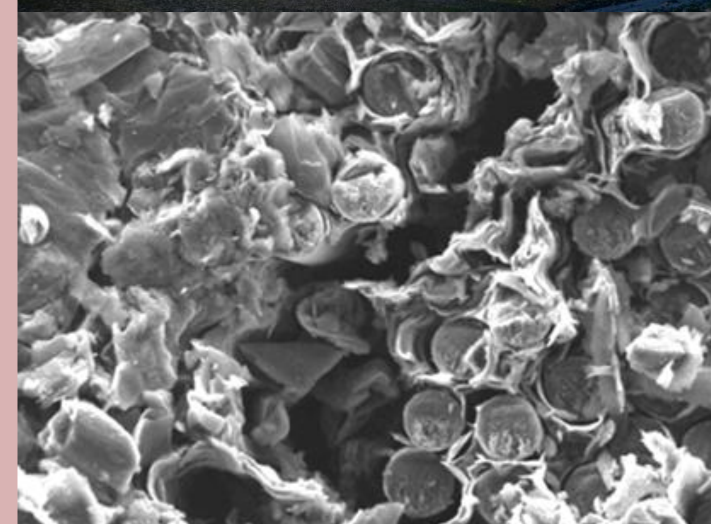
Pre-conference workshop in honor of Z.P. Bažant

Probabilistic Scaling of Fracture of Quasi-Brittle Materials

M. Salviato

William E. Boeing Department of Aeronautics and Astronautics



FraMCos-9, Berkeley (USA), May 29 2016

Motivation

$$P_f \leq 10^{-6}$$

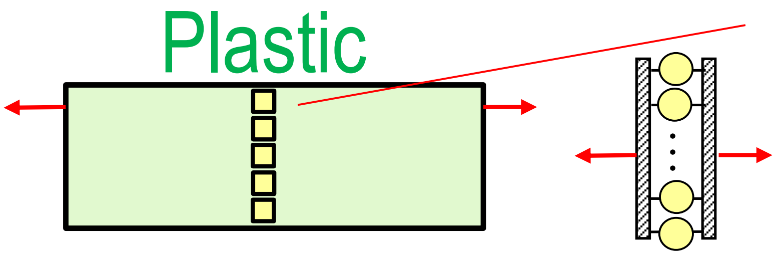
Max tolerable failure probability

Extreme value statistics problem:

To guarantee safety, the cumulative probability distribution (cdf) of structural strength must be known down to the low probability tail!!

Reliability of ductile structures

In ductile structures, failure occurs by extensive plastic deformation of the cross section



Failure load is the sum of the *random strengths* of several Representative Unit Elements (RVE)

Central limit theorem: **Gauss distribution**

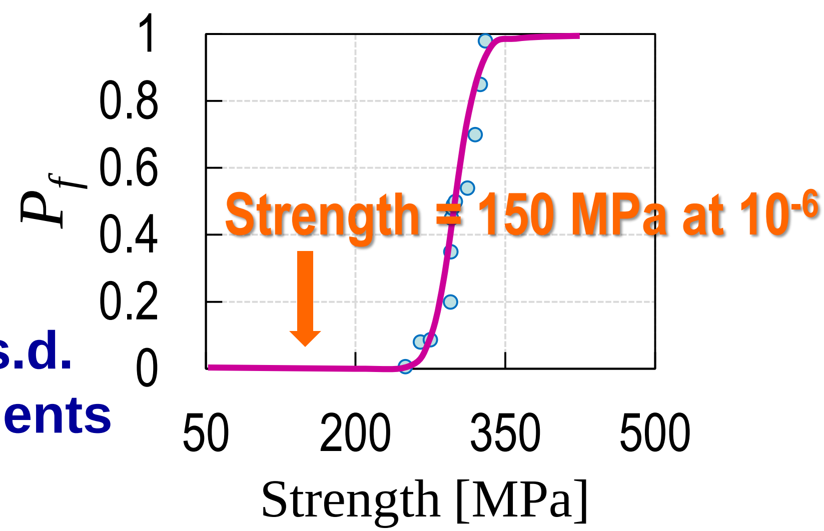
$$P_f(\sigma) = 1/\sqrt{2\pi}\int_0^\sigma \exp[-(t-\mu)^2/2\delta^2]dt$$

Mild steel

$$\sigma_N = c_n P/D^2$$

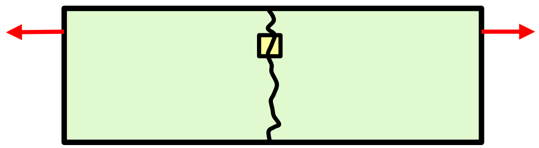
μ, δ

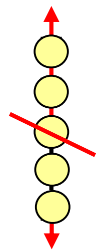
Mean and s.d. by experiments



In brittle structures, failure is highly localized: failure of one RVE = failure of the structure

Brittle



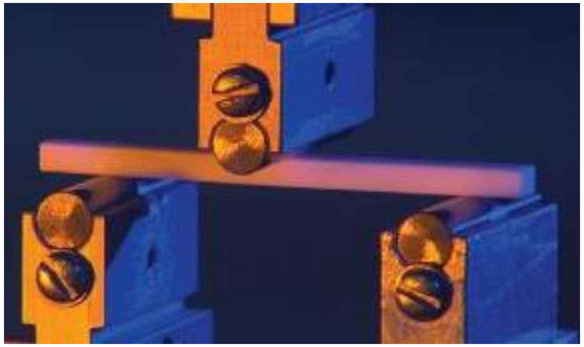


Failure load is reached when the weakest element of the structure reaches its random strength

Stability postulate (Fréchet): Weibull distribution

$$P_f(\sigma) = 1 - \exp[-\sigma^m/\sigma_0]$$

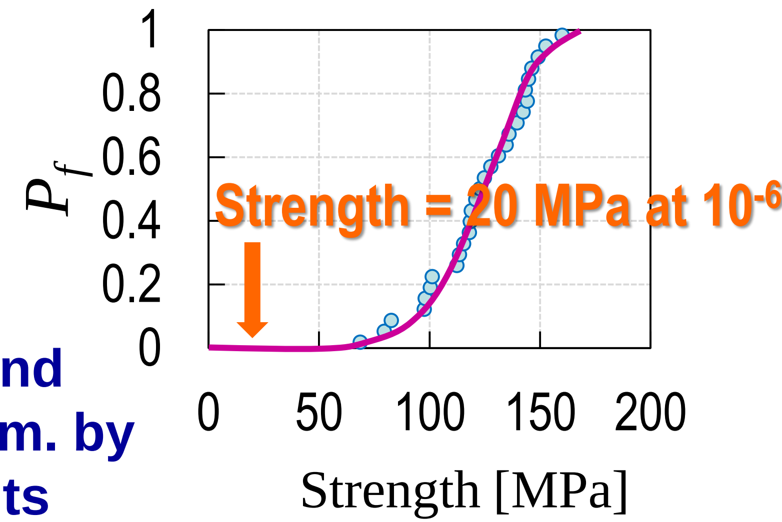
Float glass



$$\sigma_N = c_n P/D^2$$

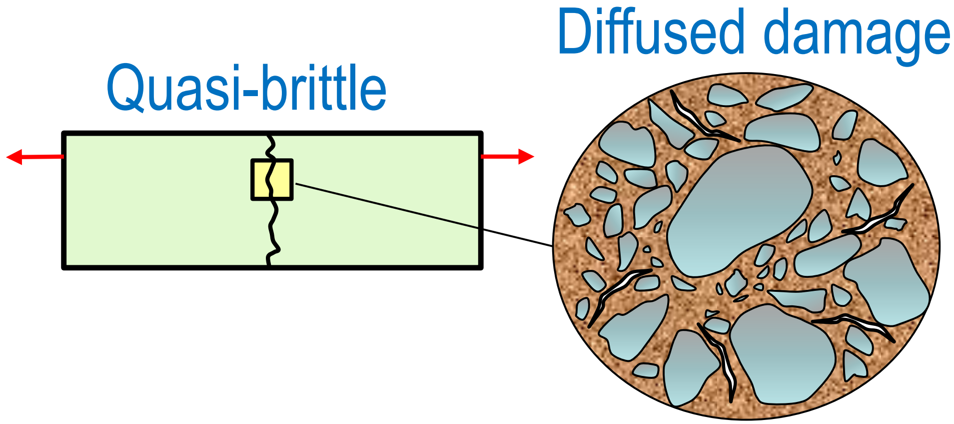
m, σ_0

Modulus and scale param. by experiments



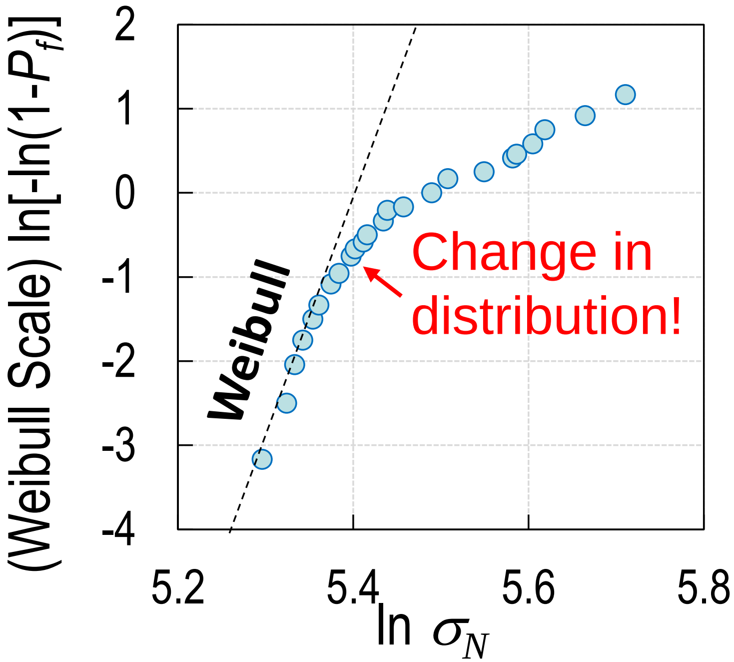
What about quasi-brittle structures?

