Modeling of multiple cracks in plain and reinforced concrete beams

K. M. Pervaiz Fathima & J. M. Chandra Kishen

Indian Institute of Science, Bangalore, India

ABSTRACT: Modeling of multiple cracks in concrete structures using available finite element packages is not a simple task. In this work, this complexity is simplified by transforming multiple cracks into an equivalent single crack or equivalent damage zone. Based on the global damage index, a method is proposed to represent multiple cracks as an equivalent single crack such that both correspond to the same value of damage index. By correlating the fracture mechanics and damage mechanics theories through an energy equivalence approach, an equivalent damage zone corresponding to multiple cracks is obtained. A finite element analysis is performed to assess the stiffness degradation of the members using both the theories and the results show that both the theories agree well with each other. Finally, it is concluded that it is easier to model discrete multiple cracks as an equivalent single crack or as an equivalent damage zone with reduced stiffness.

1 INTRODUCTION

Concrete is one of the most widely used materials for the construction of civil engineering structures. Cracks are present in concrete even before the structure is subjected to any external loading. These are the microcracks formed due to shrinkage, hydration and carbonation. When concrete is subjected to loading, these microcracks coalesce to form macrocracks, which upon further loading propagate leading to failure of the structure. It is important to model the cracks in concrete inorder to correctly analyse the structure. The methods currently available to model cracks are the smeared crack approach, discrete cracks, enhancing the continuum, enriching the element distribution function with a discontinuous field and methods where a discontinuous displacement field is used (Tano & Klisinski 1998). The smeared crack approach, introduced by Rashid (1968), replaces the crack by a continuum with altered physical properties. It is computationally convenient and remeshing is not required. On the other hand, the discrete crack models, widely used to study crack propagation in quasi-brittle materials like concrete, represent cracks by separating the nodes on the crack paths. They are usually implemented within the framework of the finite element method (FEM) or the boundary element method (BEM). Conventional FEM-based discrete crack models do not account for stress singularities, hence inaccurately calculate fracture parameters such as the stress intensity factors (SIFs), energy release rates, crack-tip stresses and opening displacements. In order to accurately assess the fracture parameters, fine meshing has to be performed at the crack tip. The stress singularities can also be represented using quarter point elements (Barsoum 1977) or hybrid singular elements (Tong 1977 & Zeng 2002). To analyse propagating cracks, remeshing procedures are required to update the finite element meshes after each crack propagation step (Xie et al. 1995 & Wawnzynek 1989). These procedures are generally sophisticated and difficult to implement. The complexity involved in the remeshing algorithm, in which fine crack tip meshes or special elements are to be implemented, further increases when propagation of multiple cracks is involved.

The computational approaches to failure use either of the two basic concepts: fracture mechanics or damage mechanics. The first one belongs to the family of discrete or discontinuous models while the other falls in the category of continuous models. The two approaches can be blended so as to overcome the limitations in both the approaches. The use of combined fracture/damage mechanics approach was proposed by Janson & Hult (1977) to obtain a more realistic assessment of capacity of a loaded structure. Mazars (1986) developed a model according to the framework of thermodynamics to describe the birth and growth of cracks, using a combination of linear elastic damage mechanics and linear elastic fracture mechanics.

Isotropic damage models are the simplest form of damage theory, wherein a single internal variable D defines the nonlinear behavior. This variable can be considered as a damage indicator and its value ranges from 0 to 1. The constitutive models based on isotropic damage concepts have been found to be more advantageous for its numerical practicability on modeling of concrete fracture compared to smeared crack applications (Köksal & Karakoç 1999).

In this work, an attempt is made to simplify the complexities involved in modeling multiple crack propagation using the existing finite element packages by introducing an equivalent approach. The equivalence is established based on a global damage index, which is a function of minimum eigenvalue of the stiffness matrix. A method is proposed to obtain equivalent single crack corresponding to multiple cracks such that damage index value is same in both the cases. The crack so obtained is further transformed into an equivalent damage zone by correlating fracture mechanics and damage mechanics approaches through an energy equivalence approach.

The equivalence is validated by computing the stiffness degradation factor using both the approaches.

