Size effect : from Irwin to Bazant and Mandelbrot

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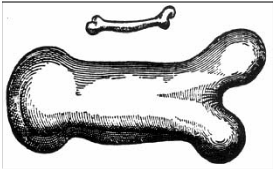
Motivation

This is about a long journey, started in 1988, searching for the broad interaction between Size Effect, theory, experimental evidence, Fractal Theory, experiment, and what does it all mean.

Still searching.

Galilei, G. (1638). *Dialogues Concerning Two New Sciences*.
Dover Publications (1954), New York, NY.
Originally published by Elzevir, The Neterlands, 1638



In the *Dialogues concerning two new sciences*, Proposition IX (on the second day), *Salviati* states:

> *...I once drew the shape of a bone, lengthened three times, and then thickened in such proportion . . . it would be necessary to either find much harder and more resistant material to form his bones*

- Galileo lacked the algebraic notation to do dimensional analysis.
- Cast in the current context:*the weight a bone carries is proportional to the animal volume (L^3) whereas the strength of this same bone is proportional to its cross section area L^2.*
- For a superficial observer (or a modern day *Simplicio*), Galileo would be the father of scaling (or dimensional) theory, and not of size effect theory.

- Paraphrase Galileo we would simply say that weight increases with L^3 and load carrying capacity (constant strength, variable cross-section) increases with L^2.
- This is exactly what the Size Effect law is all about: energy is released from a volume $\propto L^3$ and absorbed by a surface crack $\propto L^2$. As is well known by now, there are many experimental evidences supporting both Galileo and Bažant, [RILEM TC QFS, 2004].
- To that far-reaching statement, the incredulous *Simplicio* states:

> *But the immense bulks that we encounter among fishes give me grave reason to doubt whether this is so. From what I hear, a whale is as large as ten elephants; yet whales hold together.*

This questioning by *Simplicio* is not different than the incredulity of many skeptical modern time engineers, regarding the Size Effect law. But again, *Salviati* wisely replies:

> *Your doubt, Simplicio, enables me to deduce something that I did not mention before, a condition capable of making giants and other vast animals hold together and move around as well as smaller ones. That would follow if, but not only if, strength was added to the bones and other parts whose function is to sustain their own weight and that which rests on them.*

- Now, Galileo is implying that nominal strength (and not anymore mere load carrying capacity) can increase at times and that strength does not necessarily diminishes with size.
- In modern days terminology, strength does not necessarily drop to zero.

Bažant, Z. (1984). Size effect in blunt fracture: Concrete, rock, metal. *J. of Engineering Mechanics, ASCE*, 110(4):518–535

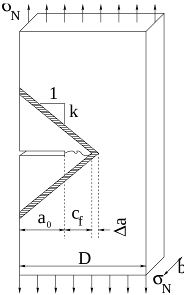
- Considering the energy exchanged during an infinitesimal crack extension in a plate of width D,

$$\underbrace{b2k(a_0+c_f)\Delta a\sigma_n^2/2E}_{\text{Released}} = \underbrace{bG_F\Delta a}_{\text{Absorbed}}$$

$$\sigma_n = \frac{Bf'_t}{\sqrt{1+\frac{D}{D_0}}}$$

$$Bf'_t = \sqrt{\frac{G_FE}{kc_f}}$$

$$\frac{D}{D_0} = \frac{a_0}{c_f} = \beta$$

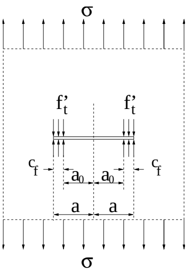


- Bazant noted that the analytical or numerical derivation of B and β is too difficult, and they are best obtained through statistical regression analysis of test data.
- Semi-analytical derivation which does not explicitly reference a plasticity and/or a linear elastic fracture mechanics solution. Yet, those two solutions are ultimately asymptotes to the derived size effect law.