The fracture behavior of quasi-brittle materials transitions from ductile to brittle increasing the structure size. **This does affect the statistics too!**



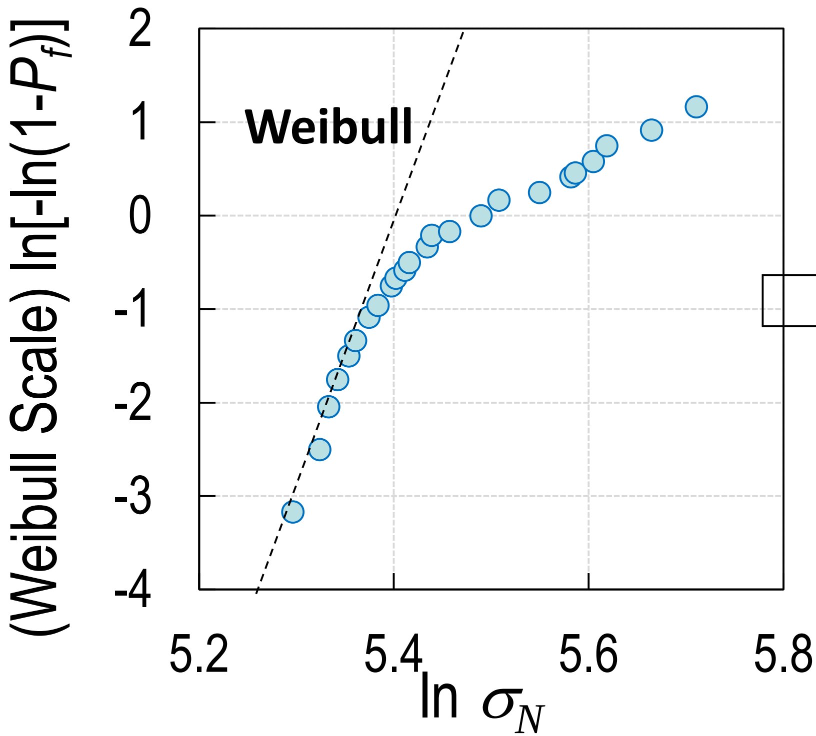
The extent of the Fracture Process Zone is not negligible compared to structure size

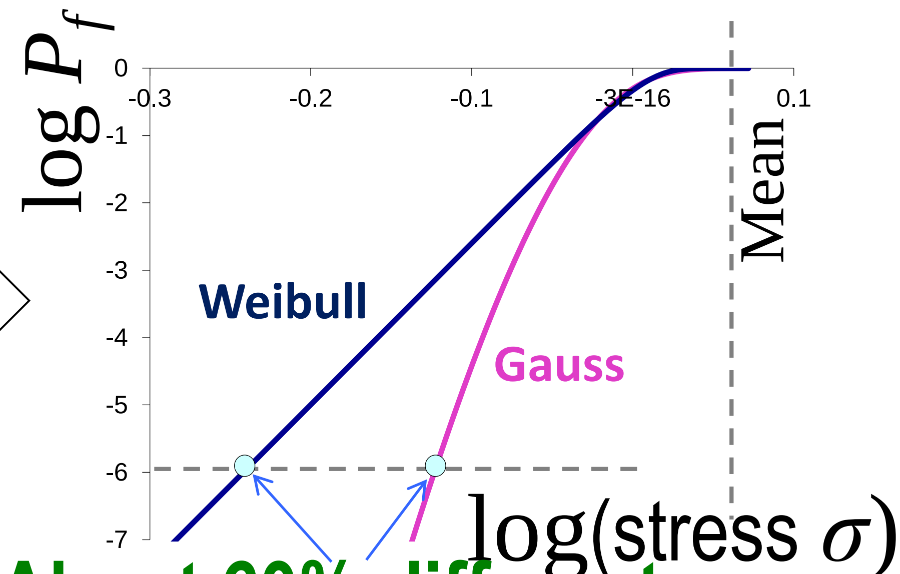
Concrete



Can we use known distributions anyway?

Different distributions have extremely different probability tails even if the mean and the CoV are the same!



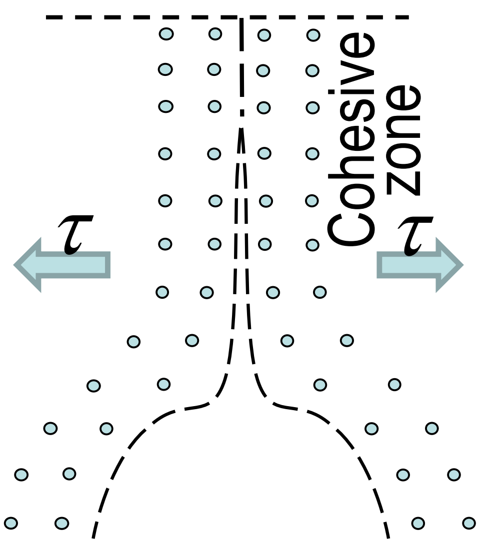


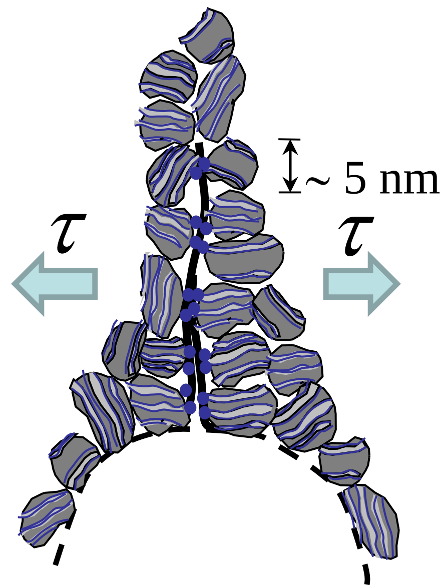
About 60% different

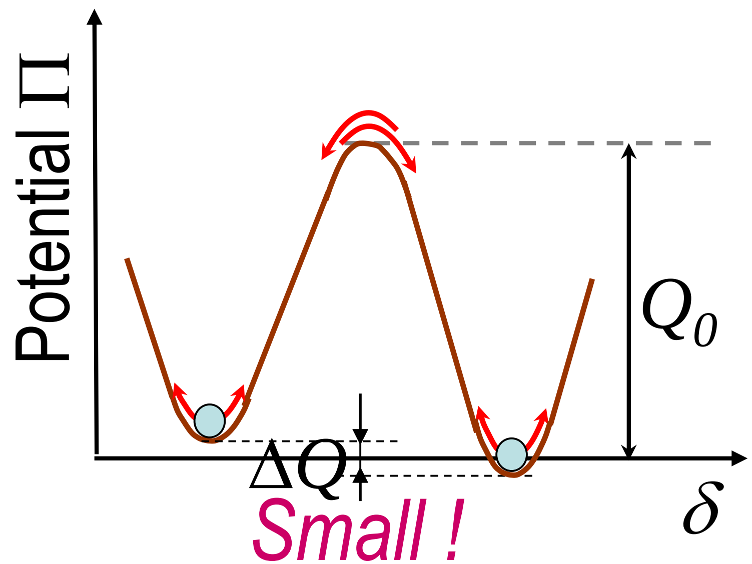
Errors can be higher than in FE simulations!!

A Multi-Scale Probabilistic Model for Quasi-Brittle Materials: I) Strength

Fracture Mechanics of a Nano-Structure







$$\Delta Q = \delta_a \left[\frac{\partial \Pi}{\partial a}\right]_\tau = \frac{V_a}{E_a}\tau^2$$

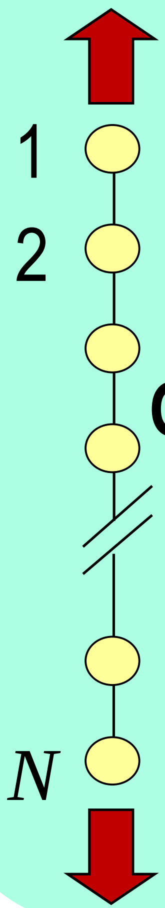
From <u>Transition-Rate Theory</u> (Kramer's formula), the **<u>frequency of crack jumps</u> is:**

$$f_b = v_a\left(e^{-(Q_0-\Delta Q/2)/kT} - e^{-(Q_0+\Delta Q/2)/kT}\right) \sim \tau^2$$

Probability P_f ~ Frequency f_b

How can we scale the CDF to the macroscale?

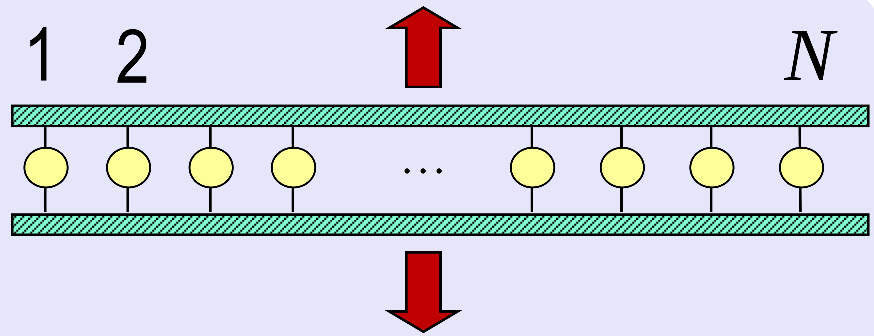
SERIES COUPLING



Once one element fails, the system fails!

Strain Localization

PARALLEL COUPLING



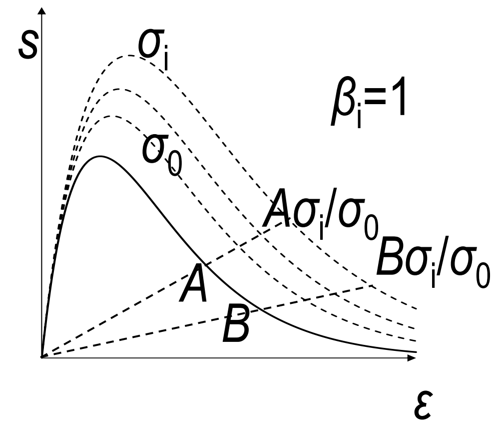
Once one element fails, the stresses get redistributed. The system fails when all the elements fail!!!

Strain Compatibility

Parallel coupling

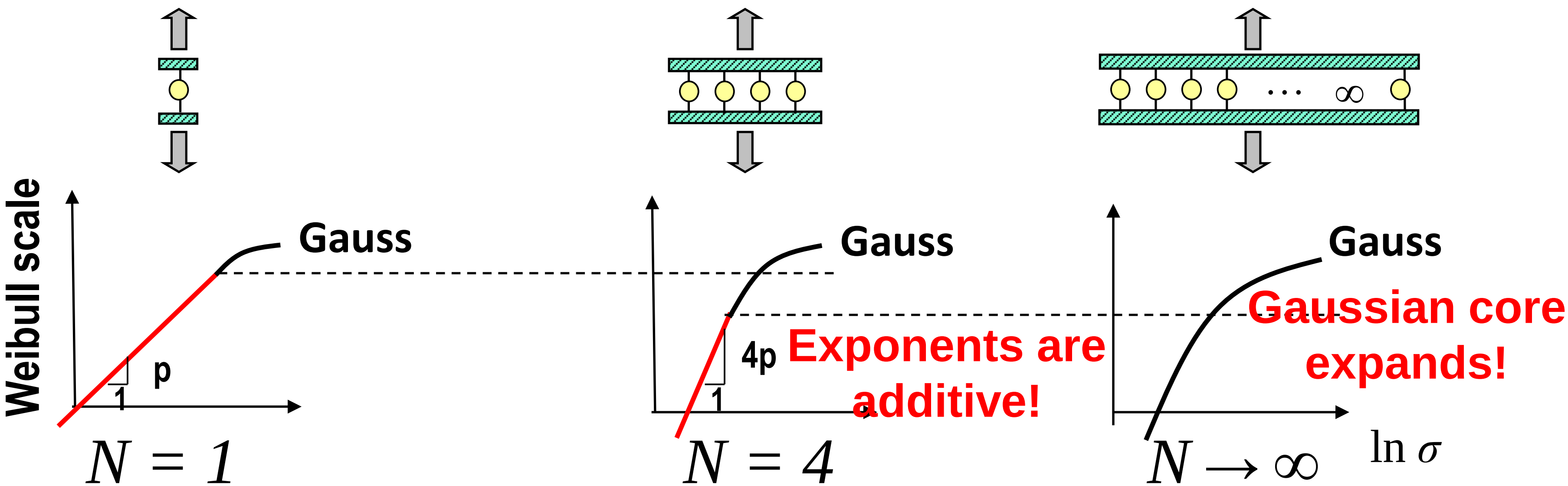
Assumptions:

- **Same deformation** for all the elements;
- **Radially affine** stress-strain constitutive law for each element.
- **Power law tail σ^p** for all the elements.