2 DAMAGE MODELS

Damage models may be classified mainly into two categories: those used mostly in seismic engineering wherein, damage indices are evaluated from parameters such as sectional forces, ductility or deformation energy of structural members; the second category is made by the continuum mechanics damage models that describe the material state at a point in the structure and are based on the principles of thermodynamics (Mazars 1986). Numerous models have been proposed to represent damage of structural members, most of which are based on empirical damage definitions. Several researchers have introduced damage indices that are a function of a few selected parameters, such as the normalized energy index (Darwin & Nmai 1986), damage ratio based on reduced stiffness (Lybas & Sozen 1977), normalized cumulative rotation (Banon & Veneziano 1982), strength drop index (Chung et al. 1989) etc. The seismic damage model by Park & Ang (1985) is also widely used for reinforced concrete members. Damage measures can also be defined in terms of structural stiffness, quantified by some characteristic parameters such as eigen frequencies and mode shapes (Kratzig et al. 2000 & Ndambi et al. 2002) etc, that reflect the structural stiffness indirectly or directly by means of eigenvalues of the global stiffness matrix (Kratzig & Petryna 2001). In the present work, a global damage index that is based on the minimum eigenvalue of the global stiffness matrix is used. It is defined as

$$D = 1 - \frac{\lambda_{\min d}}{\lambda_{\min u}} \tag{1}$$

where, λ_{minu} is the minimum eigenvalue corresponding to an uncracked beam and λ_{mind} is the minimum eigenvalue corresponding to a cracked beam. The limiting values of D may be defined as follows: when there is no crack, $\lambda_{minu} = \lambda_{mind}$, therefore D = 0corresponding to no damage; when critical crack length corresponding to failure is reached, $\lambda_{mind} \rightarrow 0$, and $D \rightarrow 1$ indicating critical damage of the member. Between these two limiting conditions, at various stages of crack propagation, the λ_{mind} value decreases monotonically with increase in crack length, and the corresponding stiffness degradation factor would fall in the range 0 < D < 1. Through this procedure, the local effects of cracking get incorporated in the global behavior of the member. By this method, the degraded structural response or the residual stiffness of the member can be computed observing the variation of local damage parameter. In other words, local damage variables act as a parameter for determining the cause of damaging event, and this result is captured through the global damage indicator (Sain & Chandra Kishen 2008).

2.1 Computation of Damage Index for Beams with Multiple Cracks

Concrete beams are one of the most regularly used structural members to study the fracture processes of concrete structures. Even, the first theoretical work on fracture of concrete (Kaplan 1961) examined the application of linear elastic fracture mechanics to concrete beams under three and four point bending. Despite the fact that a concrete beam is structurally simple, a good understanding of its failure process may well explain some of the fundamental mechanisms in the fracture of concrete. In this work, plain and reinforced concrete beams subjected to three point bending are considered.

2.2 Damage Index for Plain Concrete Beams with Multiple Cracks

In this study, damage index is computed for three geometrically similar beams of different sizes (Bazant & Xu 1991), the span to depth ratio being equal to 2.5 and the initial notch length to depth ratio equal to 1/6. The geometry and loading details of the beams are given in Table 1. A finite element analysis is carried out by discretizing the beam into 1000 beam elements and using crack beam elements at the known position of crack.

Table 1. Geometry and loading conditions of plain concrete beams.

Specimen	Width	Depth	Span	Peak Load
1	mm	mm	mm	Ν
Small	38.1	38.1	95	1815.6
Medium	38.1	76.2	191	2986
Large	38.1	152.4	381	5184

The standard stiffness matrix for beam element with two degrees of freedom (vertical deflection and rotation) at each node is used for the undamaged elements. The stiffness matrix of the cracked beam element is obtained using compliance coefficients, by partial differentiation of the total strain energy equation with respect to the nodal displacements (Tharp 1987) and is given in Equation 2.

$$\begin{bmatrix} k \end{bmatrix}_{crack} = \begin{bmatrix} \frac{1}{\lambda_{vv}} & 0 & -\frac{1}{\lambda_{vv}} & 0 \\ 0 & \frac{1}{\lambda_{mm}} & 0 & -\frac{1}{\lambda_{mm}} \\ -\frac{1}{\lambda_{vv}} & 0 & \frac{1}{\lambda_{vv}} & 0 \\ 0 & -\frac{1}{\lambda_{mm}} & 0 & \frac{1}{\lambda_{mm}} \end{bmatrix}$$
(2)

where λ_{vv} and λ_{mm} are the compliances with respect to shear and moment respectively and are given below.