Saouma, V., Natekar, D., and Hansen, E. (2003). Cohesive stresses and size effects in elasto-plastic and quasi-brittle materials. *International Journal of Fracture*, 119:287–298

- Infinite plate with a crack of length $2a$, at the tip of which we have a uniform cohesive compressive stress (Dugdale type) equal to the tensile strength f'_t
- Stress intensity factors (Cherepanov) due to the far field and cohesive stresses:

$$\begin{aligned} K_a &= \sigma\sqrt{\pi a} \\ K_b &= f'_t\sqrt{\pi a}\left(1-\frac{2}{\pi}\arcsin\frac{a-c_f}{a}\right) \end{aligned}$$



- Equating those two stress intensity factors

$$\sigma_n = f'_t\left[1-\frac{2}{\pi}\arcsin\left(1-\frac{c_f}{a}\right)\right]$$

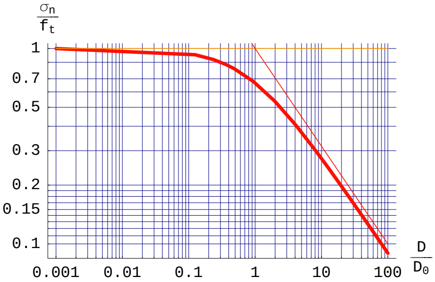
- for $a \simeq c_f$, $\sigma_n \nearrow f'_t$. For $c_f \simeq 0$, $\sigma_n \searrow 0$ as in the SEL but mathematically different.

- Taking a series expansion of the ArcSin function, and $c_f/a \to s$:

$$\sigma_n = \sigma_n = f'_t\left[1 - \frac{2}{\pi}\arcsin(1-s)\right]$$
$$\simeq \frac{2\sqrt{2}f'_t}{\pi}s^{1/2} + \frac{f'_t}{3\sqrt{2}\pi}s^{3/2} + \frac{3f'_t}{40\sqrt{2}\pi}s^{5/2} + O[s]^{7/2}$$

- Neglecting the terms of power greater than 1 (since s is at most equal to 1), and substituting $s = 1/(1+r)$ where $r = a_0/c_f$, we obtain

$$\sigma_n = \underbrace{\frac{2\sqrt{2}}{\pi}}_{B} f'_t \sqrt{\frac{1}{1+\underbrace{r}_{\beta}}}$$



- Bazant SEL recovered with the additional benefit that B is quantified

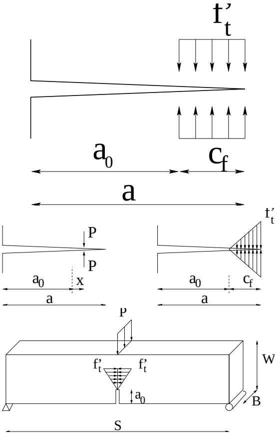
- Edge Crack: $\sigma_n = \underbrace{0.805}_{B} f_t' \sqrt{\frac{1}{1+\underbrace{r}_{\beta}}}$

- Linear cohesive Edge Crack

$$\sigma_n = \frac{0.5351 f_t'}{\sqrt{1+r}}\left(1 + \frac{0.126182}{1+r}\right) + O[s]^2$$

- Three point Bend, Linear Cohesive

$$\sigma_n = \frac{1.06738 f_t'}{\sqrt{1+r}}\left(1 + \frac{0.124401}{1+r}\right) + O[s]^2$$



- Bf'_t reported in the literature, [Bažant and Planas, 1998] assume $f'_t = 0.1f'_c$, determine B

Series	f'_c MPa	Bf'_t MPa	B	D_0 mm	Reference
A5	46.8	2.9	0.62	212.	[Walsh, 1972]
A2	35.4	2.8	0.79	157.	[Walsh, 1972]
A4	15.6	1.7	1.09	126.	[Walsh, 1972]
B1	34.1	6.0	1.76	60.	[Bažant and Pfeiffer, 1987]
A6	32.7	4.1	1.25	55.	[Walsh, 1972]
A1	23.1	4.5	1.95	36.	[Walsh, 1972]
A3	14.3	3.2	2.24	34.	[Walsh, 1972]

- Average $\mu_B = 1.39$, $\sigma = 0.61$.
- Previously derived value was 1.07.
- Note that experimentally determined B value is clearly inversely proportional to the nominal specimen size D_0.

Important Observation

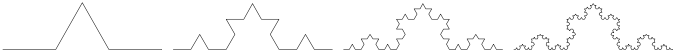
- The presence of a process zone (cohesive stresses) was a *sine qua non* condition to have a size effect exhibited.
- By extension, any material exhibiting plastic zone (i.e. metals) should also exhibit a size effect.
- This was subsequently confirmed by [Cervera and Chiumenti, 2009]

Fractals

Mandelbrot, B. (1983). *The Fractal Geometry of Nature.*

W.H. Freeman, San Francisco

- In the context of size effect, fractals may play an important role.
- Initiator of Euclidian dimension E and linear size L which can be divided into n equal smaller replicates of linear size rL. $n(rL)^E = L^E$.
- Fractal dimension D is then defined by $D = \frac{\log N}{\log \frac{1}{r}}$
- Triadic von-Koch Curve (Example of a a Self Similar Invasive Fractal)