- In parallel couplings the tail exponents **are added**.
- Parallel coupling produces **Gaussian core**.
- Parallel coupling shortens the power law tail.

M. Salviato, ZP Bažant, Prob Eng Mech 2014



Series coupling

Assumption:

- **Power law tail σ^p** for all the elements.

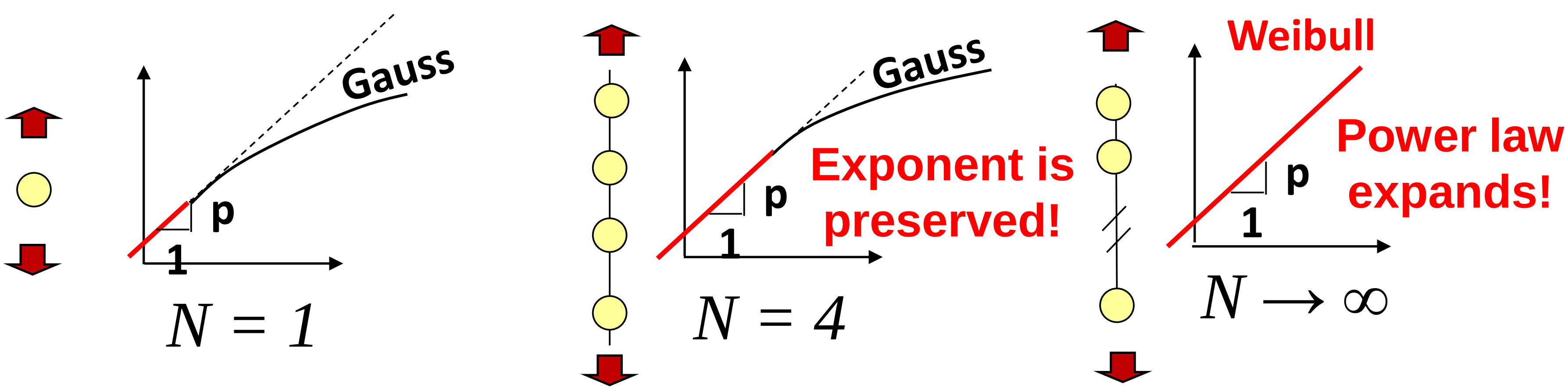
ZP Bažant, SD Pang (2007). J. of the Mechanics and Physics of Solids 55, 91-134.
J-L Le et al. (2011). J. of the Mechanics and Physics of Solids 59, 1291-1321.

- In series couplings, exponents **are preserved.**

$$P_f = 1 - (1 - P_{f1})^N \qquad P_{f1} = \text{ CDF of one element}$$

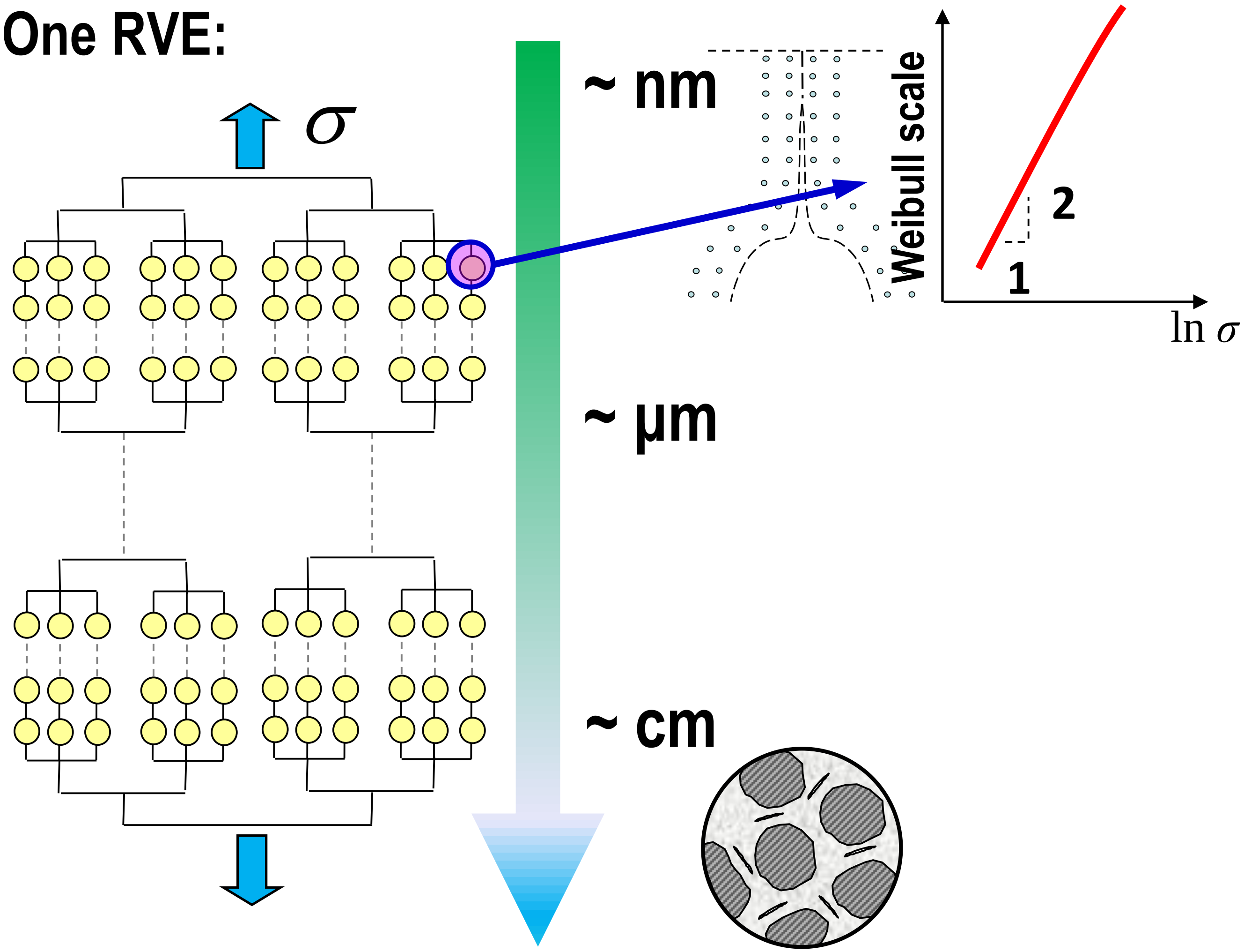
$$P_f \to 0 \quad \Longrightarrow \quad P_f = N P_{f1} = N\sigma^p$$

- From simulations, series coupling **extends the power law tail**.



Statistical Nano-Macro Transition of cdf Tail

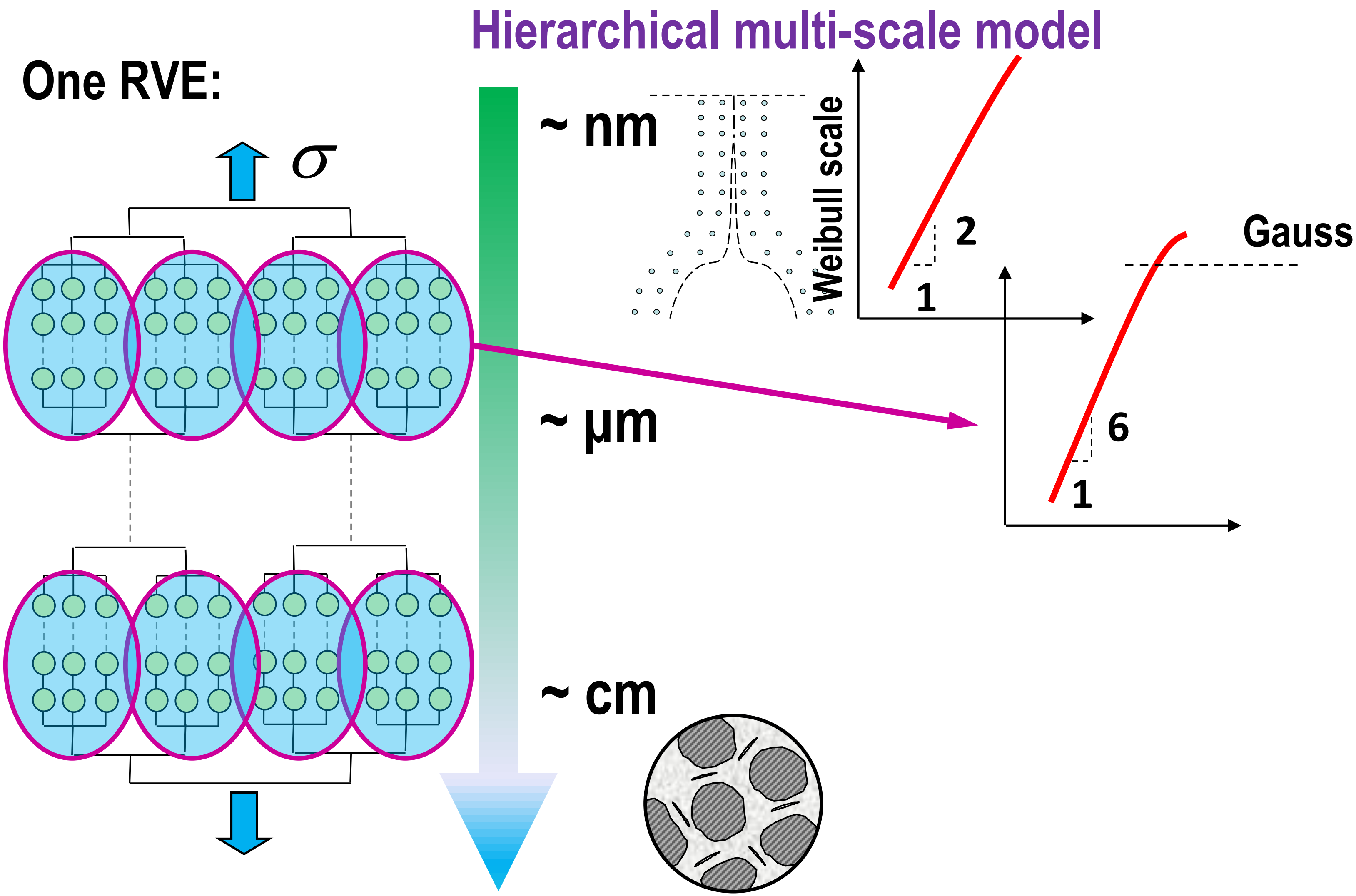
Hierarchical multi-scale model



ZP Bažant, SD Pang (2007). J. of the Mechanics and Physics of Solids 55, 91-134.

J-L Le et al. (2011). J. of the Mechanics and Physics of Solids 59, 1291-1321.