$$\lambda_{\nu\nu} = \frac{2B(1-\nu^2)}{E} \int_{0}^{a} \left(\frac{K_{11\nu}}{V}\right)^2 da$$
(3)

$$\lambda_{mm} = \frac{2B(1-\nu^2)}{E} \int_0^a \left(\frac{K_{\rm Im}}{M}\right)^2 da \tag{4}$$

where B = width of the beam; v = Poisson's ratio; a = crack length; M = Bending moment; V = Shear force; K_{Im} = Mode I SIF; K_{IIv} = Mode II SIF

Computation of damage index involves the following steps:

1. The eigenvalues of the stiffness matrix of the undamaged beam are determined.

2. The crack beam element is introduced at the known crack location and standard beam elements at other undamaged portion of the beam.

3. The global stiffness matrix for cracked member is assembled and its eigenvalues are computed.

4. The degree of global damage D is computed using the minimum eigenvalues corresponding to damaged and undamaged beams using Equation 1.

5. The crack length is incrementally increased and the above steps repeated to determine the variation of D as a function of relative crack depth (the local damage parameter). The above procedure is adopted for three point bend beams containing

(i) single crack at the midspan

(ii) multiple (five) cracks, with one crack at the

midspan and the others at a spacing of span/20 on either side of the middle one. All the cracks are assumed to be of the same length and equally spaced.

Figures 1 & 2 show the variation of damage index as a function of relative crack depth for the three different sized specimens with single crack and five cracks respectively. From these figures, it is observed that the variation of damage index with crack depth is same for geometrically similar specimens and as expected, the damage index increases with the number of cracks.



Figure 1. Variation of Damage Index with respect to relative crack depth for beams with a single crack.



Figure 2. Variation of Damage Index with respect to relative crack depth for beams with five cracks.

2.3 Damage Index for Reinforced Concrete Beams with Multiple Cracks

A reinforced concrete beam originally used by Alaee & Karihaloo (2003), whose geometrical properties are given in Table 2 is considered here.

Table 2. Details of RC beam.		
Depth (mm)	150	
Width (mm)	100	
Length (mm)	1200	
Steel area (mm^2)	113.09	
Yield stress (MPa)	544	
Young's modulus E (MPa)	35.6e3	

The procedure to compute the damage index is the same as that of plain concrete beam as discussed in the previous subsection. The difference lies in the formulation of the stiffness matrix. The concrete beam element has three degrees of freedom namely extension, vertical deflection and rotation, the reinforcement steel is superimposed over it as a bar element (only extension). Since steel induces an axial closing force, the beam may be assumed to be a flexural member subjected to axial loading as well. The stiffness matrix of such a beam element as derived by Chajes (1974) is used here.

$$\begin{bmatrix} k \end{bmatrix}_{beam} = \begin{bmatrix} k \end{bmatrix}_{std} - P[k]_{axial}$$
⁽⁵⁾

$$[k]_{ad} = \begin{bmatrix} \left(\frac{A_{.}E_{.}}{l} + \frac{A_{.}E_{.}}{l}\right) & 0 & 0 & -\left(\frac{A_{.}E_{.}}{l} + \frac{A_{.}E_{.}}{l}\right) & 0 & 0 \\ 0 & \frac{12E_{.}I_{.}}{l^{2}} & \frac{-6E_{.}I_{.}}{l^{2}} & 0 & \frac{-12E_{.}I_{.}}{l^{2}} & \frac{-6E_{.}I_{.}}{l^{2}} \\ 0 & \frac{-6E_{.}I_{.}}{l^{2}} & \frac{4E_{.}I_{.}}{l} & 0 & \frac{6E_{.}I_{.}}{l^{2}} & \frac{2E_{.}I_{.}}{l} \\ -\left(\frac{A_{.}E_{.}}{l} + \frac{A_{.}E_{.}}{l}\right) & 0 & 0 & \left(\frac{A_{.}E_{.}}{l^{2}} + \frac{A_{.}E_{.}}{l}\right) & 0 & 0 \\ 0 & \frac{-12E_{.}I_{.}}{l^{2}} & \frac{6E_{.}I_{.}}{l^{2}} & 0 & \frac{12E_{.}I_{.}}{l} \\ 0 & \frac{-6E_{.}I_{.}}{l^{2}} & \frac{2E_{.}I_{.}}{l} & 0 & 0 \end{bmatrix}$$

$$(6)$$

where $A_c = Area$ of concrete; $E_c = Modulus$ of elasticity of concrete; $A_s = Area$ of steel; $E_s = Modulus$ of elasticity of steel; I = length of element.

