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ACCURACY OF APPROXIMATION OF STRESS FIELD IN CRACKED BODIES FOR FAILURE ZONE EXTENT ESTIMATION

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Abstract: Accurate knowledge of stress/displacement field in a cracked specimen is very important for fracture analysis and for instance in the case of quasi-brittle materials, where the zone of non-linear behavior affecting the overall fracture process is larger than in other (brittle) materials, it can play a key role. In this paper, it is shown that Williams expansion can describe the crack-tip fields reliably, provided that considerably more than one or two terms of the power series are taking into account. Such an approach, referred to as multi-parameter fracture mechanics, is usually not used for a common assessment of crack behavior. The study presented shows that especially in larger distances from the crack tip the higher-order terms are crucial in order to minimize the error corresponding with use of only one or two fracture parameters (the first and second terms of Williams expansion). In other words, considering of the higher-order terms of the Williams expansion is essential if knowledge of accurate stress/displacement fields around the crack tip in the analysed specimen/structure is required.

1 INTRODUCTION

Estimation of reliability of cracked structures is a very important engineering task. Linear elastic fracture mechanics (LEFM) is widely extended and used to determine the critical (allowed) crack lengths in the case of brittle materials. The stress intensity factor is well known as the single controlling parameter for prediction of crack initiation and propagation in these materials and it is directly associated with the singular stress behavior near the crack tip.

Nevertheless, recent studies show that not only the *T*-stress (as the first non-singular term of the Williams expansion [1]) has a great effect on fracture toughness values, size and shape of the plastic zone, crack path direction, *etc.* [2−12] but also the higher-order terms can be very significant for crack behavior assessment; they can predict the constraint of elasto-plastic crack tip fields [7, 9, 13−16] and interpret (at least some aspects of) the size/geometry/boundary effect typical for quasi-brittle materials [17−21].

Furthermore, the complicated fracture processes in quasi-brittle materials do not occur exclusively in the very vicinity of the crack tip (the crack tip is practically impossible to distinguish) and therefore the stress/displacement field has to be known even in a larger distance from it. As a consequence, not only the first (singular) term of the Williams series approximation of the crack tip asymptotic field [1], but also the other (higher-order) terms have to be considered during the fracture analysis. This way, the extent of the zone around the crack tip with non-linear behavior, where the material fails, can be estimated reliably, which is essential for additional fracture response assessment.

In this paper, the higher-order terms coefficients are obtained by means of combination of numerical solution (finite elements, FE) of the problem and analytical description of the displacement field (Williams power expansion, WPE). The used approach is called as the over-deterministic method (ODM), see *e.g.* [22] for details, it represents a regression technique based on least squares formulation and its main advantage is that it does not require any special crack elements or complicated FE formulations. Contrary to other methods (boundary collocation method, hybrid crack elements method, see for instance [23−26]) it uses only the displacement field around the crack tip determined by means of the conventional FE analysis. Note that ODM represents a modification of direct methods used *e.g.* for stress intensity factor estimation.

The application of the ODM on selected mode I and mixed-mode configurations is described in the following text and the importance of consideration of the higher-order terms for stress/displacement field description is highlighted.

2 METHODOLOGY

2.1 Near-crack-tip field description

Williams [1] derived that the stress and displacement distribution can be expressed as a power series. Assuming a plane crack with traction-free faces in a linear-elastic material subjected to arbitrary remote loading, the stress/displacement field around the crack tip obtained by the Williams eigenfunction expansion technique is given as:

$$\sigma_{ij} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n \, {}^{\sigma} f_{\mathrm{I},ij}(\theta, n) + \sum_{m=1}^{\infty} \frac{m}{2} r^{\frac{m}{2}-1} b_m \, {}^{\sigma} f_{\mathrm{II},ij}(\theta, m), \tag{1}$$