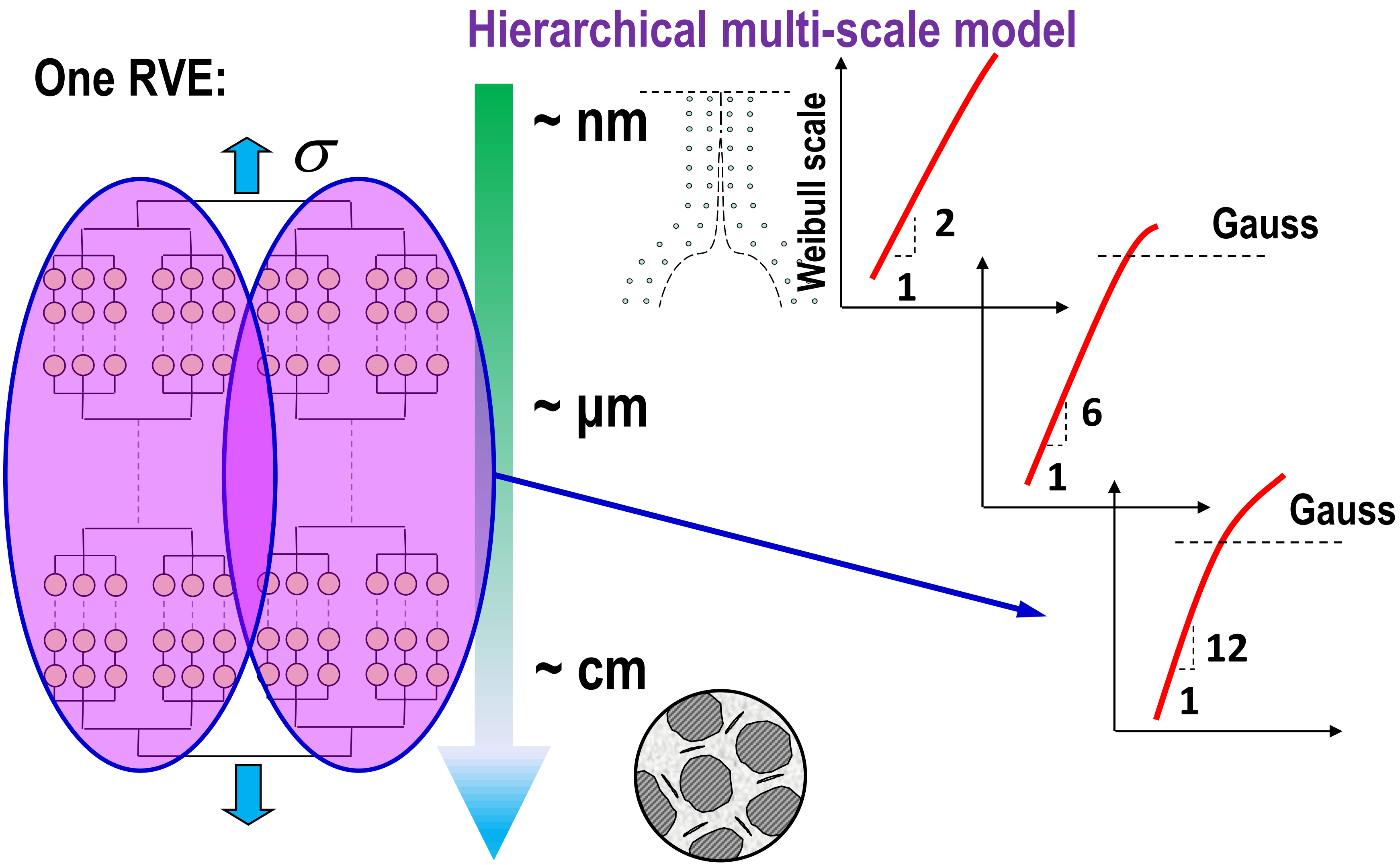
Statistical Nano-Macro Transition of cdf Tail



ZP Bažant, SD Pang (2007). J. of the Mechanics and Physics of Solids 55, 91-134.

J-L Le et al. (2011). J. of the Mechanics and Physics of Solids 59, 1291-1321.

Statistical Nano-Macro Transition of cdf Tail



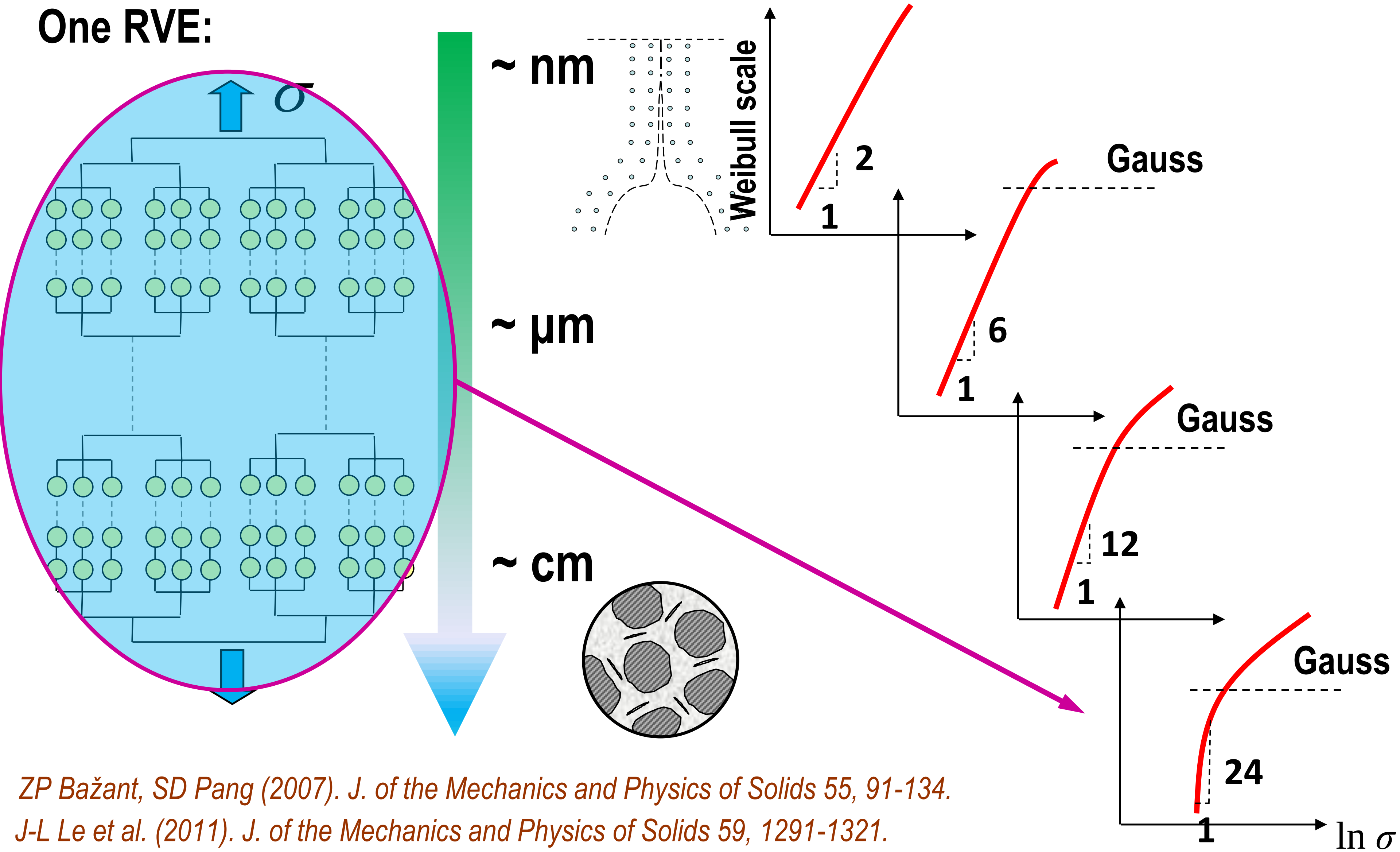
ZP Bažant, SD Pang (2007). J. of the Mechanics and Physics of Solids 55, 91-134.

J-L Le et al. (2011). J. of the Mechanics and Physics of Solids 59, 1291-1321.

Statistical Nano-Macro Transition of cdf Tail

Hierarchical multi-scale model

One RVE:



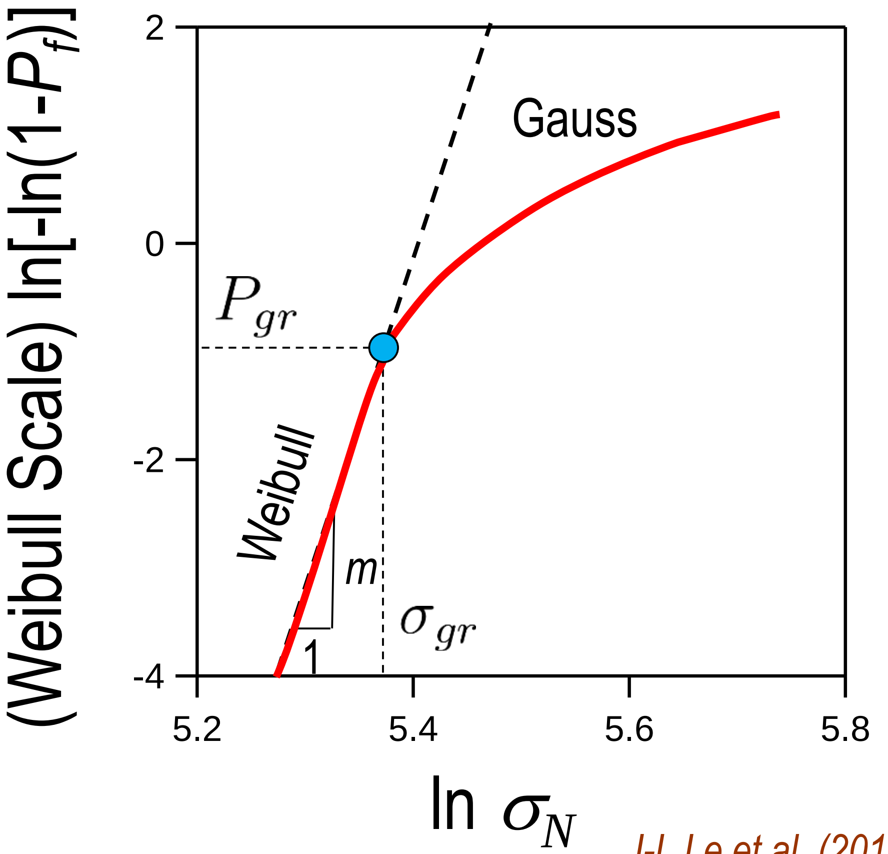
ZP Bažant, SD Pang (2007). J. of the Mechanics and Physics of Solids 55, 91-134.

J-L Le et al. (2011). J. of the Mechanics and Physics of Solids 59, 1291-1321.

Gauss-Weibull distribution of strength for one RVE

$$P_1(\sigma) = 1 - \exp(-\langle\sigma\rangle^m / b_0^m) \quad \text{for} \quad \sigma < \sigma_{gr}$$

$$P_1(\sigma) = P_{gr} + \frac{r_f}{\sqrt{2\pi}\delta_G} \int_{\sigma_{gr}}^{\sigma} e^{-(\sigma' - \mu_G)^2 / 2\delta_G^2} d\sigma' \quad \text{for} \quad \sigma \geq \sigma_{gr}$$



$P_{gr} =$ Grafting prob.

$\sigma_{gr} =$ Grafting strength

$m =$ Weibull modulus

$b_0 =$ Weibull scale parameter

$\mu_G =$ Mean of the Gauss distr.

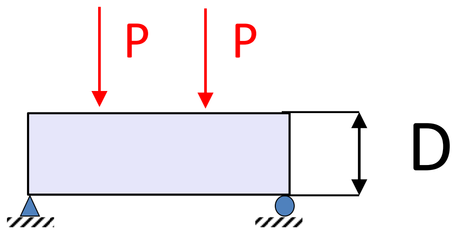
$\delta_G =$ Stand. dev of the Gauss distr.