$$\begin{bmatrix} k \end{bmatrix}_{axial} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5l} & \frac{-1}{10} & 0 & \frac{6}{5l} & \frac{-1}{10} \\ 0 & \frac{-1}{10} & \frac{2l}{15} & 0 & \frac{1}{10} & \frac{-l}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5l} & \frac{1}{10} & 0 & \frac{6}{5l} & \frac{1}{10} \\ 0 & \frac{-1}{10} & \frac{-l}{30} & 0 & \frac{1}{10} & \frac{2l}{15} \end{bmatrix}$$
(7)

Here, P is the axial force induced in the steel rod. It is computed at each crack step using the expression below which is obtained by the congruence condition i.e. by equating the rotation due to bending moment and the closing force to zero (Carpinteri 1984).

$$\frac{PH}{M} = \frac{1}{\left(\frac{1}{2} - \frac{h}{B}\right) + \gamma(\xi)}$$

where
$$\gamma(\xi) = \frac{\int_{0}^{\xi} Y_{M}(\xi) Y_{F}(\xi) d\xi}{\int_{0}^{\xi} Y_{M}^{2}(\xi) d\xi}$$
 (9)

(8)

$$Y_{M}(\xi) = 6? \left(.99\xi^{1/2} - 2.47\xi^{3/2} + 12.97\xi^{5/2} - 23.17\xi^{7/2} + 24.80\xi^{9/2} \right)$$
(10)

$$Y_F(\xi) = \left(1.99\xi^{1/2} - 0.41\xi^{3/2} + 18.70\xi^{5/2} - 38.48\xi^{7/2} + 53.85\xi^{9/2}\right) (11)$$

The stiffness matrix of the cracked beam element is given by (Tharp 1987)

$$[k]_{croct} = \begin{bmatrix} \lambda_{mm} / C & 0 & -\lambda_{mp} / C & -\lambda_{mm} / C & 0 & \lambda_{mp} / C \\ 0 & 1 / \lambda_{vv} & 0 & 0 & -1 / \lambda_{vv} & 0 \\ -\lambda_{mp} / C & 0 & \lambda_{pp} / C & \lambda_{mp} / C & 0 & -\lambda_{pp} / C \\ -\lambda_{mm} / C & 0 & \lambda_{mp} / C & \lambda_{mm} / C & 0 & -\lambda_{mp} / C \\ 0 & -1 / \lambda_{vv} & 0 & 0 & 1 / \lambda_{vv} & 0 \\ \lambda_{mp} / C & 0 & -\lambda_{pp} / C & -\lambda_{mp} / C & 0 & \lambda_{pp} / C \end{bmatrix}$$

$$(12)$$

where, λ_{vv} and λ_{mm} are defined in Equations 4 and 5; λ_{pp} and λ_{mp} compliances for extension and rotation due to axial force respectively.

$$\lambda_{pp} = \frac{2B(1-\nu^2)}{E_s} \int_0^a \left(\frac{K_{lp}}{P}\right)^2 da$$
(13)

$$\lambda_{mp} = \frac{2B(1-\nu^2)}{E_s} \int_0^a \left(\frac{K_{lp}}{P}\right) \left(\frac{K_{\rm Im}}{M}\right) da$$
(14)

and $C = \lambda_{pp} \lambda_{mm} - \lambda_{mp}^2$



Figure 3. Variation of Damage Index with respect to relative crack depth for reinforced concrete beam with 1 and 5 cracks.

Using the procedure described in the previous section, the damage index for the beam with one and five cracks is computed. Figure 3 shows the variation of damage index with the relative crack depth.