$$u_i = \sum_{n=0}^{\infty} r^{n/2} a_n \, {}^{u} f_{\mathrm{I},i}(\theta, n, E, \nu) + \sum_{m=0}^{\infty} r^{m/2} b_m \, {}^{u} f_{\mathrm{II},i}(\theta, m, E, \nu), \tag{2}$$

where $i,j \in \{x,y\}$, the coefficients a_n and b_m (in our case unknown) depend on the specimen geometry and loading conditions, and correspond to the loading mode I and II, respectively. In Eq. (1) and (2), r and θ represent the polar coordinates centered at the crack tip, see Fig. 1, E and ν are Young's modulus and Poisson's ratio and ${}^{\sigma}f_{\mathrm{I}}$, ${}^{\sigma}f_{\mathrm{II}}$ and ${}^{u}f_{\mathrm{I}}$, ${}^{u}f_{\mathrm{II}}$ symbolize known functions corresponding to the mode I/II ($f_{\mathrm{I}}/f_{\mathrm{II}}$) and stress/displacement (σ/u) distribution.

Coefficients a_n and b_m are functions of relative crack length $\alpha = a/W$, where a is the crack length and W is the specimen's width. They can be expressed as dimensionless functions (with regard to loading), g_{I} and g_{II}, respectively, as follows [27−29]:

$$g_{\mathrm{I},n}(\alpha) = \frac{a_n(\alpha)}{\sigma} W^{\frac{n-2}{2}} \quad \text{for } n = 1, 3, 4\ldots, N, \qquad g_{\mathrm{I},2}(\alpha) = t_{\mathrm{I}}(\alpha) = \frac{4a_2(\alpha)}{\sigma}, \tag{3}$$

$$g_{\mathrm{II},m}(\alpha) = \frac{b_n(\alpha)}{\sigma} W^{\frac{n-2}{2}} \quad \text{for } m = 1, 3, 4\ldots, M, \qquad g_{\mathrm{II},2}(\alpha) = t_{\mathrm{II}}(\alpha) = \frac{4b_2(\alpha)}{\sigma}, \tag{4}$$

where σ is the nominal stress caused by the applied load.

2.2 Over-deterministic method

There have been derived several methods enabling determination of the coefficients of the higher-order terms, a_n and b_m, of the Williams expansion. Most of those methods use advanced mathematical procedures and more extensive and deeper knowledge of the special elements or FE code is unavoidable. From that reason, ODM has been chosen for estimation of the coefficients of the higher-order terms in this paper. This method requires only knowledge of the displacement field data determined by means of the conventional FE analysis in a set of nodes around the crack tip. These data serve then as input for Eq. (2). As two components of displacement vector is known in each node (in the case of 2D problem), two equations are available for each node. Consequently, a system of $2k$ equations for variables a_n and b_m exist, where k represents the number of nodes selected for the ODM application. In this paper, commercial mathematical packages were used for solution of the resulting system of algebraic equations. Note that a relation between the number of nodes, k, and the number of variables of loading mode I (N: a_0, a_1,..., a_N) and of loading mode II (M: b_0, b_1..., b_M) has to satisfy the following inequality in order to meet the condition for an over-determined system of equations:

$$2k > N + M + 2\,. \tag{3}$$

More details about the ODM can be found for instance in [22]. Authors of the present paper have also devoted several publications to parametric studies on ODM accuracy, convergence, or mesh sensitivity *etc.*, see *e.g.* [30−35].

3. NUMERICAL MODELING

In order to demonstrate the significance of considering of the higher-order terms of WPE, two loading geometries have been chosen. The main goal is to compare the stress distribution obtained from the numerical simulation of the problem (by means of FE analysis) with the stress distribution defined analytically by Williams expansion, whose terms coefficients have been determined from the same FE analysis' results using ODM. The analytical approximation is done in variants with various ranges of higher-order terms of WPE.

3.1 Mixed-mode geometry − AECT

A plate with an angled edge-crack under uniaxial tension (AECT) was chosen as the mixed-mode geometry investigated. The angle between the crack and the edge normal was considered as $\beta = 30°$, see Fig. 1 and [22] for details.