Only 4 free parameters!!

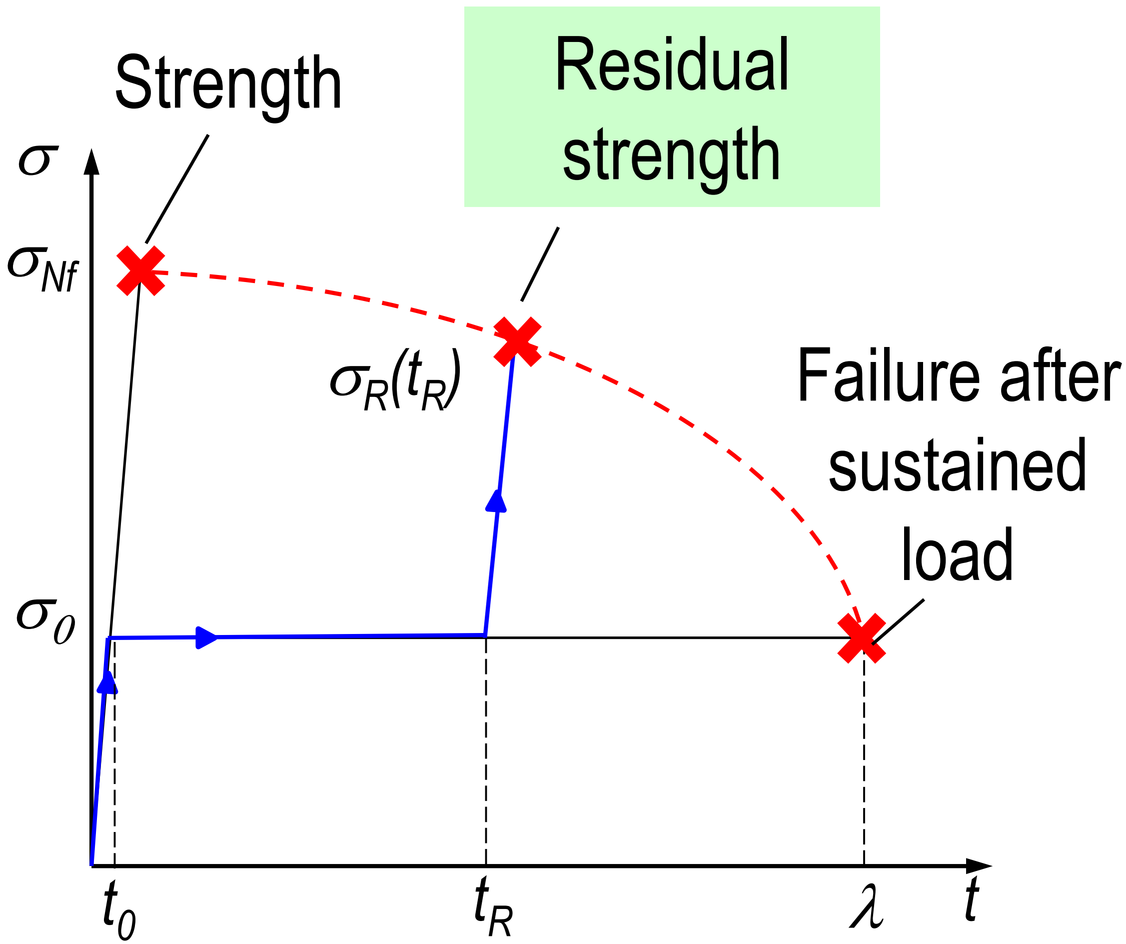
J-L Le et al. (2011). J. of the Mechanics and Physics of Solids 59, 1291-1321.

A Multi-Scale Probabilistic Model for Quasi-Brittle Materials: II) Residual Strength

Residual Strength of Structures

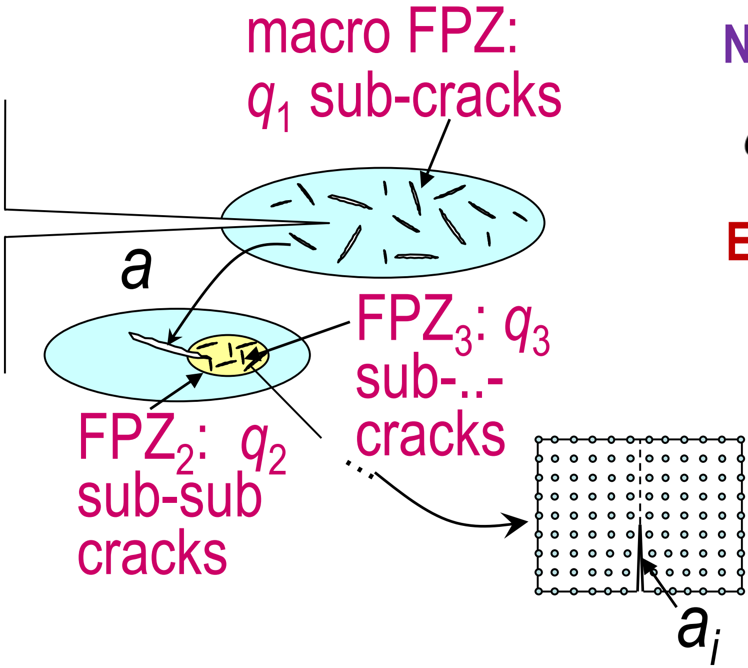


$$\sigma_N = c_n P / D^2$$



σ_{Nf}	Structural strength
σ_0	Applied constant nominal stress
λ	Lifetime
t_0	Time to reach σ_0
t_R	Time of load application
σ_R	Residual strength

Sub-critical Crack Propagation Law



Nano-crack propagation law:

$$\dot{a}_{nano} = \delta_a f_1 = v_1 \exp[Q_0/T] K_a^2$$

Equality of energy dissipation rates through scales:

$$(G\dot{a})_j = (\sum_{i=1}^{N_a} G_i \dot{a}_i)_{(j-1)}$$

.
.
.

RVE scale

Nano-macro transition based on equality of nano- and macro-dissipation:

Evans Law:

$$\dot{a} = A\, e^{-Q_0/kT} {K_1}^n$$

A = material constant

Q_0 = activation energy

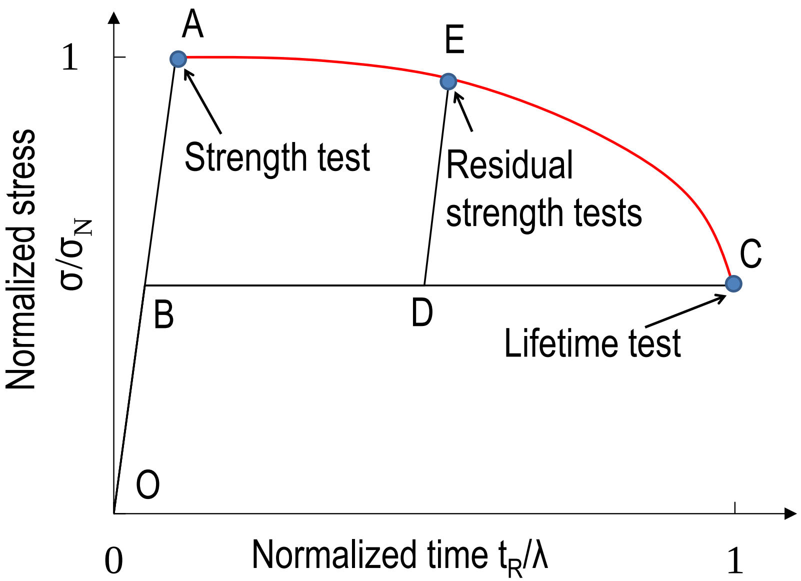
T = absolute temperature

k = Boltzmann constant

K_1 = Stress Intensity Factor for one RVE

n = Sub-critical crack grow exponent

Degradation Law of Strength for One RVE



Introducing $K_1 = \sigma\sqrt{l_0}k_1(\alpha)$ and integrating Evans law for history O-B-D-E:

Residual Strength for one RVE:

$$\sigma_R = \left[{\sigma_N}^{n+1} - \sigma_0^n(n+1)(rt_R - \sigma_0)\right]^{\frac{1}{n+1}}$$

σ_N = strength

t_R = time of load application

λ = lifetime

σ_0 = applied constant load

r = stress rate

Static Lifetime for one RVE:

$$\sigma_R \rightarrow \sigma_0 \qquad \lambda = \frac{\sigma_N^{n+1}}{r(n+1)\sigma_0^n} + \frac{n\sigma_0}{r(n+1)}$$

M. Salviato et al., JMPS 2014.

CDF of Residual Strength

$$P_{1R} = \mathrm{Prob}(\sigma_R \leq \sigma) = \mathrm{Prob}\{\sigma_{Nf} \leq [\sigma^{n+1} + f(\sigma_0, t_R)]^{1/n+1}\}$$

Composed function of the strength cdf

$$= \Phi_{G-W}\{[\sigma^{n+1} + f(\sigma_0, t_R)]^{1/n+1}\}$$

For one RVE, the **cdf** of residual strength results:

$$P_{1R}(\sigma_R) = 1 - \exp\{-[\langle \sigma_R^{n+1} + f(\sigma_0, t_R) \rangle / S)]^{m/n+1}\} \qquad (\sigma_R \leq \sigma_{R,gr})$$

$$P_{1R}(\sigma_R) = P_{gr} + \frac{r_f}{\sqrt{2\pi}\delta_G} \int_{[\sigma_{R,gr}^{n+1} + f(\sigma_0, t_R)]^{1/n+1}}^{[\sigma_R^{n+1} + f(\sigma_0, t_R)]^{1/n+1}} e^{-(x-\mu)^2/2\delta^2} \mathrm{d}x \quad (\sigma_R > \sigma_{R,gr})$$

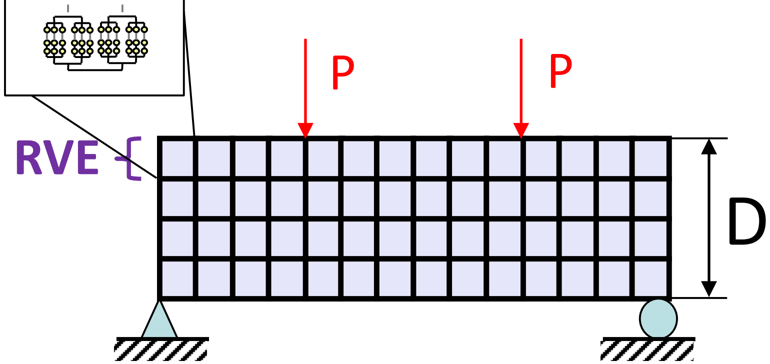
$$f(\sigma_0, t_0) = \sigma_0^n (n+1)(r t_R - \sigma_0)$$