3 TRANSFORMATION OF MULTIPLE DISCRETE CRACKS INTO AN EQUIVALENT SINGLE CRACK

In this section, a method is proposed to represent multiple cracks as an equivalent single crack using the damage index obtained in the previous section such that both correspond to the same value of damage index. The method is illustrated in Figure 4.

Figure 4 shows how five cracks are converted into an equivalent single crack based on damage index. The variation of damage index with respect to relative crack depth being known; for any a/H, for the beam with five cracks, D is known. Corresponding to this D, an equivalent single crack $(a/H)_{eq}$ is



Figure 4. Equivalent single crack corresponding to five cracks.

Table 3. Equivalent single crack length correspond-ding to multiple cracks for plain concrete beams.

Relative crack depth	Equivalent single crack
(single crack)	corresponding to 5 cracks
0.1	0.217
0.2	0.401
0.3	0.547
0.4	0.652
0.5	0.737
0.6	0.812

Table 4. Equivalent single crack length correspond-ing to five cracks for reinforced concrete beam.

Relative crack depth	Equivalent single crack
(single crack)	corresponding to 5 cracks
0.1	0.212
0.2	0.403
0.3	0.547
0.4	0.652
0.5	0.741
0.6	0.815

obtained from the plot of damage index versus relative crack depth for the beam with single crack.

The equivalent crack lengths for plain concrete beams with multiple cracks are tabulated in Table 3 and for the reinforced concrete beam in Table 4.

4 TRANSFORMATION OF DISCRETE CRACKS INTO AN EQUIVALENT DAMAGE ZONE

According to fracture mechanics theory, energy is required for an existing crack to propagate by an amount δa . This energy is commonly expressed as strain energy release rate per unit crack extension and is denoted by G. Similarly, in case of damage based analysis, the strain energy loss per unit volume of the material due to increase in damage by an amount dD is referred to as damage strain energy release rate. The idea of energy based equivalence is based on equating the energy loss due to damage, with the energy required for equivalent crack propagation within the member. In a more explanatory sense, energy based equivalence correlates two structures having the same geometry and loading condition, but different damage definitions. In a global sense they behave in the same manner when the energy dissipation corresponding to two different damage conditions become equal for the two structures (Mazars & Cabot 1996). The energy release rate δU per unit crack extension δa (which in turn is equal to the potential energy lost ($\delta \Pi$) by the applied load) is related to the stress intensity factor.

Energy release per unit crack extension is computed using

$$U(\alpha) = \frac{9}{4} \frac{P^2 L^2}{B H^2} \frac{1}{E} F(\alpha)$$
(15)

where
$$F(\alpha) = \int_{0}^{\alpha} \alpha Y^{2}(\alpha) d\alpha$$

 α is the relative crack depth and Y (α) is the geometry factor and is different for different span to depth ratios

$$Y(\alpha) = \frac{\left[1.99 - \alpha(1 - \alpha)(2.15 - 3.93\alpha + 2.7\alpha^2)\right]}{(1 + 2\alpha)(1 - \alpha)^{1.5}}$$
(16)
for $\frac{L}{H} = 2.5$

Using Equation 15, the strain energy as a function of relative crack depth α can be evaluated. The crack is replaced with an equivalent damage zone of width l_c , depth L_D and thickness B. $L_D = a + l_d$, where, a is the crack length and l_d is the length of process zone. The damage zone corresponding to the process zone length is assumed to be an equilateral triangle of side lc (Garhwal & Chandra Kishen 2008). Figure 5 shows an equivalent damage zone corresponding to a discrete crack of length a.

In three-dimensional form, the energy dissipated through damage is expressed as

$$U_D = \iint_{V \mid D(y)} (-Y) dD dV$$
(17)

where,

$$Y = \frac{\sigma_{eq}^2}{2E(1-D^2)} \left[\frac{2}{3} (1+\nu) + 3(1-2\nu) \left(\frac{\sigma_H}{\sigma_{eq}} \right)^2 \right]$$
(18)

where, σ_{eq} is the von Mises equivalent stress and σ_{H} is the hydrostatic stress.