In order to obtain the appropriate displacement field near the crack tip (which is necessary for the ODM application) a corresponding numerical model of the cracked specimen had to be created; it was done in ANSYS software [36], see the FE mesh used in Fig. 1. The specimen width, half length as well as thickness were considered to be unity, relative crack length $a/W = 0.6$, $\beta = 30°$, applied stress $\sigma = 1$. Note that all units used were self-consistent and therefore no units are presented. Because of the independence of the coefficients of the WPE on material properties, values of Young's modulus and Poisson's ratio were chosen as $E = 1$ and $\nu = 0.25$. The crack-tip singularity was modeled through the first row of elements made of the so-called crack elements with shifted mid-side nodes, whereas standard 8-node isoparametric elements were used for modeling of the rest of the specimen. Plane stress conditions were met in accordance with [22] whose results were employed for (partial) verification of presented results.

For evaluation of the higher-order terms coefficients the displacements of the fifth ring of nodes around the crack tip were used as inputs for the ODM procedure programmed in Mathematica package [37]. For purposes of this paper, the first ten higher-order terms coefficients of the both loading modes (a_n and b_m) were determined.

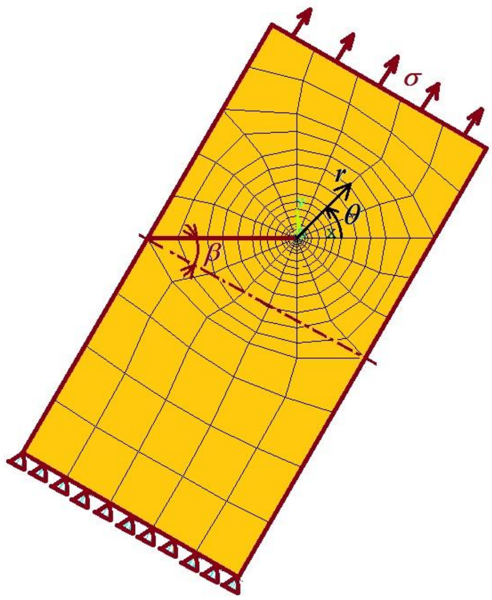

Figure 1: Numerical model of the plate with an angled edge-crack under uniaxial tension, $\beta = 30°$, with applied boundary conditions used for the stress/displacement fields comparative study

3.2 Mode I geometry – WST

A cube-shaped wedge-splitting test (WST) specimen was selected as the mode I geometry. In this paper, the WST specimen's size, shape and details of the boundary conditions (*e.g.* loading platens, load decomposition, dimensions of the groove for inserting of the platens *etc.*) were considered in accordance with prepared experimental campaign and corresponding numerical analyses which preceded the testing preparation [32, 33, 38–41]. Note that the basic dimension of the cube-shaped specimen is equal to 100 mm; the value of the splitting component of the loading force (the horizontal one in Fig. 2) was considered equal to 1 kN, the angle of the loading wedge equal to 15°. Elastic constants of the material model for concrete and steel platens were set to $E = 35$ GPa, $\nu = 0.2$ and $E = 210$ GPa, $\nu = 0.3$, respectively. For other details see *e.g.* [33]. The model was created similarly to the above-mentioned mixed-mode case, the same FEM software was used [36]. Plane strain conditions were used here as the breadth of the specimen is significant. FE model of the symmetrical half of the test specimen, including the detail of the mesh around the crack tip is depicted in Fig. 2.