Note: ***The failures before the overload create a threshold.***

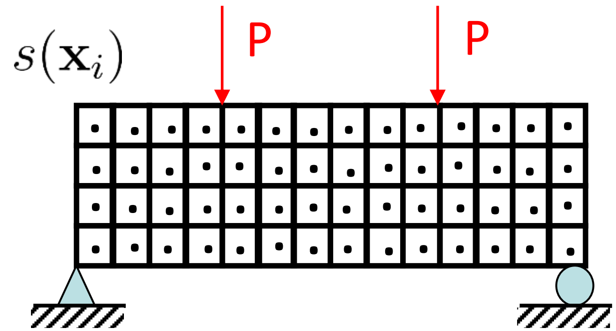
Scaling from RVE to large structures

CDF of Residual Strength for Structures of any Size

1. Discretization

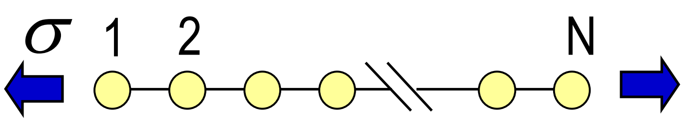


2. Stress analysis



3. Finite Weakest Link Theorem:

σ = first principal stress



Equivalent chain of *N* RVEs

$$P_f(\sigma_N) = 1 - \prod_{i=1}^{N} \{1 - P_{1,R}[\langle \sigma_N s(\mathbf{x}_i) \rangle]\}$$

$P_f =$ Structural cumulative probability of failure

$P_1 =$ Cumulative probability of failure of one RVE

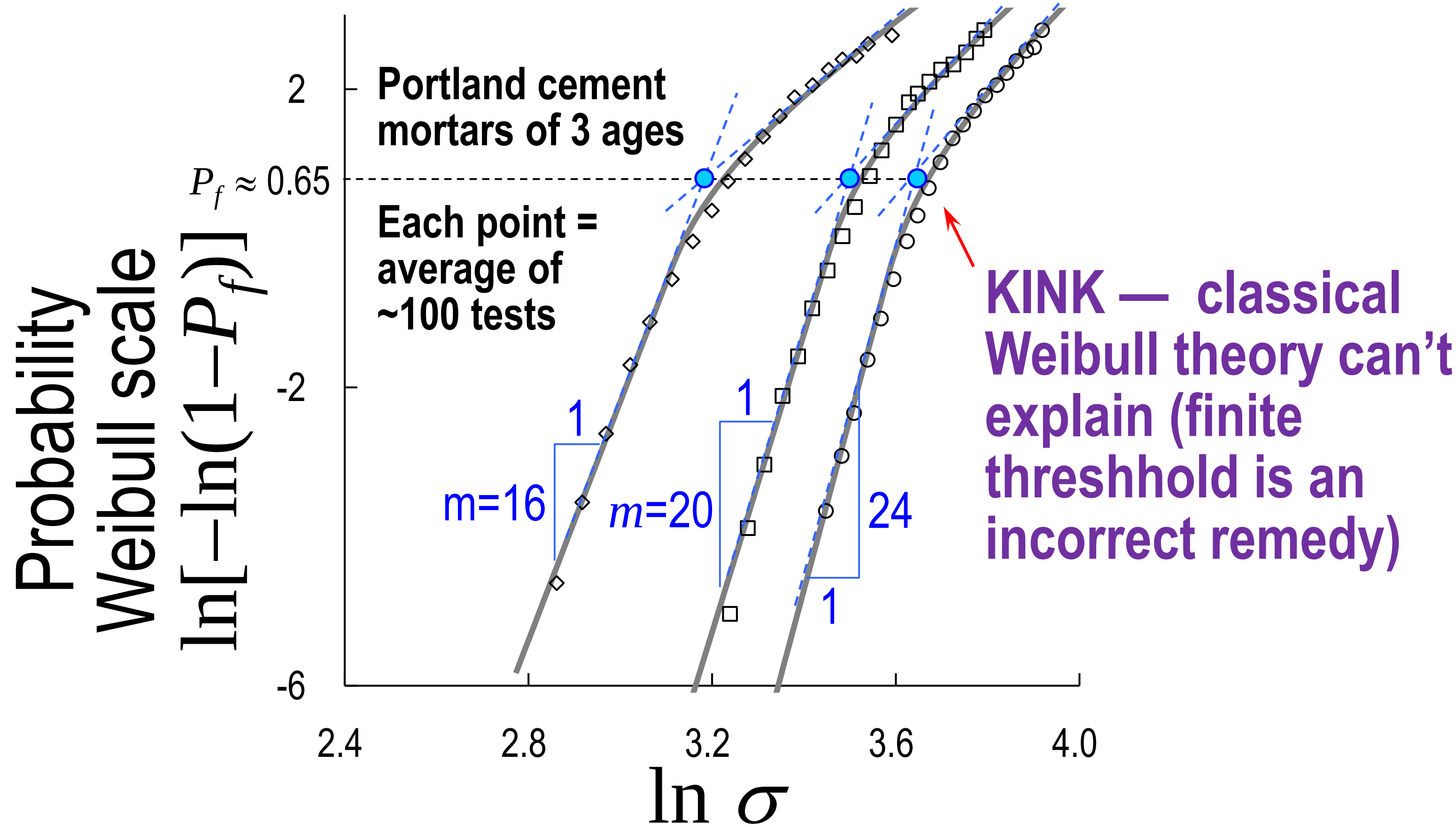
$\sigma_N = c_n P / D^2 =$ Nominal applied load

$s(\mathbf{x}_i) =$ Dimensionless stress field of the *i*-th RVE

Calibration and verification against experimental data: Strength

Experimental validation

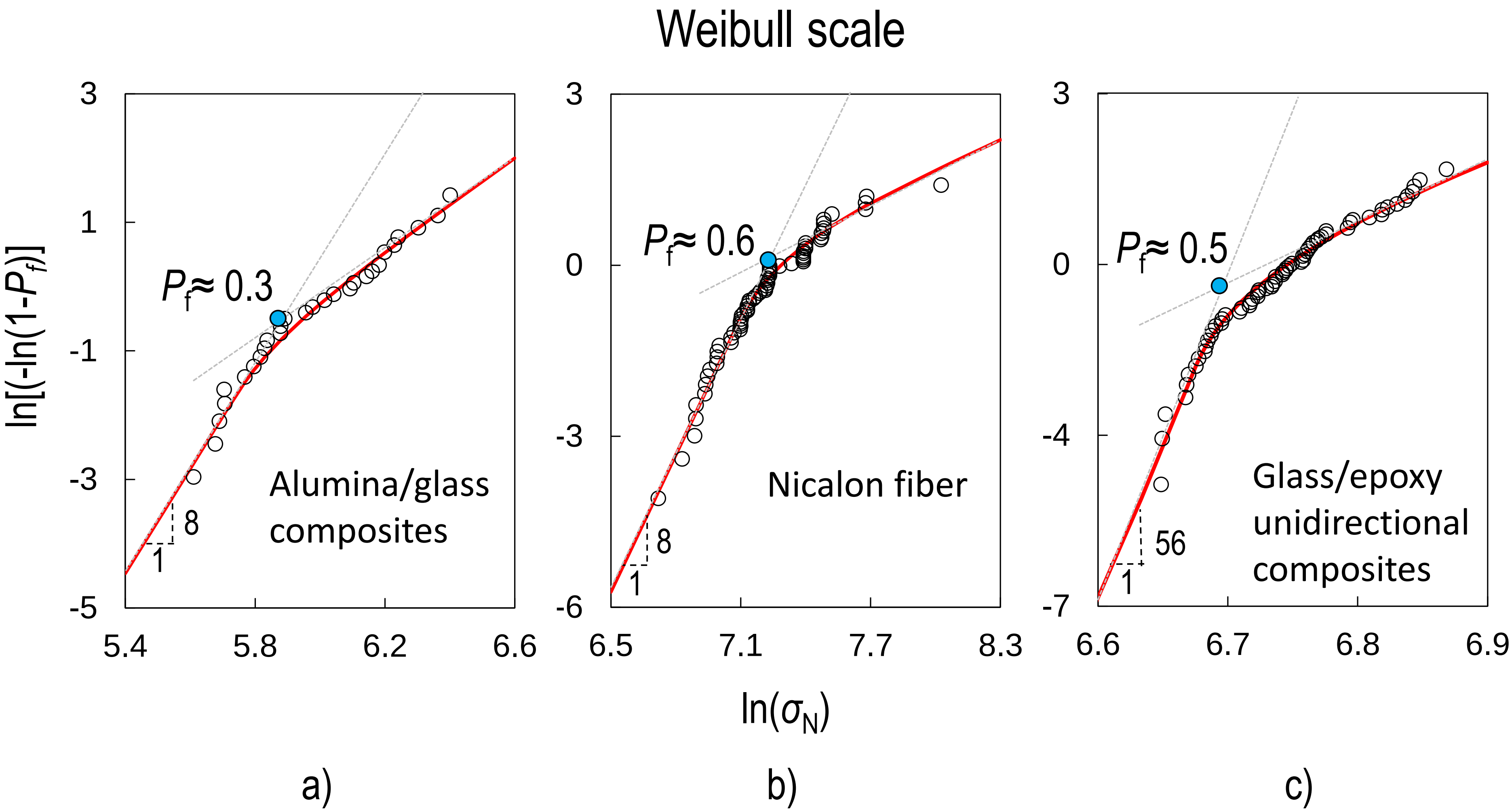
VALIDATION AND CALIBRATION:
Optimum fit of Weibull's (1939) tests by a finite chain with a zero threshold



RVE size ~ 0.6-1.0 cm; Specimen vol. ~ 100-3000 cm^3

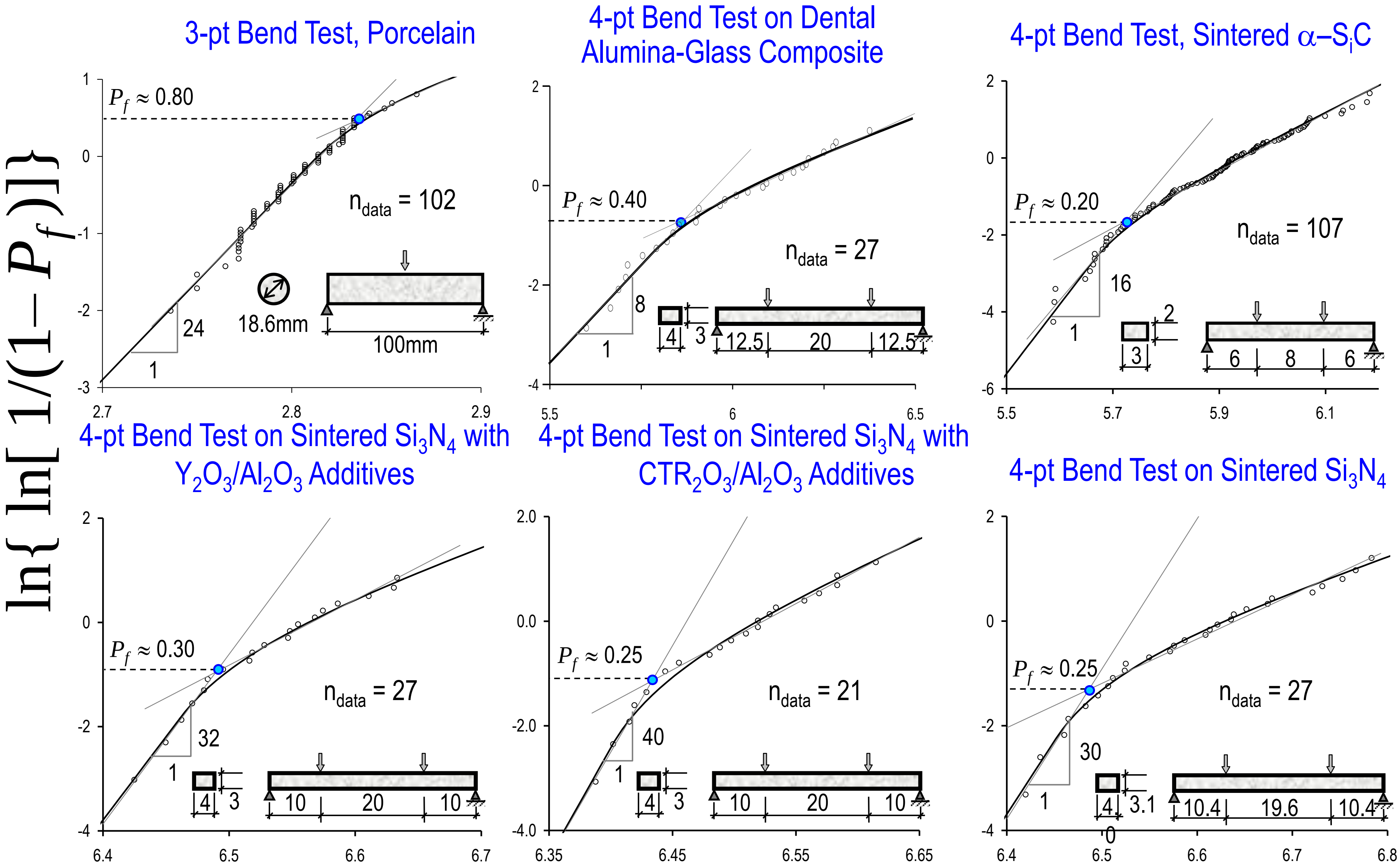
$\Rightarrow \; P_{gr} \approx \mathbf{0.001}$

