bending specimen is defined as,

$$K_I = \frac{P}{B\sqrt{D}} f(\alpha) \quad (3)$$

where P, B, D are the applied load, specimen width and depth respectively. $f(\alpha)$ is the geometry factor, written as,

$$f(\alpha) = \frac{(1 - 2.5\alpha + 4.49\alpha^2 - 3.98\alpha^3 + 1.33\alpha^4)}{(1-\alpha)^{3/2}} \quad (4)$$

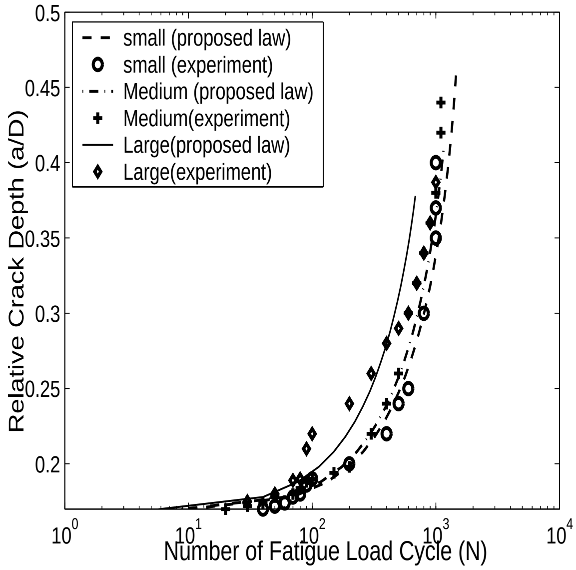

Figure 1: Fatigue Crack Propagation (LEFM law)

In Figure 1, the proposed model is compared with the experimental results of Bazant and Xu on three point bend specimens. It is seen in this figure that initially, the crack growth rate is moderate, and as the stress intensity factor approaches the fracture toughness, crack growth becomes faster finally leading to failure, which is represented by the asymptotic nature of the crack propagation curve. For all the three specimens, a good agreement is obtained between the proposed model and experimental results.

3 RELIABILITY ANALYSIS USING MONTE CARLO SIMULATION

As described earlier, the fatigue life of the member can be evaluated by integrating Equation 1 over the crack length a_0 to a_c, as

$$\int_0^{N_f} dN = \frac{(K_{Ic})^p}{C} \int_{a_0}^{a_c} \frac{da}{(K_{Imax})^m (\Delta K_I)^n} \quad (5)$$

or,

$$N_f = \frac{(K_{IC})^p}{C} \int_{a_0}^{a_c} \frac{da}{(K_{Imax})^m (\Delta K_I)^n} \quad (6)$$

$$= \frac{(K_{IC})^p f}{C'} \int_{a_0}^{a_c} \frac{da}{(K_{Imax})^m (\Delta K_I)^n}$$

where a_0 and a_c are the initial and final notch length; N_f is the fatigue life of the member. For simplification, constant amplitude fatigue loading is considered; hence K_{Isup} is neglected. Further, to incorporate the effect of loading frequency explicitly, constant C has been split as C'/f; where f is the frequency of external loading and the new constant C' is expressed in terms of mm/sec.
The parameters involved in Equation 5 are stress intensity factors K_I which are functions of stress history, notch length and specimen geometry. Considering all these parameters to be randomly distributed

with known statistical distributions, the performance function for the proposed analysis may be defined as,

$$g := N_f - N_c \tag{7}$$

where N_C is the failure cycle to be obtained from experiments. In the present analysis N_C is considered to be a design parameter, which will govern the limit state. If the computed failure cycle N_f considering the variability of all the input parameters, becomes lesser than the design life cycle of the member, then the structure is assumed to fail. To compute the failure probability of a fatigue specimen based on the above mentioned theory, Monte Carlo simulation technique is used. The simulation procedure can be described through an algorithm as follows:

1. Knowing the distribution and statistical parameters of the random variables involved in the computation (such as a_0, a_c f etc.), random sample is generated using Latin Hypercube sampling technique. Let us say the sample size for each random variable is n_T.