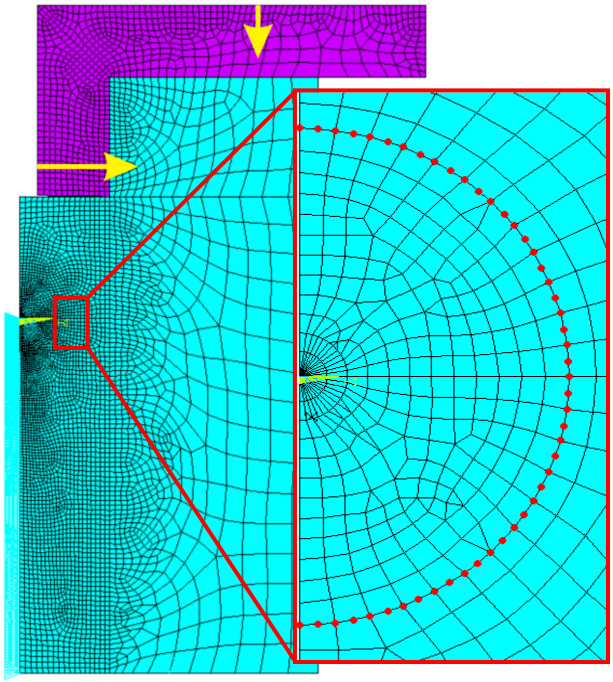

Figure 2: Numerical model of the WST on cube-shaped specimen (symetrical half) with detail of the FE mesh around the crack tip (nodes considered for the higher-order terms coefficients computation using ODM are emphasized)

Table 1: Values of the dimensionless expression of the higher-order terms coefficients of the loading mode I, $g_{\mathrm{I},n}$, and the loading mode II, $g_{\mathrm{II},m}$, used for the comparative analysis

n, m	$g_{\mathrm{I},n}$ [-]	$g_{\mathrm{II},m}$ [-]
1	1.394	-3.785×10^{-1}
2	5.971×10^{-1}	-6.660
3	-1.272	-2.212×10^{-1}
4	2.477×10^{-1}	-1.931×10^{-1}
5	-5.435×10^{-1}	1.817×10^{-1}
6	2.707×10^{-1}	-6.539×10^{-1}
7	-2.817×10^{-1}	2.730×10^{-1}
8	1.188×10^{-2}	-5.623×10^{-1}
9	3.614×10^{-1}	3.942×10^{-1}
10	-3.846×10^{-1}	-8.561×10^{-1}

4 RESULTS AND DISCUSSION

4.1 AECT

Values of the coefficients a_n and b_m determined by ODM and then used for the stress evaluation are introduced in Tab. 1. Note that values of the first five coefficients corresponding to both loading modes have been verified with data published in literature and their mutual agreement is very good, see [31] for details.

In order to present the influence of the higher-order terms on the stress distribution, the stress tensor components σ_x, σ_y, τ_{xy} have been investigated in several directions around the crack tip: $\theta = 0°$ (a path ahead of the crack tip), $\theta = 45°$ and $\theta = 180°$ (a path along the stress-free crack faces). The last angle enables to verify the numerical model and estimate the region in the very vicinity of the crack tip subjected to numerical errors. Note that similar dependences can be observed also in other directions.

In Fig. 3, the opening stress σ_y is plotted for the angle of 180°, *i.e.* along the crack faces. Dependences in Fig. 3 prove the functionality of the numerical model used for the calculations: the value of the opening stress on the crack faces (that should be stress-free) is zero, as also the stress values from the Williams expansion show. The non-zero σ_y values emerging in the small region in the very vicinity of the crack tip represent the typical numerical errors. Therefore, the stress values obtained in this area both in other directions and for other stress components should be considered as inaccurate/invalid.

The typical singular stress behavior can be observed in Fig. 4, where for instance the stress component σ_x in dependence on the relative distance from the crack tip ahead of the crack tip ($\theta = 0°$) is presented (dependences for the other stress components are similar).

Fig. 4 shows that there exist differences between the numerical solution of the stress distribution and the stress values calculated by means of the WPE. The ratio between the values can be about 10 ÷ 20 in a large distance from the crack tip if only one or two higher-order terms of the Williams expansion are used for calculation of the corresponding stress component.