Comparison with experimental data



M. Salviato, Z.P. Bažant, Prob Eng Mech 2014

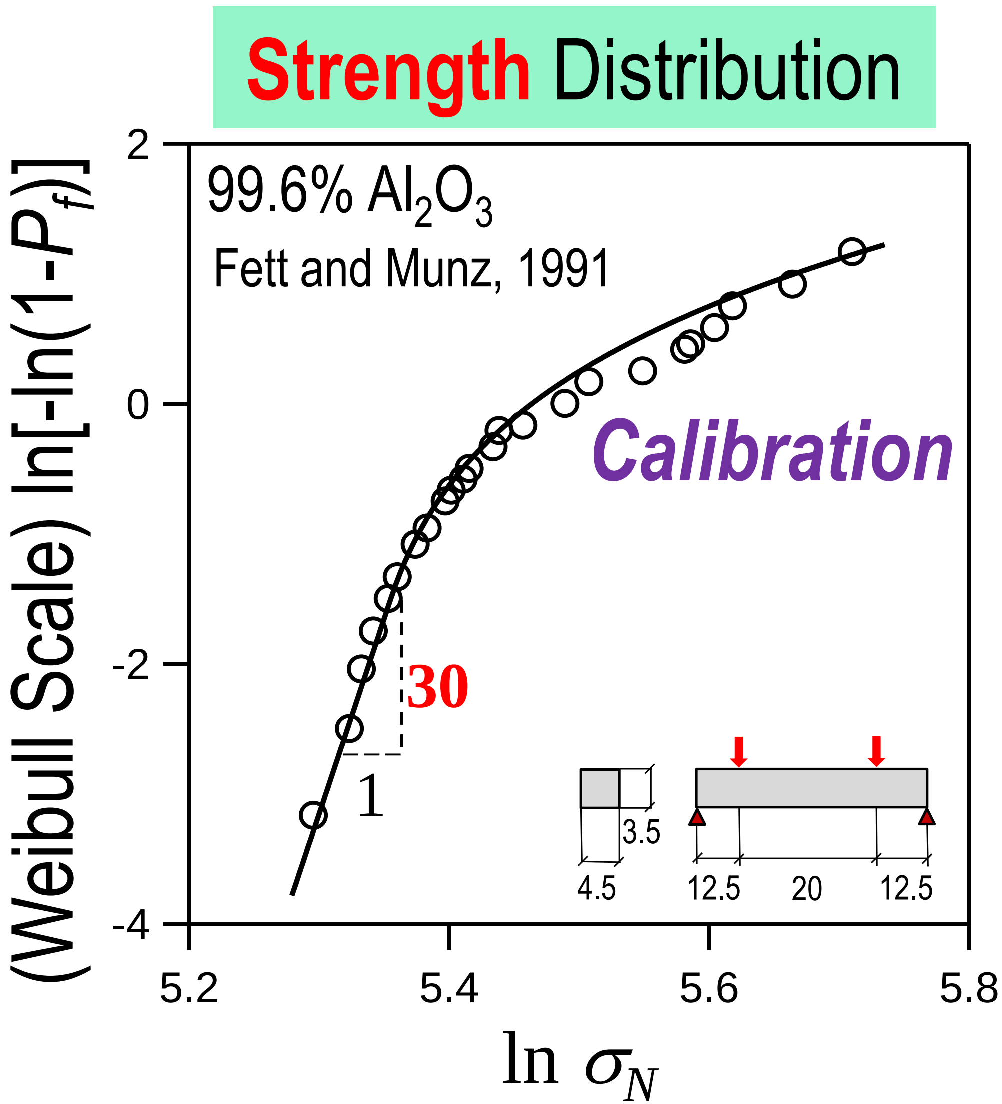
Histograms of Ceramics—Finite Chain, Zero Threshold

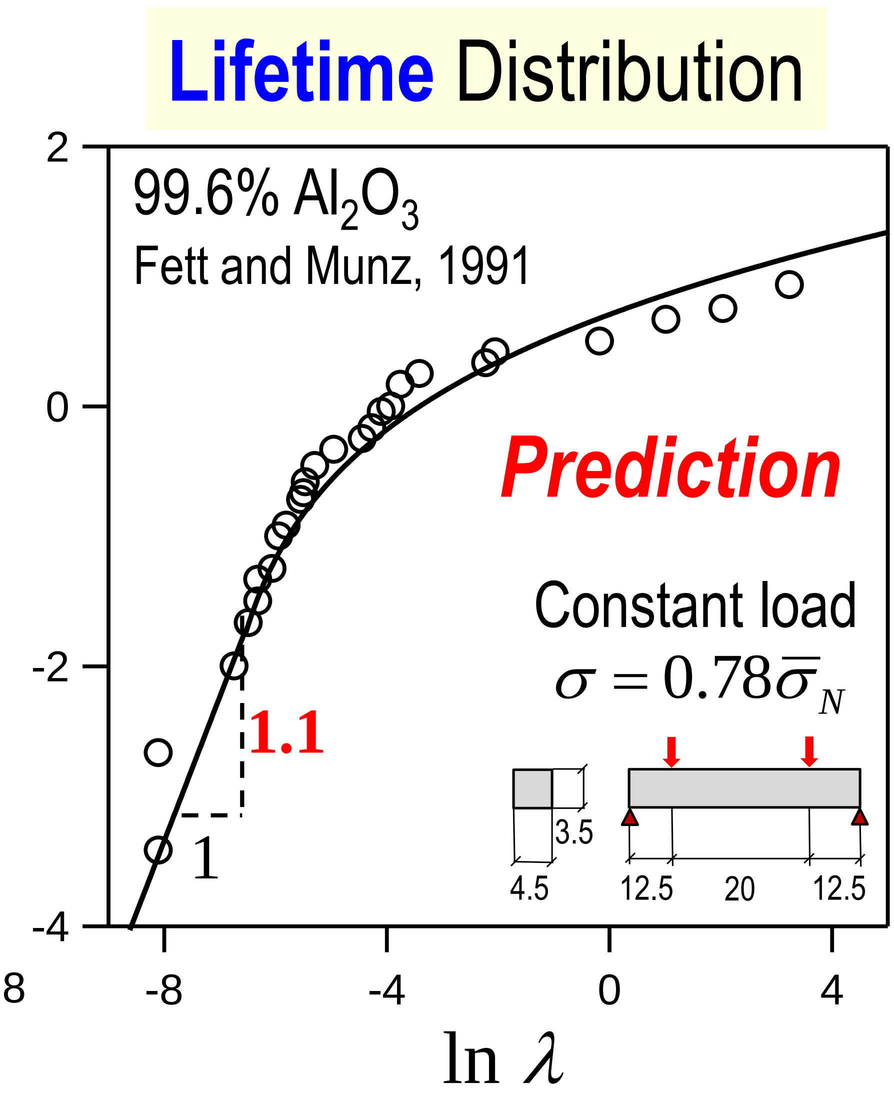


Le and Bažant, (2009), Le et al. (2011). JMPS

Calibration and verification against experimental data: Residual Strength

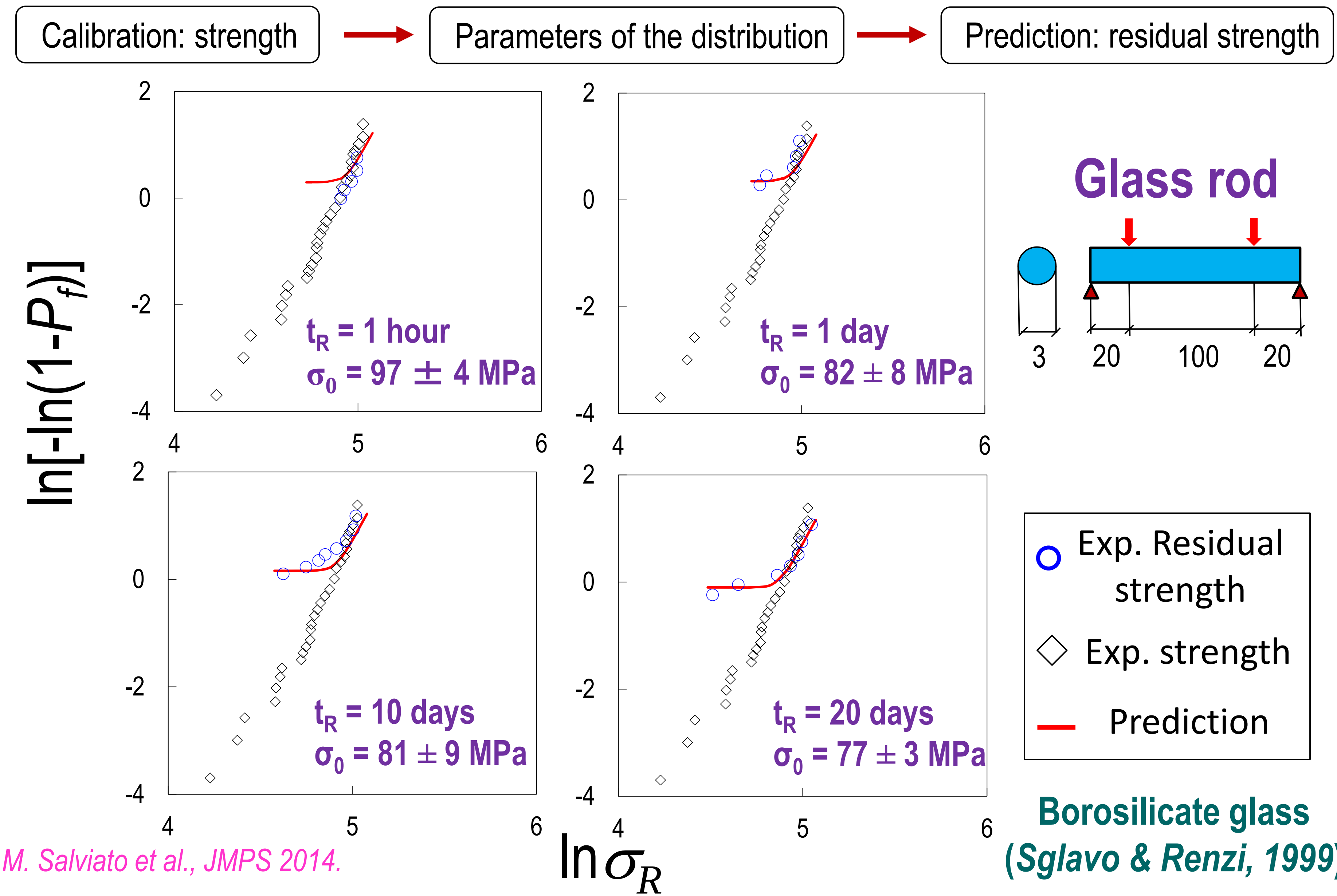
Analysis of Predicted Lifetime and Residual Strength Distributions (1)