2. The number of fatigue load cycles required to reach a_c is computed using Equation 5.

3. The computed life cycles is compared with the desired life N_C through Equation 7.

4. The percentage reliability is computed as

$$R(\%) = \left(1 - \frac{n_f}{n_T}\right) 100\% \tag{8}$$

where n_f is the number of simulation run for which the limit state function turns out to be zero.

In the present study Monte Carlo simulation is preferred over other methods for obtaining the failure probability because of the following reasons. Firstly, the proposed method involves a number of random variables associated with the method described above. The limit state function is a highly nonlinear implicit function of those random variables. Hence, finding out a closed form expression for the probability density function of the output variable is not straightforward. Secondly, if the design parameter N_C is also considered to be a random variable (which is reasonably possible in real situation), the performance function will be even more complicated. Therefore, simulation is the only solution in the present case. As it is mentioned earlier, for generating the sample runs for the limit state conditions, random samples are generated as a matrix of $k \times n_T$, where k is the number of random variables and n_T is the number of samples generated for each random variable. The nonlinear performance function is to be evaluated for each

Table 2: Input Random Variables

Variable	Distribution	Mean	Std. Dev.
α_0	Log-normal	0.1	0.05
α_c	Log-normal	0.4	0.2
f	Log-normal	0.03	0.021

of those k values, which becomes prohibitively expensive in terms of computational power. Therefore, Latin Hypercube sampling (LHS) technique is used to reduce the sample size. The technique employs stratified sampling on each of k input variables. The range of each input variable is exhaustively divided into n disjoint intervals of equal probability. For each input variable, one observation is randomly drawn from each interval. The n values of the first variable generated by this process are paired at random without replacement with the n values of the second variable. The process continues with each successive input variable until nk- tuples are formed. The benefit that arises from using LHS rather than simple random sampling is that the statistical estimates of the output values from the simulation will almost always have more precision than with the later one (Hora and Helton 2003). Hence, with fewer sampling values computation time will be substantially saved.

4 CASE STUDIES

The proposed procedure is applied for computing the fatigue reliability of the three-point bend specimens as mentioned earlier. Three different size of specimens are considered. The specimen details are already listed earlier in Table 1. It is assumed that the elastic modulus E and fracture energy G_f values of the concrete remains constant. Hence, Equation 2 suggests a constant value of C'; whereas C will be random depending on the distribution of f. In the first step of the analysis, the stress history is considered to be deterministic. Further the initial notch length a_0, critical crack length a_c (or in other ways the relative initial and final crack length as α_0 and α_c) are also assumed to be random. The empirical constants m, n, p are to be determined from experiments. Therefore, it is considered to be deterministic for a particular specimen and loading set. The input random quantities and their statistical distributions are tabulated in Table 2.

In the present study, two different set of external loading history are considered as mentioned below:
Set 1: Minimum fatigue load level P_{min} is maintained to be zero. Maximum load level P_{max} is increased in certain steps to vary the cyclic stress range $\Delta\sigma$.
Set 2: Maximum load is kept fixed at a constant value, minimum load level is altered to obtain same set of stress range $\Delta\sigma$ as in Set 1.
The objective is to study the influence of stress range, *i.e* the effect of minimum load level on the cyclic be-