The extent of the stress deviations depends on two factors. It holds that the larger distance from the crack tip, the higher difference between the numerical solution and the WPE approximation. The other dependence (on the number of the higher-order terms considered) behaves less exactly than one would expect in this interpretation of the results. Thus, it seems it is generally impossible to state which number of the higher-order terms of WPE brings the best results. This dependence is not as unambiguous as in the previous case.

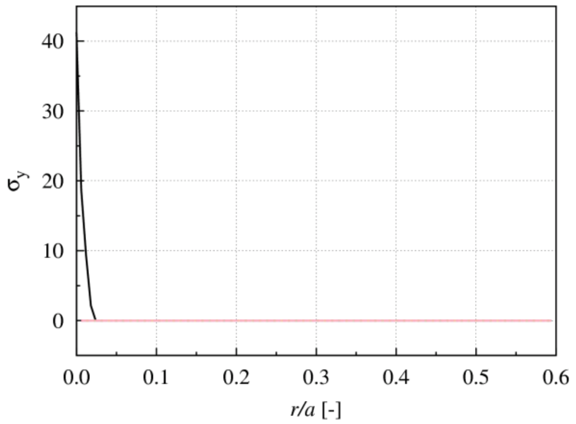

Figure 3: Demonstration of the extent of the region with numerical errors occuring in FE solution by means of displaying of the opening stress σ_y values on the crack stress-free faces ($\theta = 180°$)

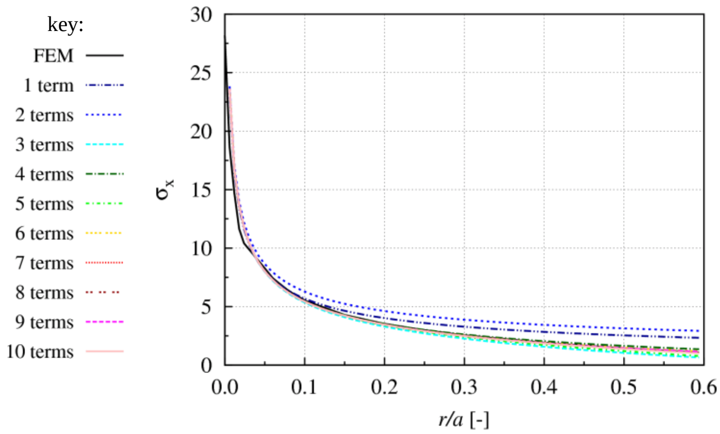

Figure 4: Comparison of values σ_x determined numerically and values calculated from Eq. (1) under consideration of different numbers of the higher-order terms in WPE; angle of the path investigated $\theta = 0°$

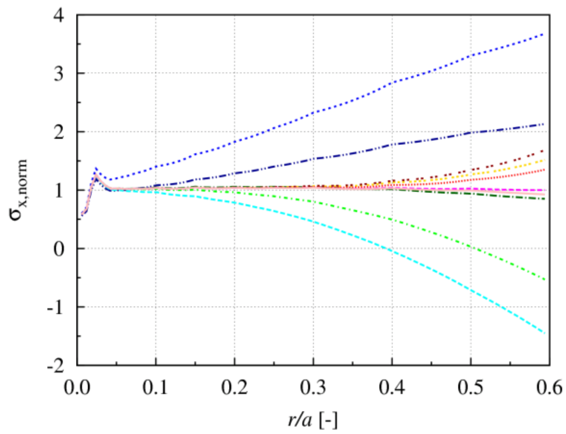

Figure 5: Normalized $\sigma_{x,\mathrm{norm}}$ values calculated from WPE for various ranges of the higher-order terms; angle of the path investigated $\theta = 45°$