The energy dissipated due to progressive damage near the location of the crack results in the gradual change of damage variable D from $0 \rightarrow 1$ and is given by,

$$U_{D} = BI_{c} \begin{cases} a \left(1 - \frac{y}{a}\right) \\ \int \int (-YdD) dy \\ 0 & 0 \end{cases} + \frac{BI_{c}}{2} \begin{cases} a + I_{d} \left(1 - \frac{y}{a + I_{d}}\right) \\ + \int \int (-YdD) dy \\ a & \left(1 - \frac{y}{a}\right) \end{cases}$$

$$(19)$$

Equating the two energy terms in Equations 15 and 19, we can solve for the unknown dimension of the damage zone, L_D through a trial and error procedure, after defining L_D in terms of l_c . The above procedure is adopted for the three plain concrete specimens (Table 1) and the reinforced concrete beam (Table 2) with single and multiple (five) cracks. In the case of multiple cracks, they are first converted into an equivalent single crack. The variation of the damage zone length L_D with respect to crack length is shown in Figures 6 & 7 for plain concrete beam.



Figure 5. Equivalent damage zone corresponding to a discrete crack of length a.



Figure 6. Variation of the equivalent damage zone length LD with respect to relative crack depth for plain concrete beams with a single crack.



Figure 7. Variation of the equivalent damage zone length L_D with respect to relative crack depth for plain concrete beams with five cracks.



Figure 8. Variation of the equivalent damage zone length L_D with respect to relative crack depth for reinforced concrete beam with one and five cracks.

5 VALIDATION OF THE ENERGY EQUIVALENCE CONCEPT

In this study, the stiffness reduction in concrete beams is determined using both fracture mechanics and damage mechanics approaches. The stiffness reduction factor, is defined as (Lybas & Sozen 1977),

$$D_r = \frac{K_0}{K_r} \tag{20}$$

where, K_0 is the initial stiffness and K_r is the reduced secant stiffness associated with maximum displacement.

The finite element package FRANC2D (Cornell Fracture Group 1997) is used to model the beams. The reduced secant stiffness is computed using the maximum displacement as a function of increasing crack length for fracture based analysis and as a function of increasing damage zone length for damage based analysis. In the fracture mechanics based analysis, a rosette of singular crack tip quarter point elements are used to model the crack tip. The damage zone, in the damage mechanics based analysis is modeled using reduced value of modulus of elasticity. According to the theoretical definition of damage, the modulus of elasticity of the elements in the damage zone should approach zero. In this finite element study, a trial value of 1/100 of the undamaged modulus of elasticity E is considered in order to avoid the numerical difficulties arising from using a zero value. Thus, by computing the maximum displacement from the finite element analysis, the reduced secant stiffness Kr is computed. In case of fracture mechanics approach, the crack tip is specified and the crack length is increased incrementally by moving the crack tip. In case of damage mechanics approach, damage zone size is increased. The



Figure 9. Stiffness reduction factor as a function of crack length for beam (small) with a single crack.



Figure 10. Stiffness reduction factor as a function of equivalent crack length for beam (small) with five cracks.

variation of stiffness reduction factors with increasing relative crack depth are shown in Figures 9 & 10 for small specimen.

It is seen from these figures, that the stiffness reduction factor match closely with each other for fracture and damage mechanics based approaches, thereby validating the energy equivalence concept used.

6 CONCLUSIONS

In this study, it is shown that multiple cracks can be represented as an equivalent single crack using damage index. The damage index, defined using the minimum eigenvalue of the stiffness matrices is independent of the size of the specimen for geometrically similar specimens. An energy based equivalence approach is proposed to model multiple discrete cracks in the form of a distributed damage zone. The stiffness reduction factor is computed using both fracture mechanics and damage mechanics theories and the results show that both the theories agree well with each other. Treating the multiple cracks as an equivalent single crack highly simplifies the complexity involved in modeling multiple cracks in concrete structures. By representing multiple cracks as an equivalent damage zone, that is reducing the modulus of elasticity of that zone, the modeling becomes much simpler as there is no need to consider the stress concentrations occurring at the crack tip and hence more efficient.

REFERENCES

- Alaee, F. & Karihaloo, B. 2003. Fracture model for flexural failure of beams retrofitted with CARDIFRC. J Eng Mech ASCE 129(9): 1028–1038.
- Banon, H. & Veneziano, D. 1982. Seismic safety of reinforced

concrete members and structures. *Earthquake Engineering* and structural Dynamics 10: 179–193.