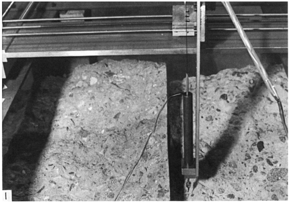


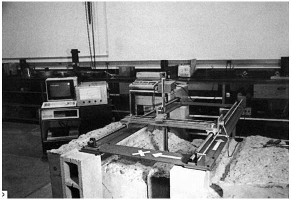
- $N = 4$, $r = 3$, and thus the fractal dimension is $D = ln4/ln1/3 = 1.2619$.
- Sierpinsky carpet, (Example of a Self Similar Lacunar Factal)

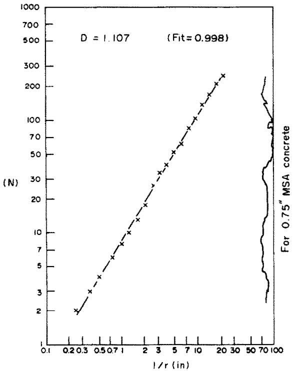
- Here, $N = 8$, $r = 3$, $E = 2$ and thus the fractal dimension is $D = ln8/ln1/3 = 1.8927$.
- In both examples we do have fractal objects as the Hausdorf-Besicovitch dimension D strictly exceeds the topological dimension $D_T = 1$.

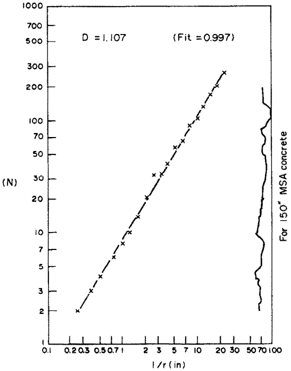
Fractality of Cracked Concrete Surfaces

Saouma, V., Barton, C., and Gamal-El-Din, N. (1990). Fractal Characterization of Cracked Concrete Surfaces. *Engineering Fracture Mechanics Journal*, 35(1):47–53

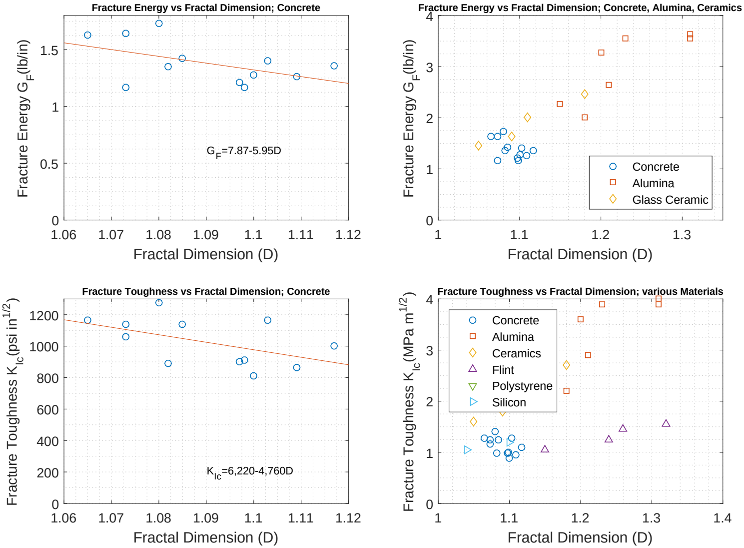








- Nearly identical fractal dimension for concrete with MSA 0.75 and 1.5 inches
- Fractal dimension of profiles normal to the direction of cracking appear to be slightly smaller than for the profiles taken in the direction of cracking.