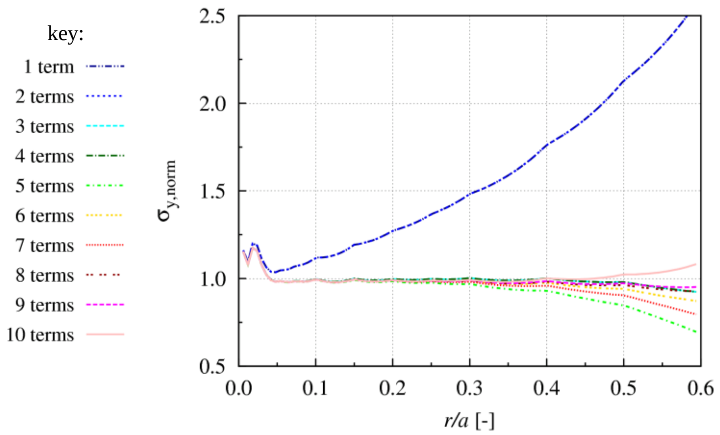

Figure 6: Ditto Fig. 5 for $\sigma_{y,\mathrm{norm}}$ values

The stress profile reconstructed by the WPE considerably fluctuates with changing number of terms of the series; however, the trend of improving accuracy with increasing number of terms is apparent. Both of the conclusions presented above can be better seen in Fig. 5 and Fig. 6, where the dependences of the normalized $\sigma_{x,\mathrm{norm}}$ and $\sigma_{y,\mathrm{norm}}$ components for the angle of 45° are displayed. The stress components values were normalized via the corresponding stress values obtained from the numerical solution: $\sigma_{ij,\mathrm{norm}} = \sigma_{ij}/\sigma_{ij,\mathrm{num}}$ and $i,j \in \{x,y\}$, *i.e.* in the ideal case $\sigma_{ij,\mathrm{norm}} = 1$, which means that the stress value calculated by WPE equals the stress value obtained numerically.

It can be seen in Fig. 5 and 6 that although adding of another higher-order term not always leads to better (more accurate) results, using only one or two terms of Williams expansion can cause considerably different stress values than there are in reality in the cracked specimen. If such stresses are subsequently used in a fracture criterion or any other analysis, the error can arise much more. Therefore, estimation of the coefficients of the higher-order terms of WPE is strongly recommended in order to avoid inaccuracies especially in larger distances from the crack tip.

4.2 WST

Similar analysis was conducted also in the case of the mode I geometry – the WST configuration. Here, the values of the higher-order terms coefficients were determined for wide range of relative crack lengths in order to fit the results by suitable functions and thus create formulas enabling analytical recon-struction of the stress/displacement fields in cracked specimens for various crack lengths. The higher-order terms coefficients were evaluated from results of computations with several models varying in the near-crack-tip FE mesh shape. The reason of that was to introduce variances in the selection of (the number and the position of) nodes from which the displacements and coordinates were taken to enter the ODM and thus investigate the accuracy of the method. Detailed description of the conducted study related to this issue can be found in [33]; here coefficients, a_n, of the first twelve terms of the WPE for selected three relative crack length values are presented (Tab. 2). Note that the values of coefficients correspond to the load magnitude mentioned above. Again, normalized values of the first five coefficients have been compared with published data [28, 29] and good agreement was observed [32].

In the case of the WST the ODM procedure programmed alternatively also in Mathcad mathematical package [42] was used.

Table 2: Dimensionless values of the higher-order terms coefficients, g_n, for WST geometry

n	g_n [-]		
	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.85$
1	2.105	4.309	2.896×10^{1}
2	-1.482×10^{-1}	4.670	3.674×10^{1}
3	−1.796	−6.126	-1.251×10^{2}
4	2.014	1.140×10^{-1}	3.796×10^{1}
5	−4.490	6.220×10^{-1}	-2.101×10^{2}
6	-4.725×10^{-1}	−1.616	1.547×10^{2}
7	1.797×10^{1}	-5.726×10^{-1}	-6.032×10^{2}
8	-3.678×10^{1}	2.142	5.692×10^{2}
9	4.048	−5.065	-1.897×10^{3}
10	2.177×10^{2}	1.009	1.990×10^{3}
11	-4.084×10^{2}	−3.960	-6.233×10^{3}
12	-6.369×10^{2}	1.353×10^{1}	6.743×10^{3}

The calculated coefficients of the higher order terms were subsequently used for approximation of the stress field in the cracked body by means of WPE, similarly to the previous example. However, here the entire stress fields (*i.e.* the functions of two variables), not only their profiles (*i.e.* particular cuts), are investigated.