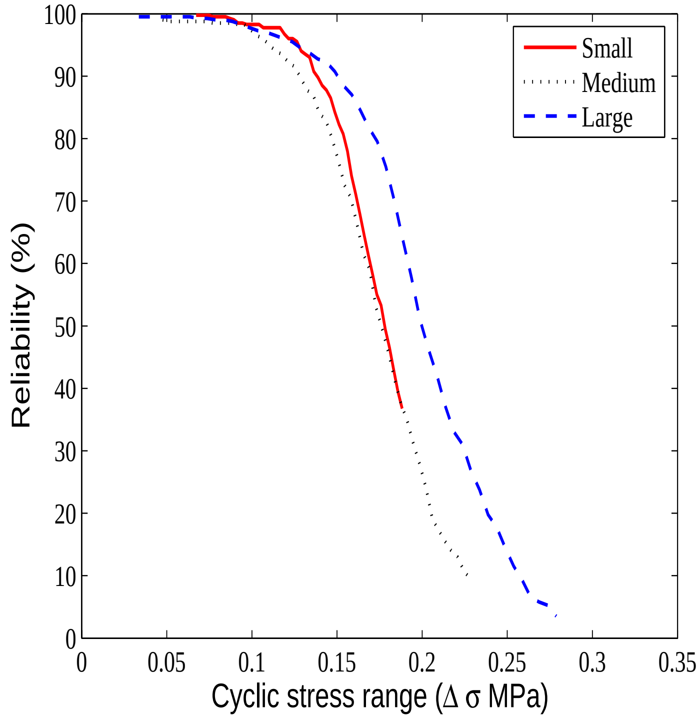

Figure 2: Reliability Measure for Flexural Fatigue (a) Set 1

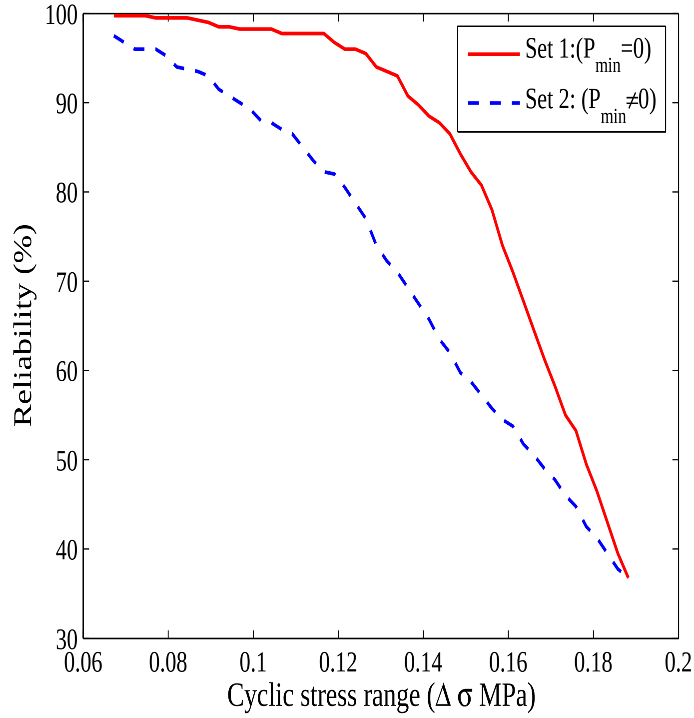

Figure 3: Reliability Measure for Flexural Fatigue (Comparison between Set 1 and 2): Small specimen

havior. In both the cases, the maximum load level is restricted upto 80% of the monotonic failure load.

The reliability index in percentage is computed using the above method for small, medium and large beam specimen for Set 1 type of loading condition, and plotted in Figure 2. It is seen that upto cyclic stress range of ≈ 0.1 MPa, reliability is nearly 100% for all the specimens. Beyond this stress range, the reliability starts decreasing along with increase in stress range value, and the reduction rate is quite fast. Percentage reliability drops down to 10% for $\Delta\sigma = 0.2MPa$. Further, the percentage reliability is computed for Set 2 loading condition and compared against the value obtained for Set 1 condition. Figure 3 shows the reliability index computed considering Set 1 and Set 2 conditions for small beam specimen. It is observed that for a fixed cyclic stress range value, reliability is higher in Set 1 case, compared to Set 2. This phenomenon can be described in the light of the proposed fatigue law used for the reliability calculation. In Equation 1, crack propagation rate per cycle depends on ΔK as well as on K_{max}. Hence, the cyclic stress range value together with the maximum stress in a cycle eventually effects the crack propagation rate.
Under the loading condition of Set 1, maximum stress was increased from an initial lower value to a maximum limit, keeping $P_{min} = 0$; whereas in Set 2, P_{max} was always equal to the limiting value of $1400N$, and P_{min} is varied to maintain the same stress range as Set 1. Therefore, in Set 2 condition, effect of K_{Imax} is higher throughout the loading range than in Set 1, except at the maximum value of upper load level, where K_{Imax} is same for both the cases. Hence, crack propagation rate is also faster in Set 2 condition, resulting in lower reliability in terms of fatigue life. At the ultimate value of P_{max}, $R(\%)$ turns out to be the same for both the sets, as explained earlier. It can be concluded from the present study, that reliability computation depends on the maximum load level in a fatigue cycle; since failure strain is dependent on the maximum stress level. If the applied cyclic stress is also random, one has to be careful while assessing the failure proability, as the $\Delta\sigma$ and σ_{max} both influence the failure cycle.