Fracture Energy vs Fractal Dimension; Concrete
Fracture Energy G_F(lb/in)
Fractal Dimension (D)
G_F=7.87-5.95D
Fracture Energy vs Fractal Dimension; Concrete, Alumina, Ceramics
Fracture Energy G_F(lb/in)
Fractal Dimension (D)
Concrete
Alumina
Glass Ceramic
Fracture Toughness vs Fractal Dimension; Concrete
Fracture Toughness K_Ic(psi in^1/2)
Fractal Dimension (D)
K_Ic=6,220-4,760D
Fracture Toughness vs Fractal Dimension; various Materials
Fracture Toughness K_Ic(MPa m^1/2)
Fractal Dimension (D)
Concrete
Alumina
Ceramics
Flint
Polystyrene
Silicon

Saouma, V. and Barton, C. (1994). Fractals, fractures and size effects in concrete.
Journal of Engineering Mechanics of the American Society of Civil Engineers, 120(4):835–854

Specimen	Profile direction	Profile Distance from Centerline				Average	σ_D%
		1 in.	3 in.	5 in.	7 in.		
3-ft specimens of 1.5-in. MSA							
A	0°	1.096	1.118	1.098	1.113	1.106	1.1
	90°	1.087	1.115	1.101	1.088	1.098	1.1
	+45°	1.096	1.100	1.104	1.123	1.106	1.1
	−45°	1.073	1.109	1.090	1.097	1.092	1.1
B	0°	1.096	1.118	1.064	1.112	1.097	2.2
	90°	1.109	1.100	1.088	1.133	1.107	1.7
	+45°	1.112	1.111	1.073	1.125	1.105	2.0
	−45°	1.094	1.085	1.096	1.085	1.090	0.5
C	0°	1.130	1.128	1.094	1.115	1.117	1.5
	90°	1.092	1.098	1.126	1.106	1.105	1.3
	+45°	1.108	1.087	1.122	1.105	1.105	1.3
	−45°	1.113	1.127	1.101	1.102	1.111	1.1
3-ft specimens of 3.0-in. MSA							
A	0°	1.099	1.107	1.084	1.097	1.097	0.9
	90°	1.123	1.107	1.087	1.129	1.111	1.7
	+45°	1.084	1.071	1.089	1.114	1.089	1.7
	−45°	1.148	1.115	1.094	1.123	1.120	2.0
B	0°	1.147	1.096	1.123	1.069	1.109	3.0
	90°	1.116	1.127	1.100	1.165	1.127	2.5
	+45°	1.104	1.111	1.083	1.110	1.102	1.2
	−45°	1.094	1.087	1.107	1.115	1.101	1.1
C	0°	1.118	1.098	1.113	1.106	1.109	0.8
	90°	1.115	1.101	1.088	1.098	1.100	1.0
	+45°	1.100	1.104	1.123	1.106	1.108	0.9
	−45°	1.109	1.090	1.097	1.092	1.097	0.8

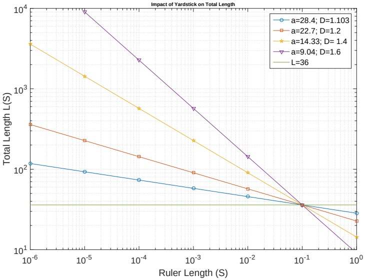
Specimen	Distance from centerline 1 in.	3 in.	5 in.	7 in.	Fractal dimension Average	σ_D%	G_F, [lb/in]	K_{Ic} [psi$\sqrt{\text{in}}$] Average	σ_K%
3-ft specimens of rounded 1.5-in. MSA									
S32A	1.096	1.094	1.096	1.112	1.100	0.8	1.28	812.	10.0
S32B	1.096	1.118	1.064	1.112	1.098	2.2	1.17	909.	5.5
S32C	1.130	1.128	1.094	1.115	1.117	1.5	1.36	1,004.	11.3
Average	1.107	1.113	1.085	1.113	1.105		1.27	908	
3-ft specimens of rounded 3.0-in. MSA									
S33A	1.099	1.107	1.084	1.097	1.097	0.9	1.21	901.	6.3
S33B	1.147	1.096	1.123	1.069	1.109	3.0	1.26	862.	6.4
S33C	1.118	1.098	1.113	1.081	1.103	1.5	1.40	1,166.	7.7
Average	1.121	1.100	1.107	1.082	1.103		1.29	976	
3-ft specimens of 1.5-in. MSA (subangular basalt aggregate)									
SS32A	1.084	1.089	1.096	1.052	1.080	1.8	1.73	1,274.	9.6
SS32B	1.090	1.077	1.096	1.076	1.085	0.9	1.42	1,137.	12.3
Average	1.087	1.083	1.096	1.064	1.083		1.57	1,206.	
5-ft specimens of rounded 1.5-in. MSA									
S52A	1.085	1.074	1.064	1.069	1.073	0.8	1.17	1,058.	6.9
S52B	1.066	1.070	1.072	1.050	1.065	0.9	1.63	1,164.	6.1
S52C	1.082	1.067	1.082	1.059	1.073	1.1	1.64	1,138.	3.5
Average	1.078	1.070	1.073	1.059	1.070		1.48	1,120.	
5-ft specimens of rounded 3.0-in. MSA									
S53A	1.091	1.077	1.088	1.073	1.082	0.8	1.35	893.	13.7
S53B	1.037	1.049	1.060	1.048	1.049	0.9	Not applicable		
Average	1.064	1.063	1.074	1.061	1.065				
5-ft cold joint specimens									
CJ52B	1.050	1.045	1.051	1.027	1.043	1.1	0.46	457.	3.3
CJ53A	1.051	1.064	1.056	1.050	1.055	0.6	0.76	643.	3.9
CJ53C	1.073	1.087	1.070	1.062	1.073	1.0	0.56	567.	2.6
Average	1.058	1.065	1.059	1.046	1.057		0.59	494	