A procedure in Java was developed for handling and displaying of the reconstructions of the fields over the area of the specimen. The procedure is a subroutine of an application being developed for estimation of the size and shape (and possibly other relevant features) of the fracture process zone evolving around the tip of a propagating crack [43−45].

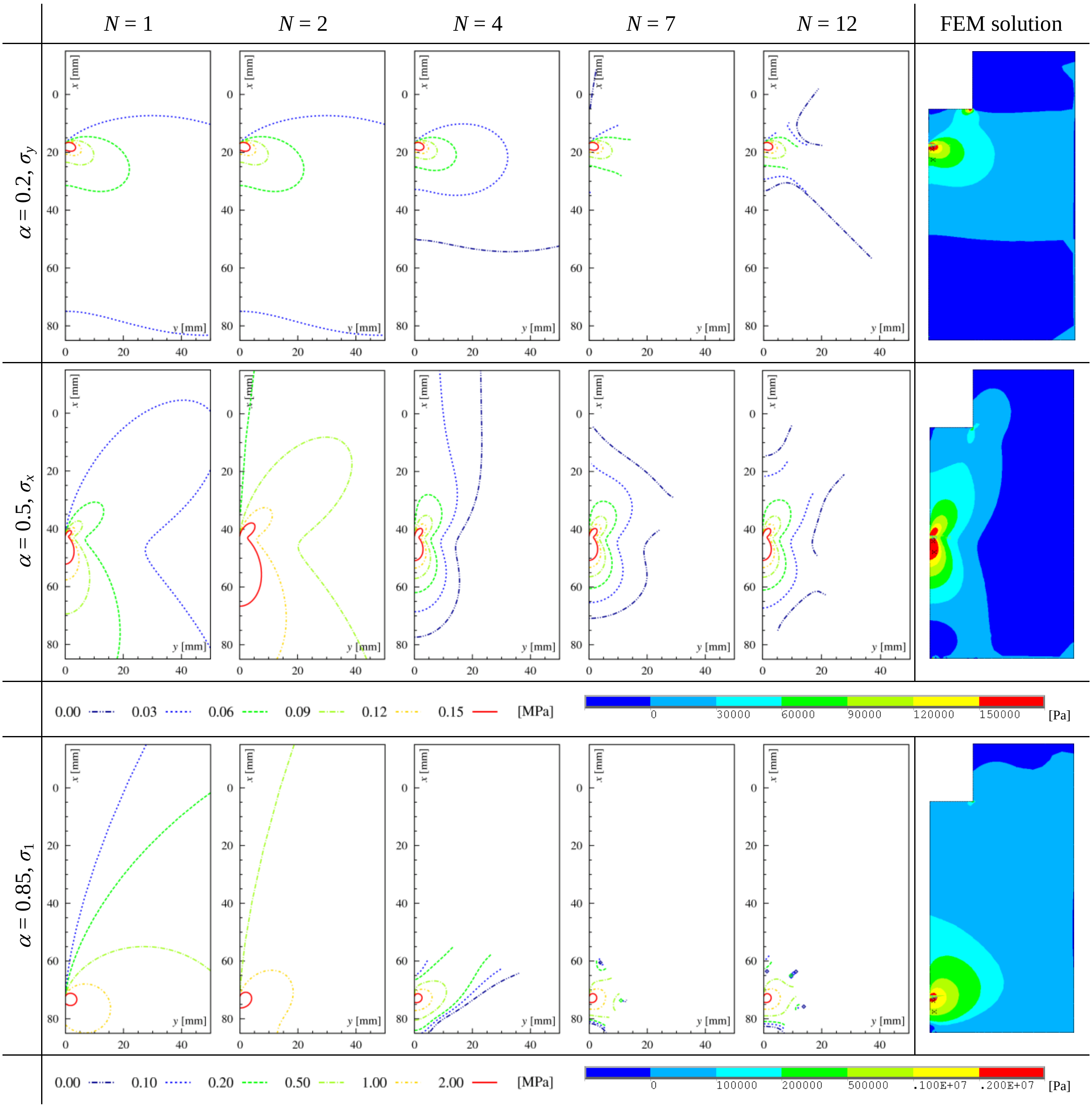

Figure 7: Comparison of stress fields (from top σ_y, σ_x, σ_1) reconstructed by means of WPE using various ranges of terms of the series (from left 1, 2, 4, 7, and 12, respectively) with the numerical solution (considered as the correct one)

In the present study, a tool of the application which enables determination of the plastic zone contours (or simply isolines at some defined levels of stress) is employed; the search for points on the plastic zone boundary is performed via Newton's method. A number of failure criterions are implemented within the application; however, here only the basic stress tensor components used for the plasticity condition are presented.

Selected results of the study on description of the stress fields using finite number of terms of the WPE, N, are shown in Fig. 7. For three relative crack lengths and three components of stress tensor, namely $\alpha = 0.2$ and σ_y, $\alpha = 0.5$ and σ_x, and $\alpha = 0.85$ and σ_1 (from top), respectively, the fields are analyticaly recon-structed via WPE using different ranges of its initial terms, namely $N = 1$, 2, 4, 7, and 12 (from left), respectively. These fields reconstructions are interpreted through isolines (the scale is identical for σ_x and σ_y, but differs for σ_1, see the scales on the left) and can be compared to FE solution ploted as isoareas (contours between different colours corres-pond to values for isolines of the analytical solution, see the scales on the right). Note that the FE solution is considered as exact and its results served as inputs to the ODM from which the values of the higher order terms were calculated.

In some cases, mostly for the higher values of the last considered term, N, certain regions (sectors of certain range of angle, θ) are observed where no solution of the plasticity condition was found within a distance from the crack tip, r, smaller than that of the specimen boundary. In these cases the stress isolines, particularly for larger distances from the crack tip, are not continuous. This effect is extremely visible in the case of long crack ($\alpha = 0.85$), high value of N and low value of σ_1, when only discrete points were found as the solution.