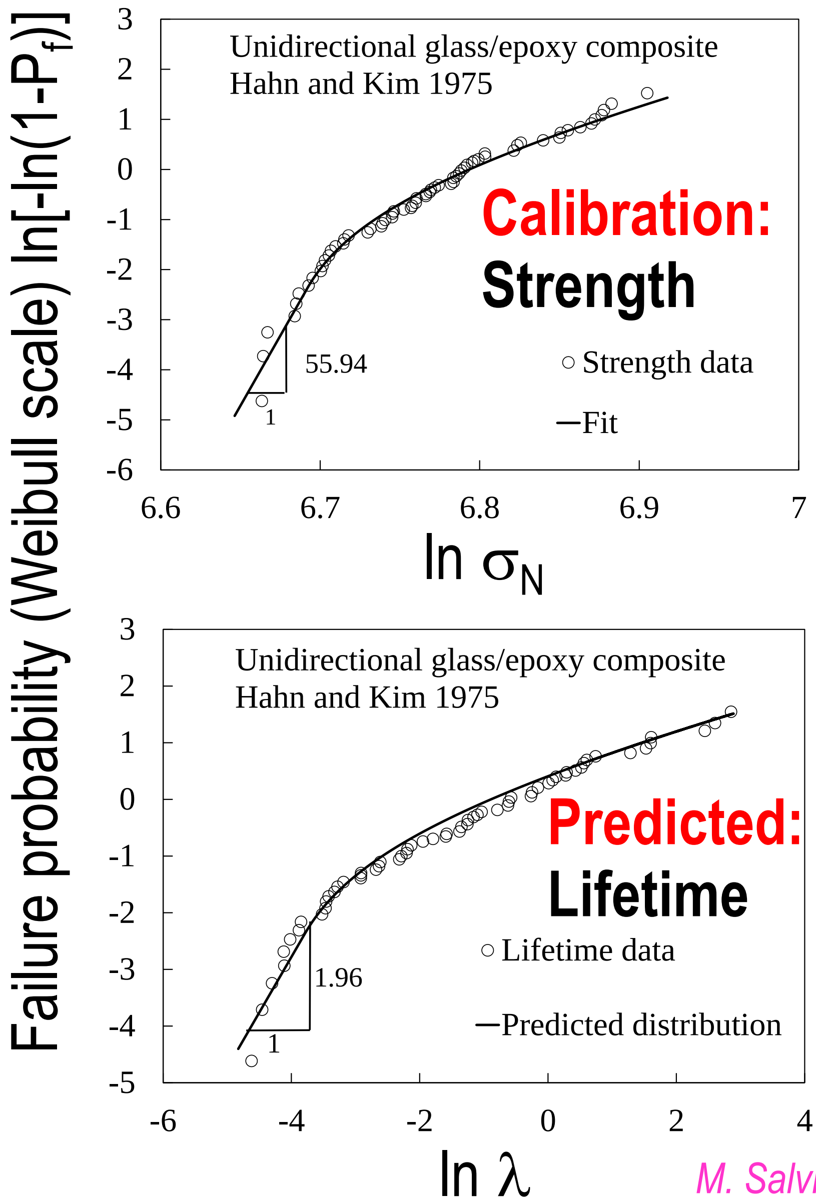
M. Salviato et al., JMPS 2014.

Optimum Fits of Residual Strength Histograms of Borosilicate glass

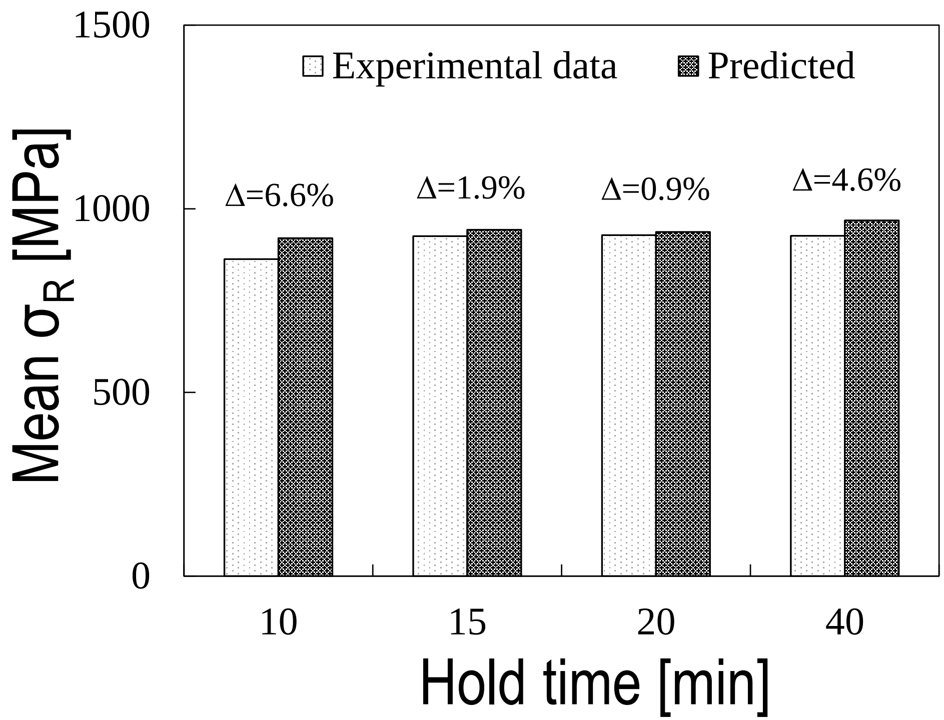


Borosilicate glass
(*Sglavo & Renzi, 1999*)

M. Salviato et al., JMPS 2014.



Predicted: Mean Residual Strength

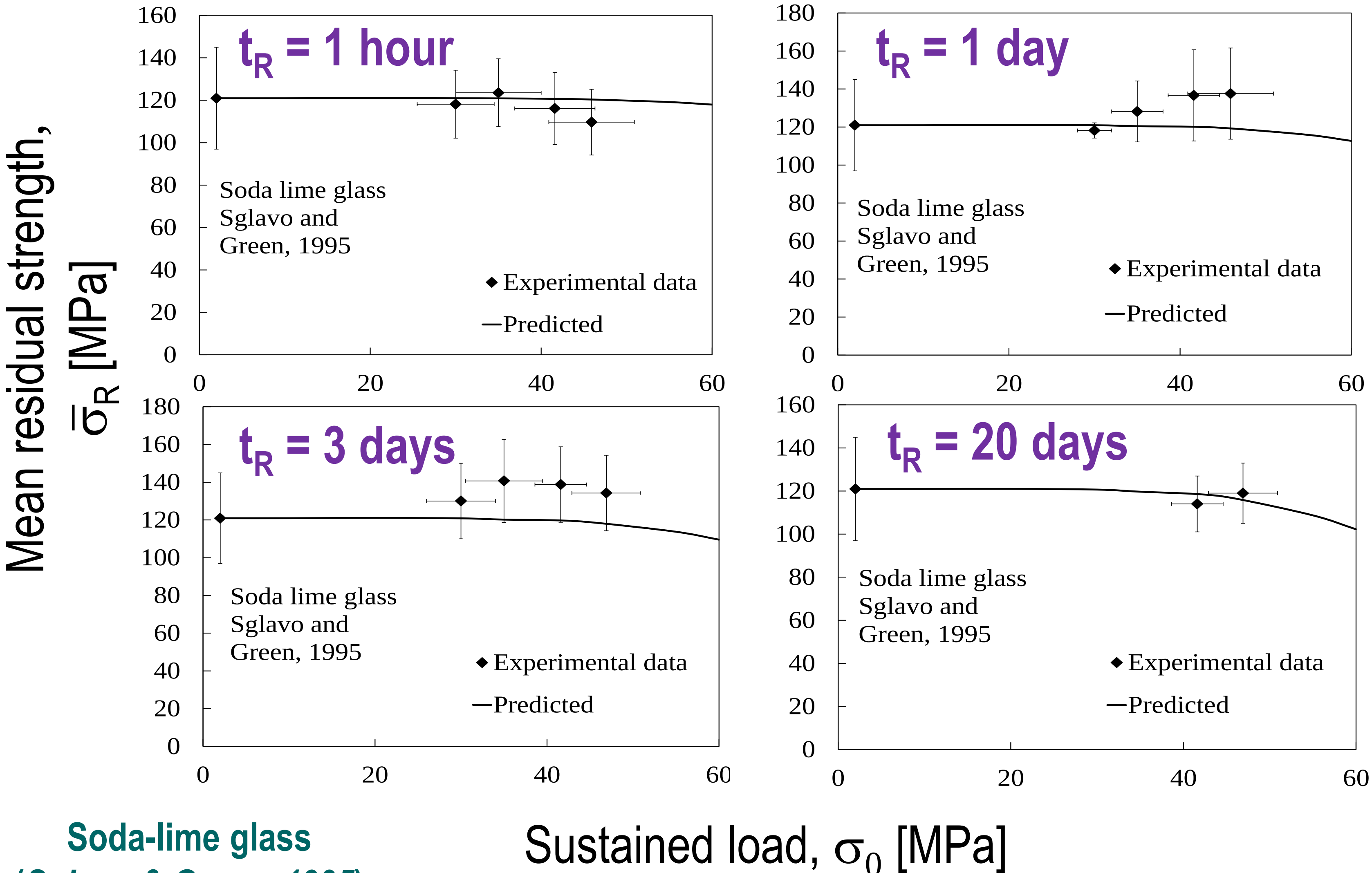


M. Salviato et al., JMPS 2014.

(Unidirectional glass/epoxy composites *Hahn & Kim, 1975*)

Statistical Nano-Macro Transition of cdf Tail

Calibration: strength → Parameters of the distribution → Prediction: mean residual strength



Soda-lime glass
(*Sglavo & Green, 1995*)

Conclusions

- Safe design is a problem of extreme value statistics. The knowledge of the cdf of failure down to the law probability tail is fundamental;
- For quasi-brittle structure, the cdf is different from the well-known distributions for brittle and ductile materials. **Errors in the evaluation of the tails can be higher than errors in FE simulations**.
- From an analysis of atomic bond ruptures at the nanoscale and then the application of a hierarchical multi-scale model, it is demonstrated that the cdf of strength is a graft of a Gauss and a Weibull distributions;
- The grafting point changes with the size and geometry of the structure. Increasing the structure size the Weibullian part extends, i.e. brittle behavior;
- From the cdf of strength, the cdf of residual strength and static lifetime can be easily derived and used for design;
- Good agreement with the existing test data on a wide range of quasi-brittle materials is demonstrated.

Thank you!