- There is no apparent correlation between the fractal dimension and the profile orientation.
- D, G_F, and K_{Ic} tested higher for concrete than for the cold-jointed specimen.
- Specimens prepared with subangular basalt aggregate had generally higher G_F and K_{Ic} and lower D.
- Linear regression

$$
\begin{aligned}
G_F &= 7.87 - 5.95D \qquad \text{lb/in;} \qquad \chi^2 = 0.258 \\
K_{Ic} &= 6,220 - 4,766D \qquad \text{psi}\sqrt{\text{in}}; \qquad \chi^2 = 0.267
\end{aligned}
$$

  Note low goodness of fit. in using the above equations.
- A lower fractal dimension is synonymous with a higher fracture toughness because there is mostly aggregate rather than bond failure.

- SEL and and a fractal analysis bear many similarities: a) Log-Log plot; b) Size/Scale dependency: Lack of a unique value for length/area or strength, dependency on either ruler or specimen size; c) Singularity of results for very small ruler or very large specimen sizes.
- It can be easily shown that $L(S) = aS^{1-D}$ where$S$ is the ruler length and $L(S)$ the total length $L(S)$.
- For $L = 3$ ft, S =0.1 in, $D = 1.103$ $a = \frac{L}{S^{1-D}}$ =28.40.



- Correct profile length $L^* = (\frac{S^*}{S_0})^{(1-D)} L_0$ where L_0 is the length measured with S_0.
- Hypothesize that $\underbrace{\lim_{d\to\infty} \mathcal{P}^{\text{SEL}}}_{\text{Multiple specimen sizes}} = \underbrace{\lim_{S\to 0} \mathcal{P}^{\text{F}}}_{\text{multiple yardsticks}}$
- This leads to $G_F^* = \left(\frac{S_0}{S^*}\right)^{2(1-D)} G_F$

Comparison between "corrected" G_F^* and G_c values based on [Swartz and Kan, 1992]							
Specimen	Aggregate type	Water/cement	E_c psi $\times 10^6$	K_{Ic} psi$\sqrt{\text{in}}$	G_F lb/in.	G_c lb/in.	G_F^* lb/in.
NC-.64	Crushed Limestone	0.64	4.5	922	0.565	0.190	0.187
HC-.64	Crushed Quartzite	0.64	5.08	1,206	0.824	0.286	0.273
NP-.64	Crushed & Polished Limestone	0.64	4.74	980	0.570	0.203	0.189
NP-.30	Crushed & Polished Limestone	0.30	5.46	1,266	0.727	0.293	0.241
HC-.30	Crushed Quartzite	0.30	5.54	1,523	0.952	0.419	0.315
NC-.30	Crushed Limestone	0.30	6.03	1,308	0.679	0.280	0.225

Wnuk, M. and Yavari, A. (2003). On estimating stress intensity factors and modulus of cohesion for fractal cracks.

Engineering Fracture Mechanics, 70:1659–1674

- K for a fractal crack of length $2a$ and characterized by a fractal singularity α in an infinite plate subjected to a uniform far field stress σ

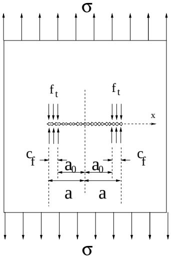
$$K_\sigma^F = \frac{\sigma}{\pi^\alpha a^{\alpha-1}} \underbrace{\int_0^1 \frac{(1+s)^{2\alpha}+(1-s)^{2\alpha}}{(1-s^2)^\alpha} ds}_{\chi(\alpha)}; \qquad s = x/a$$

$$K_{coh}^F = \frac{2f_t'}{\pi}(\pi a)^{1-\alpha}(1-m)^{1-\alpha} \underbrace{\int_0^1 \frac{G(\lambda,\omega,n)}{\{(1-\lambda)[1+(1-m)\lambda+m]\}^\alpha} d\lambda}_{\Gamma(\omega,n,\alpha,m)}$$