- Barsoum, R.S. 1977. Triangular quarter-point elements as elastic and perfectly-plastic crack tip elements. *International Journal for Numerical Methods in Engineer ing* 11: 85–98.
- Bazant, Z. P. & Xu, K. 1991. Size effect in fatigue fracture of concrete. ACI Materials Journal 88(4): 390–399.
- Carpinteri, A. 1984. Stability of Fracturing Process in RC Beams. Journal of Structural Engineering ASCE 110: 427-437.
- Chajes, A. 1974. Principles of structural stability theory. Prentice Hall, Inc. Englecliffs, N.J.
- Chung, Y. S., Meyer. C. & Shinozuka, M. 1989. Modeling of concrete damage. ACI Structural Journal 86(3): 259–271.
- Cornell Fracture Group 1997. FRANC2D, A two dimensional fracture analysis code. *Cornell University, Ithaca, NY.*
- Darwin D. & Nmai C. 1986. Energy dissipation in reinforced concrete beams under cyclic load. ASCE 112(8): 1829– 1846.
- Garhwal V. & Chandra Kishen J.M. 2008. Correlation between fracture and damage for quasi-brittle bi-material interface cracks. *Engg. Fracture Mechanics* 75: 2208-2224.
- Janson, J. & Hult, J. 1977. Fracture mechanics and damage mechanics—a combined approach. J. Appl. Mech. 1 (1): 69–83.
- Kaplan, M. 1961. Crack propagation and the fracture of concrete. *ACI Journal* 58: 591–610.
- Köksal, H. O. & . Karakoç, C. 1999. An isotropic damage model for concrete. Materials and Structures 32: 611-617.
- Kratzig, W.B. & Petryna, Y.S. 2001. Assessment of structural damage and failure. Arch Appl Mechan 71:1–15.
- Kratzig, W.B., Petryna, Y.S. & Stangenberg, F. 2000. Measures of structural damage for global failure analysis. Int J Solids Struct 37: 7393–407.
- Lybas, J. & Sozen, M. 1977. Effect of beam strength and stiffness on dynamic behaviour of reinforced concrete coupled walls. *Civil Engineering. Studies, Structural Research Series (44). University of Illinois, Urbana.*

- Mazars, J. 1986. A description of micro- and macro-scale damage of concrete structures. *Journal of Engineering Fracture Mechanics* **25:** 729–737.
- Mazars, J. & Cabot, G. P. 1996. From damage to fracture mechanics and conversely: a combined approach. *Physics Review* 33: 3327–3342.
- Ndambi, J.M., Vantomme, J. & Harri, K. 2002. Damage assessment in reinforced concrete beams using eigen frequencies and mode shape derivatives. Engng Struct 24: 501–15.
- Park Y. & Ang, H. 1985. Mechanistic seismic damage model for reinforced concrete. J Struct Div ASCE 111: 722–39.
- Rashid, Y.R. 1968. Ultimate strength analysis of prestressed concrete pressure vessels. *Nuclear Engineering and Design* 7(4): 334-344.
- Sain, T. & Chandra Kishen, J.M. 2008. Damage indices for failure of concrete beams under fatigue, *Engg. Fracture Mechanics* 75: 4036-4051.
- Tano, R. & Klisinski M. 1998. Crack modelling of concrete, rock and soil with inner softening bands. Engineering Geology 49: 271-276.
- Tharp, T. M. 1987. A finite element for edge-cracked beam columns. *International Journal for Numerical Methods in Engineering* 24: 1941–1950.
- Tong, P., Pian, T.H.H. & Lasry, S.J. 1977. A hybrid-element approach to crack problems in plane elasticity. *International Journal for Numerical Methods in Engineering* 11:180–4.
- Wawrzynek, P.A. & Ingraffea, A.R. 1989. An effective approach to local remeshing around a propagation crack. *Finite Elements in Analysis and Design* 5: 87–96.
- Xie, M., Gerstle, W.H. & Rahulkumar, P. 1995. Energy-based automatic mixed-mode crack-propagation modelling. *Jour*nal of Engineering Mechanics, ASCE 121: 914–23.
- Zeng, D., Katsube, N., Zhang, J. & Soboyejo, W. 2002. Hybrid crack-tip element and its applications. *Finite Element in Analysis and Design* 38: 319–35.