From the mutual comparison of the stress fields' reconstructions for different N and the comparison of those with the FE solution, following points are worth emphasizing:

- Description of the stress fields using WPE under consideration of only the first or the first two terms is feasible only in very small region around the crack tip.
- With increasing of the distance from the crack tip also the number of terms necessary for keeping reasonable accuracy of the description increases.
- In larger distances from the crack tip, the stress field reconstruction provided by high number of terms of WPE may start behave in rather uneven manner, which complicates its utilization within fracture analysis.
- Suitable number of terms of the WPE which should enter the fracture analysis depends on mutual relations of the specimen size and shape, applied load and strength limit. In other words, on the proportion of the size/shape of the region where the failure takes place to the specimen boundary. Of course, the type failure condition influences the choice as well.

5 CONCLUSIONS

It has been found out that higher-order terms of the Williams power expansion derived for the description of the stress/displacement field in a cracked specimen play a key role if a knowledge of accurate stress/displacement fields not only very close to the crack tip is required. Sufficient number of higher-order terms necessary for accurate stress and displacement field description within a body with a crack depends on the size of the region in question; the studies presented show that probably more than one or two terms, which are used conventionally as the well-known one- or two-parameter fracture mechanics, should be taken into account. For instance in the case of quasi-brittle materials, where the stress distribution has to be known also farther from the crack tip, the use of the higher-order terms of Williams expansion can contribute to more accurate and reliable fracture analysis and prediction of structural behaviour.

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