- For small scale yielding, $m \to 1$, $K_{coh}^F \to K_\sigma^F$
- For the Euclidian crack, $\alpha = 1/2$, $\chi(1/2) = \pi$.

$$K_\sigma = \sigma\sqrt{\pi a}$$

$$K_{coh} = \left[\frac{\sqrt{2}f_t}{\pi}\sqrt{\pi a}\right]\sqrt{1-m}\cdot\Gamma(\omega,n,\frac{1}{2})$$



Size Effect in Fractal Cracks; Derivation

Saouma, V. and Fava, G. (2006). On fractals and size effects.

International Journal of Fracture, 137:231–249

- As before $K^F_\sigma + K^F_{coh} = 0 \Rightarrow \frac{\sigma}{f'_t} = -2\left(\frac{1}{1+r}\right)^{1-\alpha} \frac{\Gamma(\omega,n,\alpha,m)}{\chi(\alpha)}$
- We recover Bažant's original size effect law for $\alpha = 1/2$, and size effect law is clearly independent of the cohesive stress distribution $\Gamma(\omega, n, \alpha, m)$.
- For small scale yielding $m = \frac{a_0}{a} \to 1 \Rightarrow \frac{\sigma}{f'_t} = -2^{1-\alpha}\left(\frac{1}{1+r}\right)^{1-\alpha} \frac{\Gamma(\omega,n,\alpha)}{\chi(\alpha)}$
- Complex form, $\chi(\alpha)$ can not be explicitly evaluated, take a series expansion with respect to $s = 0$, i.e. for $c_f \ll a$

$$\begin{aligned}
\frac{\sigma}{f_t} &= -\frac{2^{-2\alpha}(-1-r)\left(\frac{1}{1+r}\right)^{2\alpha}\left[2^{2\alpha}\sqrt{\pi}\,\Gamma(1-\alpha) - 2B_{\frac{2r+1}{2(r+1)}}(1-\alpha, 1-\alpha)\Gamma(3/2-\alpha)\right]}{\Gamma(3/2-\alpha)} \\
\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t \\
B_r(z,w) &= \int_0^r t^{z-1}(1-t)^{w-1}\mathrm{d}t
\end{aligned}$$

are the Gamma and Beta functions.

Size Effect in Fractal Cracks; Interpretation

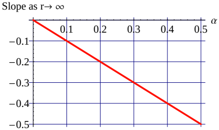
Saouma, V. and Fava, G. (2006). On fractals and size effects.
International Journal of Fracture, 137:231–249

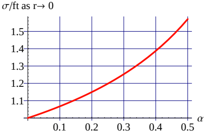
- Slope as $r \to \infty$ is equal to α,
- The -1/2 asymptotic slope of Bažant's original size effect law is recovered, and the strength of the size effect law is reduced for fractal cracks.
- Asymptotic value of σ/f'_t as $r \to 0$,

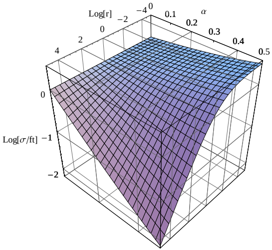
$$\left.\frac{\sigma}{f_t}\right|_{r \longrightarrow 0} = \frac{\sqrt{\pi}\,\Gamma(1-\alpha)}{2\Gamma\left[\frac{3}{2}-\alpha\right]}$$

which for $\alpha = 1/2$ is equal to $\frac{\pi}{2}$.

- This limit value is problematic, as one would have expected to retrieve the value of 1. This discrepancy may be attributed to the approximation of $\chi(\alpha)$; nevertheless this discrepancy requires further investigation.







Motivation

This is about a long journey, started in 1988, searching for the broad interaction between Size Effect, theory, experimental evidence, Fractal Theory, experiment, and what does it all mean.

Still searching.

(Preliminary) Conclusions/Observations

- Most of my research has sought to ultimately seek practical applications.
- The applicability of this presentation is ~ 0 but has been most challenging and rewarding.
- Would not have been possible without the inspirational challenge of Prof. Bažant.

Thank you Ždenek!

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