5 SENSITIVITY ANALYSIS

The variability influence of the maximum number of loading cycles N_f, that has reached corresponding to critical crack length, on the variability of input random quantities was studied by means of stochastic sensitivity analysis. Two different well known approaches are followed for the calculation of sensitivity coefficients. The concept is based on the assumption that there will be higher correlation degree of the output in case of the input parameters, which are relatively more sensitive to the output. The first method is based on the comparison of sensitivity coefficients p_i defined on behalf of variation coefficients by the

Table 3: Sensitivity Coefficient (Case 1)

Variable	p_i	r_i
P_{max}	57.022	-0.995
α_0	3.89	-0.4
α_C	7.16	0.7412
f	4.9	0.5243

Table 4: Sensitivity Coefficient (Case 2)

Variable	p_i	r_i
P_{max}	10.34	-0.8942
$\Delta\sigma$	3.23	-0.926
α_0	4.22	-0.417
α_C	8.19	0.783
f	3.8	0.45

relation (Kala 2006):

$$p_i = 100\frac{v_{yi}^2}{v_y^2} \tag{9}$$

where v_{yi} is the variation coefficient of the output quantity, assuming that all the input quantities except the i^{th} one are considered to be deterministic ones (during the simulation, they are equal to the mean value); where $i = 1, 2, \cdots M$; M being the number of input variables. v_y is the coefficient of variation of the output quantity, assuming that all the input quantities are considered to be random ones.

The second method is to determine the Spearman rank-order correlation r_i frequently applied in the framework of a simulation method. The Spearman rank-order correlation can be defined as:

$$r_i = 1 - \frac{6(\sum_j k_{ji} - l_j)^2}{N(N^2-1)} \tag{10}$$

where r_i is the order representing the value of random variable X_i in an ordered sample among N simulated values applied in the j^{th} simulation (the order k_i equals the permutation at LHS), l_j is the order of an ordered sample of the resulting variable for the j^{th} run of the simulation process, $(k_{ji} - l_j)$ is the difference between the ranks of two samples). If the coefficient r_i has value near to 1 or -1, it would suggest a very strong dependence of the output on the input. The sensitivity coefficients are calculated for small beam specimen using both the methods and compared. In this analysis also, two different cases are considered. In the first case, P_{min} is zero, and $P_{max}(\mu = 1000; \sigma = 500)$ is a log-normally distributed variable. The sensitivity co-efficients are tabulated in Table 3. It is seen that the most sensitive parameter turns out to be P_{max}, followed by α_C, f and α_0 respectively. In the second case, P_{min} is also taken to be a normally distributed random variable with a mean of 100 MPa and standard deviation of 50 MPa. Hence, one more parameter $\Delta\sigma$ is also taken into account in the sensitivity calculation. Table 4 represents the sensitivity measures obtained for the random variables. In this case also, the first method based on variability coefficients results in higher sensitivity for P_{max}; whereas Spearman correlation predicts $\Delta\sigma$ to be the most sensitive parameter.

6 EFFECT OF FRACTURE PROPERTY ON RELIABILITY

In the earlier part of this study, the inherent fracture property K_{Ic} of concrete is assumed to be deterministic. In this section, the computation of reliability index is further refined considering the fracture toughness K_{Ic} to be randomly distributed. To make the computation easier, once the sensitivity of the other external parameters are determined, as discussed in previous section, the parameters which are predominantly sensitive, could be assumed as randomly distributed. Other less sensitive parameters such as α_0, f and $\Delta\sigma$ can be considered to be deterministic. Hence the random variables turns out to be α_C and K_{Ic}. The statistical distributions are reported as in Table 5. For a specific value of the samples α_C and K_{Ic}, the failure strength σ_{max} can be obtained from the condition of unstable fracture as follows:

$$\frac{(P_{max})_C}{B\sqrt{D}} f(\alpha_C) = K_{Ic} \tag{11}$$

Further, the parameter C is a function of characteristic length l_{ch} as seen in Equation 2, which in turn depends on the fracture energy G_f (or K_{Ic}) as given by the following relation:

$$l_{ch} = \frac{EG_f}{f_t^2} = \left(\frac{K_{Ic}}{f_t}\right)^2 \tag{12}$$

Hence, C will also vary depending on the distribution of K_{Ic}. The reliability index is computed through the above described method as given in Equation 8, using the Monte Carlo Simulation technique. Figure 4 shows the reliability index computed for small medium and large beam specimens. The reliability index is much lower in this case than the value obtained considering K_{Ic} to be deterministic. Also the value of the percentage reliability reduces in a much faster rate, even in the low cyclic stress range. Therefore, fracture toughness property can be considered as one of the most influencing parameter in fatigue reliability computation.

7 CONCLUSIONS

In the present study, a probabilistic framework is suggested for predicting the fatigue life of plain concrete member considering modified LEFM based fatigue

Table 5: Random Variables (Effect of K_{Ic})

Variable	**Distribution**	**Mean**	**Std. Dev.**(σ)
K_{Ic}	log-normal	1	0.5
α_C	log-normal	0.4	0.2

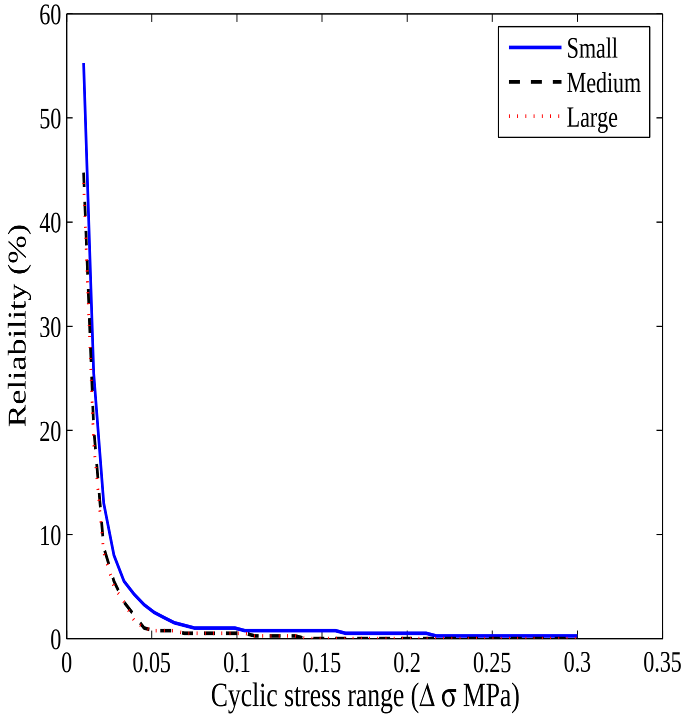

Figure 4: Reliability Measure for Flexural Fatigue (Effect of K_{Ic})

law. The external parameters which drive the fatigue process such as, stress history, frequency of the cyclic loading, and the notch length have been considered as random variables. The reliability index are computed using Monte Carlo simulation technique, considering two different loading scenarios. The Latin Hypercube sampling technique is used to reduce the computational time. It is seen from the case studies that the reliability depends on the maximum stress level as well as the cyclic stress range. The stochastic sensitivity analysis is performed to determine the predominant factor amongst the input variables, which influences mostly the fatigue reliability prediction. It is observed that reliability is mostly sensitive to maximum cyclic stress value, followed by critical crack length, frequency of applied loading and the initial notch length in that order. Further, the reliability computation is refined considering the fracture toughness to be random variable, with log-normal distribution. It is seen from the analytical results, that reliability reduces considerably compared to the earlier one, where K_{Ic} is assumed to be deterministic. Thus fracture toughness is one of the highly influencing parameter in fatigue reliability computation.

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